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SPECTRAL METHODS IN ECONOMICS: A DISCUSSION

BY

Tichaendepi R. Masaya

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Abstract

The role of spectral and cross spectral analysis of time series is emphasized and the mathematical presentation pursued step by step in a simplified fashion. The relationship between this method and econometric methods is pointed out where appropriate.

Problems of non-period phenomena take a large share of the content of the paper.

SPECTRAL METHODS IN ECONOMICS: A DISCUSSION

1 INTRODUCTION

Spectral methods are associated with the analysis of data observed over a period of time. Such data is dependent on time t and a series of readings at each epoch $t_1 < t_2 \ldots < t_n$ is called time series.

Traditionally the variable X_t is decomposed into trend, cycle, seasonal and iregular components for the purpose of studying one or more of these after isolating the rest by using the well known additive or multiplicative models.

The purpose of this paper is to attempt to simplify and clarify the mathematical presentation, to show where certain aspects of spectral analysis are related to regression analysis and to introduce the subject to the interested public.

2. TIME DOMAIN

We may be interested in studying how values of X_t at epoch t are related to future or past epochs $(t+\tau)$, or $(t-\tau)$ respectively, where $\tau = 0 \pm 1 \pm 2 \ldots$ and duration τ is called the lag of the function. For a single series such information is conveyed by the auto-covariance function.

$$(2.1) \quad C_{xx}(\tau) = \frac{1}{n} \sum_{\tau=1}^{n-\tau} (x_t - \bar{x})(x_{t+\tau} - \bar{x})$$

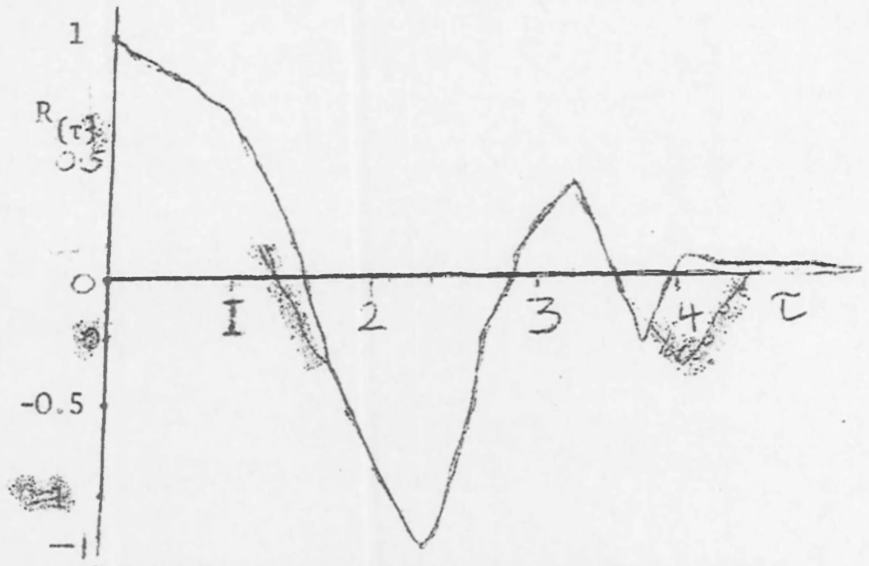
where $\bar{x} = \frac{1}{n} \sum_{t=1}^{n} x_t$. Clearly $C_{(o)}$ is the variance of the sample. When $C_{(\tau)}$ is normalized by dividing each covariance by $C_{(o)}$ the result is auto-correlation function and a plot of

$$(2.2) \quad C_x(\tau) / C_{(o)} = R_x(\tau) \quad (\tau = 0,1, \ldots M)$$ against τ is known as the correlogram. M is the maximal lag or the truncation point.

The correlogram starts with a value of 1 at $\tau = 0$ and vanishes to a zero value as we move into distant past or unforeseeable future. This is in line with historical or natural facts, namely that memory is lost as we detatch ourselves more and more from the past. Figure 1 is a typical, example.

Fig. 1 correlogram

(Auto-correlation function)



Taking τ as one of one year spacing the interpretation of Fig.1 would be that the variation of the series about its long term trend has a strong positive association between values observed for one year and three years apart and negative association at two and three and
a half years apart. There is negligeable correlation after four years.

