A Simplified Formalism for Computing Growth Rates for Perturbations in