On studying two (or more) series X_t and Y_t a similar procedure is employed except that

(2.3) $C_{yx}(\tau) = {}^1/_n \sum_{\tau=1}^{n-\tau} (x_t - \bar{x})(y_{t+\tau} - \bar{Y})$ and

(2.4) $C_{xy}(\tau) = \frac{1}{n} \sum_{\tau=1}^{n-\tau} (Y_t - \bar{Y})(x_{t+\tau} - \bar{x})$ are now called cross covariances between X_t and Y_t while the normalized

(2.5) $$\frac{C_{yx}(\tau)}{\left[\{C_{x(o)}\}.\{C_{y(o)}\}\right]^{\frac{1}{2}}} = R_{yx}(\tau) \text{ or } \frac{C_{xy}(\tau)}{\left[\{C_{x(o)}\}\{C_{y(o)}\}\right]^{\frac{1}{2}}} = R_{xy}(\tau)$$

is now known as the cross correlogram. $R_{yx}(\tau)$ is not usually the same as $Rxy(\tau)$ since the former implies that X_t causes Yt and the latter case Yt causes X_t.

When the above method is used we say a time series is described in the time domain. The usual procedure is to detrend the data before subjecting the residuals to the method just described.

3. FREQUENCY DOMAIN

If X_t is a periodic series depending on time with a period $P = 2\pi/\omega$ it can be expanded as series of quantities varying harmonically in the form

$$(3.1)\quad f(x_t) = a_0/_2 + \sum_{t=1}^{\infty} (a_t \operatorname{Cos} \omega t + b_t \sin \omega t)$$

or ignoring the mean value (and the phase) as

$$(3.2)\quad f(x_t) = \sum_{t-1}^{M} (a_t \operatorname{Cos} \omega t + b_t \operatorname{Sin} \omega t) \text{ where}$$

$$\omega = \frac{2\pi}{T} \quad \text{for } P = T.$$ Thus the function is completely defined by its Fourier coefficients a_t and b_t and its angular frequency ω.

If X_t is a stationary zero mean Gaussian stochastic process the probability law governing this random process is time invariant. This means, among other things, that the auto-correlation function depends only on lag τ and not on epoch t. This is confirmed as follows; using (3.2) we assume

$$(3.3)\ (a)\quad E(a_j a_k) = E(b_j b_k) = \sigma_j^2 \quad \text{if } j = k$$

$$= 0 \quad \text{if } j \neq k$$

$$(b)\quad E(a_j b_j) = E(a_k b_j) = E(a_j b_k) = 0.$$

The auto-covariance of X_t is

$$(3.3)\quad (c)\quad C(\tau) = E\,(x_t x_{t+\tau})$$

$$= E\left[\sum_{j-m}^{m}\sum_{k=-m}^{m}\left\{a_j \operatorname{Cos}\omega_j t + b_j \operatorname{Sin}\omega_j t\right\}\left\{a_k \operatorname{Cos}\omega_j (t+\tau) + b_k \operatorname{Sin}\omega_k (t+\tau)\right\}\right]$$

$$= E\left[\sum_{j=-m}^{m}\sum_{k=-m}^{mm}\left\{a_j \operatorname{Cos}\omega_j t \left(a_k \operatorname{Cos}\omega_k (t+\tau) + b_k \operatorname{Sin}\omega_k (t+\right)\right.\right.$$

$$= E\left[\sum_{j=-m}^{m}\sum_{k=m}^{m}\left\{\alpha_j\,\alpha_k \operatorname{Cos}w_j t\left(\operatorname{Cos}w_k(t+\tau)\right)+\alpha_j\,b_j \operatorname{Cos}w_j t \operatorname{Sin}w_k(t+\tau)\right.\right.$$

$$+ b_j \operatorname{Sin}w_j t \quad \alpha_k \operatorname{Cos}w_k(t+\tau)+b_k \operatorname{Sin}w_k(t+\tau)$$

$$\left.\left.+b_j\alpha_k \operatorname{Sin}w_j t \operatorname{Cos}(t+\tau) + b_j b_k \operatorname{Sin}w_j t\left(\operatorname{Sin}w_k(t+\tau)\right)\right\}\right]$$

$$= \left[\sum_{j=-m}^{m}\sum_{k=-m}^{m}\left\{\sigma_j^2 \operatorname{Cos}w_j t\left(\operatorname{Cos}w_k(t+\tau)\right)+\sigma_j^2 \operatorname{Sin}w_j t\left(\operatorname{Sin}w_k(t+\tau)\right)\right\}\right]$$

$$= \left[\sum_{j=-m}^{m}\sum_{k=-m}^{m}\left\{\sigma_j^2 \operatorname{Cos}w_j t\left(\operatorname{Cos}w_k t \operatorname{Cos}w_k\tau-\operatorname{Sin}w_k t \operatorname{Sin}w_k\tau\right)\right.\right.$$

$$\left.\left.+\operatorname{Sin}w_j t\left(\operatorname{Sin}w_k t \operatorname{Cos}w_k\tau+\operatorname{Cos}w_k t \operatorname{Sin}w_k\tau\right)\right\}\right]$$

$$= \left[\sum_{\substack{j=-m\\ j=k}}^{m}\sigma_j^2\left\{\operatorname{Cos}^2 w_j t \operatorname{Cos}w_j T-\operatorname{Cos}w_j t \operatorname{Sin}w_j t \operatorname{Sin}w_j\tau\right.\right.$$

$$\left.\left.+\operatorname{Sin}^2 w_j t \operatorname{Cos}\omega_j\tau + \operatorname{Sin}w_j t \operatorname{Cos}w_k t \operatorname{Sin}w_k\tau\right\}\right]$$

$$= \left[\sum_{j=-m}^{m}\sigma_j^2 \operatorname{Cow}_j\tau\left\{\operatorname{Cos}^2 w_j t+\operatorname{Sin}^2 w_j t\right\}\right]$$

$$= \sum_{j=-m}^{-m}\sigma_j^2 \operatorname{Cos}\omega_j\tau .$$

This expression depends is independent of time as was claimed.We may write $\{3.3\ (c)\}$ as

(3.4) $$C_{(\tau)} = \sum_{j=-m}^{m}\sigma_j^2 \operatorname{Cos}\omega_j\tau = \sum_{j=-m}^{m}\sigma_j^2 e^{i\omega_j\tau} .$$

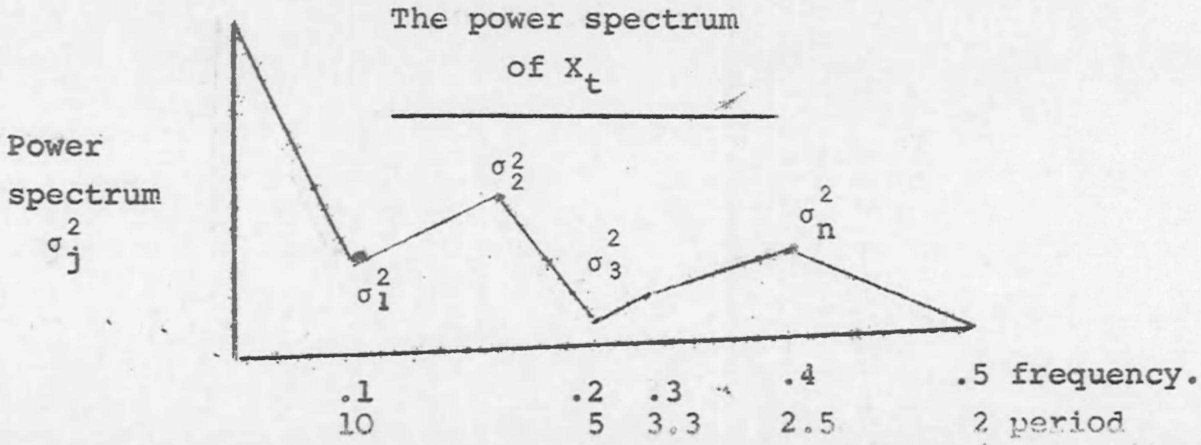
or

(3.5) $$\sigma_j^2 = \sum_{j=-m}^{m} C(T)\, e^{-i\omega_j\tau} .$$

σ_j^2 is an amplitude associated with frequency j and a plot of σ_j^2 against each frequency is known as the power spectrum.

Time series analyzed using this methom is said to have been described in the frequency domain. The spectrum may be as shown in figure 2.

Fig 2.
The power spectrum
of X_t



It is shown by (3.4) that at zero frequency the spectrum is total variance. This indicates that, essentially, studying the spectrum of a time series is an attempt to investigate the contribution to total variation by different frequencies.

Formulae (3.4) and (3.5) bring out the fact that the autocovariance function and the spectrum form a Fourier transform pair and one gets as much information from the former as from the latter. However spectral analysis in a wider sense has more advantages than its counter part.

For the actual estimates of the spectral densities one would re-write (3.5) in the form

$$(3.6)\quad \hat{f}_{xx}(w_j) = \frac{1}{T} \sum_{\tau=-m}^{m} C(\tau) e^{-i\omega_j \tau}$$

$$= \frac{1}{\pi}\left[\sum_{\tau=-M}^{M}\left\{C(\tau)\left(\mathrm{Cos}\,\omega_j\tau - i\,\mathrm{Sin}\,\omega_j\tau\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[\sum_{\tau=-M}^{1} C(\tau)\mathrm{Cos}\,\omega_j T + C(\tau)\sum_{\tau=-M}^{1} \mathrm{Sin}\,\omega_j\tau + C(o)\right.$$

$$\left. + C(T)\sum_{\tau=1}^{M} \mathrm{Cos}\,\omega_j\tau - \sum_{\tau=1}^{M} C(\tau)\mathrm{Sin}\,\omega_j\tau\right]$$

$$= \frac{1}{2\Pi}\left[C(o) + 2\sum_{\tau=1}^{M} C(\tau)\ \mathrm{Cos}\,\omega_j\tau\right]$$

$$= \frac{c(o)}{2\,\Pi} + \frac{1}{\pi}\sum_{\tau=1}^{M} C(\tau)\mathrm{Cos}\,\omega_j\tau \qquad \text{Since } \mathrm{Cos}\,\omega_j T$$

is an even function and $\mathrm{Sin}\,\omega_j$ is odd while $\mathrm{Sin}\,\Pi = 0$, $\omega_j = \frac{2\Pi j}{2M}$

$= \frac{\Pi j}{M}$, $j, \tau = o, 1, \ldots M$

M is the number of covariances computed.

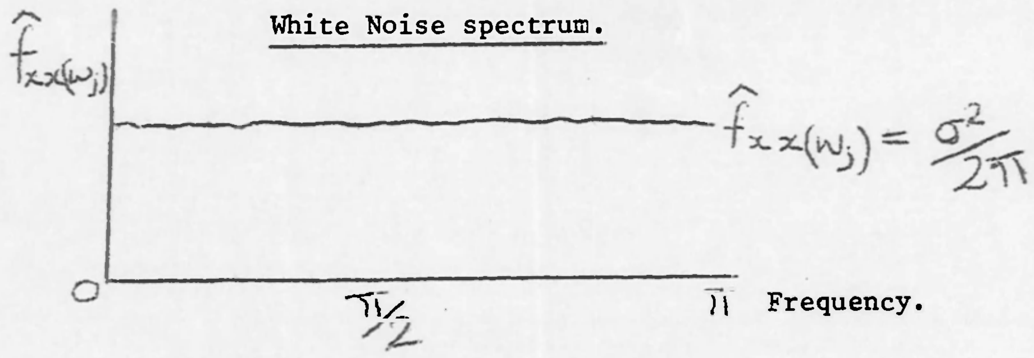
Looking at (3.6) we observe that when the observations are not correlated the second term on the right will vanish and the spectrum becomes (3.7) $f_{xx}(\omega_j) = Co/_{2\Pi} = \frac{\sigma^2}{2\Pi}$ at all frequencies.

We call (3.7) white noise because of its similarity to background noise when listening to a programme on a radio.

The picture is as in figure 3.

Fig. 3

White Noise spectrum.



4. AVERAGING THE SPECTRUM

The spectral density function just define is only applicable to strictly periodic functions. Hardly do we experience any repetition of economic phenomena lasting for a long time. Unemployment, for instance, is usually corrected by public policy instruments before it is allowed to pursue its own course. Even in the realm of natural setting do events repeat themselves indefinitely. It is true that occations do arise . when the outcome of events may well be represented by a sine-cosine model during the period of the experiment (and possibly for sufficiently long periods before and after this experiment) before the forces causing the phenomena are qualitatively transformed into another physical being governed by different laws of behavior.

Suppose that x_t represents the pressure on a commodity due to random arrival (and decisions) of the public at a market place, x_t will vary in a random fashion overtime. We are therefore unable to predict in advance (except for the trend) the quantity assumed by $\bar{x}_t$ at any epoch. It is also true that any relationship between the magnitude of the quantity and time, measured during a certain duration, will never recur in any other. However, a large class of stochastic processes can be described as completely as possible if they possess stationary properties.

This calls for an adjustment of (3.6). The formula as it stands is equivalent to the famous chuster's periodogram[1]; which when used under the assumption that X_t is periodic, sometimes showspeaks even for processes known to be of white Noise form. Thus if we used the

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1. A.Schuster: "The Periodogram of the Magnetic Declination as obtain from the Records of the Greenwich Observatory during the year 1811-1895." Transactions of the Cambridge Philosophical Society. Vol. 18 (1899).

Periodogram - to estimate the "running" spectral densities such as the one associated with the data the economist is accustomed to, it will turn out that although the estimates are unbiased asymptotically, they are not consistent. It has been shown by research workers in this field that as sample size approaches infinity the variance of the ordinates of the periodogram tend to $\sigma^4 f_{xx}^2(\omega_j)$. It is this behavior that partially accounts for the existence of peaks to what otherwise would have been a flat spectrum.

G.M. Jenkins (1961) has given an example in which it is always found that when estimating a probability density function using grouped data "the estimated density function becomes very erratic if the group interval for the histogram is too small". Since on the average the periodogram is the same as the power spectrum, improvements can be made by choosing the correct group interval with the frequency of interest as the mid-point. This is equivalent to averaging power within each frequency band with more weight given to the power with frequencies near the central frequency. These aims are realized either in the realm of time or that of frequency. A weighting scheme related to the former is known as the LAG WINDOW and its inverse transform related to the latter case SPECTRAL WINDOW.

Many windows have been suggested but one by Parzen has certain statistical advantages. These are (in the time domain):

$$(4.7)\quad \lambda_\tau = \begin{cases} 1-6\dfrac{\tau^2}{M}\left(1-\dfrac{\tau}{M}\right) & \text{for } 0 \leq \tau \leq \dfrac{M}{2} \\ 2\left(1-\dfrac{\tau}{M}\right)^3 & \text{for } \dfrac{M}{2} \leq \tau \leq M \\ 0 & \text{else where} \end{cases}$$

and (4.8) $A(\omega) = \frac{3}{8\pi M}3\left[\frac{\text{Sin}(M\omega/4}{\text{Sin}(\omega/4}\right]$ in the

frequency domain. M is the truncation point.

Formula (3.6) now becomes

$$(4.9)\quad \hat{f}_{xx}(\omega_j) = \lambda_o C(o)/2\pi + \frac{1}{\pi}\sum_{\tau=1}^{M} \lambda_\tau C(\tau)\,\text{Cos}\,\omega_j \tau$$

with λ_τ as defined in (4.7 and $\lambda_o = 1$.

The estimates are now both unbiased and consistent.

It may be asked how many lags should be used in the estimation of the spectrum. The answer is there is no prescribed number. It has been suggested that the number should be about 20 to 30 percent of the sample size. This indeed is a rule of thumb.

One would do well by observing the manner in which the auto-or cross covariance function dies out with time Jenkins (1961) p. 159 . It should be rembered, however, that the more the number of lags used the better is the forecasting power of the estimates. These contradictions have to be compromised.

5 CROSS SPECTRUM AND RELATED TOPICS

The above discussion confines itself to spectral analysis of one series. This is helpful if one is interested in just building an econometric model of the type commonly known as auto-regressive scheme.

In most cases it is known before hand that two series X_t and Y_t are related. When this is true it would be inadequate to treat them separately using spectral methods as formulated above. The method used to study the relationship between two series is known as CROSS SPECTRAL analysis.

As a starting point we follow Nerlove (1964). The problem is as follows: a time series X_t is governed by the trend, cyclical, seasonal and irregular components. The presents of the trend "destroys" the assumption of stationarity. Further, the long term component introduces excessive power in the lower frequency area which leaks into the neighbouring frequencies and thereby distort spectral estimates. The thing to do is to eliminate the trend using one of the available time-invariant linear filters so that the residual series can be treated as a second order stationary stochastic process. Further, it pays to compute the mean of the residuals and deduct it from each observation. The whole procedure is known as filtering or pre - whitening of the original series X_t.

The pre-whitened series may now be considered as a new series Y_t, the output series, related to X_t, the input series. Cross spectral analysis investigates this relationship. Before we go into this it is necessary to define the meaning of a time invariant linear filter. A filter is considered linear if when applied to an input x_{1t} results in an output y_{1t} and a second input x_{2t} has a corresponding output y_{2t} and for arbitrary constants α and B, $\alpha x_{1t}+Bx_{2t}$ has an output $\alpha y_{1t} + By_{2t}$. In addition the same filter is time invariant if we have the input $\{x_t : t\varepsilon(-\infty, +\infty)\}$ corresponding to the output $\{Y_t : t\varepsilon(\infty,+\infty)\}$ and another observed output $\{x_{t+\tau} : t\varepsilon(-\infty, +\infty)\}$ has the output $\{Y_{t+\tau} : t\varepsilon(-\infty,+\infty)\}$, where τ is a fixed time delay. Verbally this means that the probability structure of x_t remains stable over a sufficient length of time.

We now wish to see how the input function is related to the output function Let $k(\tau)$ be the kernel of the filter then (5.10) $Y_t = \int_{-\infty}^{\infty} k(\tau)\, x(t-\tau)\, d\tau$. Expressing the input function as a complex harmonic amplitude A, zero phase and angular frequency $\omega = \frac{2\pi}{p}$ where p is the period of the sinusoid, we have

(5.11) $x_{t-\tau} = Ae^{\,i[\omega(t-\tau)+x=o]} = Ae^{i\omega(t-\tau)}$.

Substitution (5.11) into (5.10) the result is

(5.12) $Y_t = \int_{-\infty}^{\infty} k(\tau)\, Ae^{i\omega(t-\tau)}\, d\tau$

$$= \int_{-\infty}^{\infty} k(\tau) A^{i\omega t\ -i\omega\tau}\, d\tau.$$

$$= \int_{-\infty}^{\infty} k(\tau) Ae^{i\omega t}\, e^{-i\omega\tau}\, d\tau$$

$$= Ae^{i\omega t}\int_{-\infty}^{\infty} k(\tau) e^{-i\omega\tau}\, d\tau$$

$$= \phi(\omega)\, Ae^{i\omega t}$$

(5.13) $\phi(\omega) = \int_{-\infty}^{\infty} k(\tau) e^{-i\omega\tau}$ is the frequency response function of the filter which is a Fourier transform of the kernel of the filter and is a complex valued function of frequency ω. Hence, the effect of a filter in producing the output function is to amplify or attinuate the amplitude of the input function by a Fourier transform of the kernel of the filter. This fact is brought out more clearly by combining (5.11) and (5.12) into (5.14) $A\phi(\omega)e^{i\omega t} = A\phi(\omega)X_{t-\tau=o} = A\phi(\omega)X_t$ which is the input series at epoch t. For a descrete series (3.13) is (5.15) $\phi(\omega) = \sum_{\tau=-M}^{M} k(\tau)e^{-i\omega\tau}$

The square of the frequency response function is the transfer function associated with the filter; namely

(5.16) $T(\omega) = |\phi(\omega)|^2$ This becomes useful once the method of pre-whitening has been specified. One of the methods used is of the form. (5.17) $\Delta_B x_t = x_t - Bx_{t-1}$ and $0<B<1$. clearly, (5.17) is a first difference equation if B=1. When B is as defined in (5.17) it becomes a first quasi-difference Nerlove (B 61).

However the order and the value of B within the specified limits are to be chosen by the user.

We now proceed to re-write (5.17) as

(5.18) $x_t - Bx_{t-1} = x_t - BL^{-1}x_t = (1-BL^{-1})x_t$, where L^{-1} is a linear operator.

(5.19) $x_t - Bx_{t-1} = Y_t$ from (5.19) and therefore (5.20) $Y_t = (1-BL^{-1})x_t$

where $(1-BL^{-1})$ is the kernel of the filter at $\tau = o$ and $A=1$.
This is a result of the first order quasi-differencing. For the K^{th} order quasi-difference we have

(5.21) $$\left(1-BL^{-1}\right)^k = \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}L^{-\tau} = \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}x_{t-\tau}$$
$$= \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}e^{i\omega(t-\tau)} = \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}e^{i\omega t}e^{-i\omega\tau}$$

This may further be written as

(5.22) $$\left(1-BL^{-1}\right)^k e^{-i.\omega t} = \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}e^{-i\omega\tau}.$$

The right hand side of this formula looks more like (5.13) with the intergral sign replaced by the summation sign.
We now formerly write the frequency response function of (5.13) for the K^{th} order quasi-difference filter as

(5.23) $$\phi(\omega) = \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}e^{-i\omega\tau} = \sum_{\tau=o}^{k}\binom{k}{\tau}(-B)^{\tau}\left[e^{-i\omega}\right]^{\tau} = \left[1-Be^{-i\omega}\right]^k$$

By (5.16) the transfer function associated with this kind of filter is

(5.24) $$T(\omega) = \left[\phi(\omega)\right]^2 = \left[1-Be^{-i\omega}\right]^k\left[1-Be^{i\omega}\right]^k$$
$$= \left[\left(1-Be^{-i\omega}\right)\left(1-Be^{i\omega}\right)\right]^k$$
$$= \left[1-Be^{i\omega}-Be^{-i\omega}+B^2e^{-i\omega}e^{i\omega}\right]^k$$
$$= \left[1-B(e^{i\omega}+e^{-i\omega})+B^2\right]^k$$
$$= \left[1-B\cdot(\text{Cos}\omega+i\sin\omega+\text{Cos}\omega-i\text{Sin}\omega)+B^2\right]^k$$
$$= \left[1-B(2\text{Cos}\omega)+B^2\right]^k$$
$$= \left[1-2B\text{Cos}\omega+B^2\right]^k.$$

One may ask whether or not the hard road along which we have travelled to get to (5.24) is worth the effort. The answer is in the affirmative for it can be shown that at the jth frequency the spectrum of the output series $f_{yy}(\omega_j)$ is related to that of the input series $f_{xx}(\omega_j)$ by

(5.25) $f_{yy}(\omega_j) = T(\omega_j)\, f_{xx}(\omega_j)$. This shows that when the spectrum of the prewhitened series has been estimated it is possible to recover the spectrum of the original series by dividing the former by the transfer function. The spectral estimates are said to have been recoloured. The spectrum of x_t is thus found by (5.26) $f_{xx}(\omega_j) = \left[T(\omega_j)\right]^{-1} f_{yy}(\omega_j)$. The transfer function is forthcoming from (5.24) if that method of filtering has been used.

Extending our knowledge of a single series spectrum to two stochastic variables x_t and y_t we present their cross spectrum as

(5.27) $\hat{f}_{yx}(\omega_j) = \frac{1}{\Pi} \sum_{\tau=-M}^{M} \frac{1}{2}\left(C_{yx}(\tau) + C_{xy}(\tau)\right) e^{-i\omega_j\tau}$

$$= \frac{1}{\Pi} \sum_{\tau=-M}^{M} \frac{1}{2}\left(C_{yx}(\tau) + C_{xy}(\tau)\right)\left(\operatorname{Cos}\omega_j\tau - i\operatorname{Sin}\omega_j\tau\right)$$

$$= \frac{1}{2\Pi}\left[\frac{1}{2}\left\{C_{yx}(o) + C_{xy}(o)\right\} + 2\sum_{\tau=1}^{M} \operatorname{Cos}\omega_j\tau \right.$$

$$\left. -i\,2\sum_{\tau=1}^{M} \frac{1}{2}\left\{C_{yx}(\tau) - C_{xy}(\tau)\right\}\operatorname{Sin}\omega_j\tau\right]$$

$$= \frac{1}{4\Pi}\left[C_{yx}(o) + C_{xy}(o)\right] + \frac{1}{2\Pi}\sum_{\tau=1}^{M}\lambda_\tau\left[C_{yx}(\tau) + C_{xy}(\tau)\right]\operatorname{Cos}\omega_j\tau$$

$- \frac{1}{2\Pi}\sum_{\tau=1}^{M}\lambda_\tau\left[C_{yx}(\tau) - C_{xy}(\tau)\right]\operatorname{Sin}\omega_j\tau$, where $C_{yx}(\tau)$ and $C_{xy}(\tau)$ are as defined in (2.3) and (2.4) respectively, while remembering that if past values of x_t cause future values of y_t it may not follow that the reverse is the same. Thus the difference between pairs of these two will not all assume zero values.

The power spectrum in (5.27) is divided into two parts:

(5.28) $\hat{C}_{yx}(\omega_j) = \lambda o/4\Pi\left[C_{yx}(o) + C_{xy}(o)\right] + \frac{1}{2\Pi}\sum_{\tau=1}^{M}\lambda_\tau\left[C_{xy}(\tau) + C_{yx}(\tau)\right]\operatorname{Cos}\omega_j\tau$

and

(5.29) $\hat{q}_{yx}(\omega_j) = \frac{1}{2\Pi}\sum_{t=1}^{M}\lambda_\tau\left[C_{xy}(\tau) - C_{yx}(\tau)\right]\operatorname{Sin}\omega_j\tau$ where λ_τ is a lag window. (5.28) is called the cospectrum of x_t and y_t and measures the spectrum of the in phase spectrum of the two random variables. (3.29) is the quadrature spectrum measuring the spectrum of the same random variables 90 degree out of phase.

In the analysis of two time series it is of great interest to know:

(1) to what extent are the two variables related at each frequency?

(2) What share does x_t take of y_t at a given frequency?

(3) By how many units does one variable lag behind the other?

The answers to these questions are important for policy-making,

The measure used for (1) is the coherency and is equivalent to the coefficient of determination in regression analysis, (2) is the gain equivalent to regression coefficient and (3) is the phase angle-.

The coherency is given by

$$(5.30) \quad \hat{k}_{xy}(\omega_j) = \frac{\hat{C}_{yx}(\omega_j)^2 + \hat{q}_{yx}(\omega_j)^2}{\left[\hat{f}_{xx}(\omega_j)\right]\left[\hat{f}_{yy}(\omega_j)\right]}$$

To show that the gain is similar to the regression coefficient we multiply (5.25) by $\hat{f}_{xx}(\omega_j)$ on both sides to get

$$(5.31) \quad \hat{f}_{yy}(\omega_j)\hat{f}_{xx}(\omega_j) = T(\omega_j) \left[\hat{f}_{xx}(\omega_j)\right]^2 \text{ and}$$

$$(5.32) \quad T(\omega_j) = \frac{\hat{f}_{yy}(\omega_j)\hat{f}_{xx}(\omega_j)}{\left[\hat{f}_{xx}(\omega_j)\right]^2}$$

This looks much like the regression coefficient in which x_t is the independent variable. The square root of the transfer function is the gain namely

$$(5.33) \quad \hat{g}(\omega_j) = \left[T(\omega_j)\right]^{\frac{1}{2}} = \frac{\left[\hat{f}_{xx}(\omega_j)f_{yy}(\omega_j)\right]^{\frac{1}{2}}}{\hat{f}_{xx}(\omega_j)}$$

$$= \frac{\left[\hat{C}_{yx}(\omega_j)^2 + \hat{q}(\omega_j)^2\right]^{\frac{1}{2}}}{\hat{f}_{xx}(\omega_j)}$$

The phase angle is computed from the estimates of the cospectrum and the quadrature spectrum;

$$(5.34) \quad \hat{\chi}(\omega_j) = \arctan\left[-\frac{\hat{q}_{yx}(\omega_j)}{\hat{C}_{yx}(\omega_j)}\right] = \tan^{-1} \frac{\hat{q}_{yx}(\omega_j)}{\hat{C}_{yx}(\omega_j)}$$

6. CONCLUSION

This discussion has attempted to show that time series may be analyzed both in the time domain and frequency domain. It became clear that the latter has more advantages than the former although the advantages are more appreciated during concrete application. Spectral analysis is of further help to the econometrician interested in model building.

An attempt was also made to show where certain aspects of spectral analysis were related to regression analysis and at the same time trying to simplify the mathematical presentation.

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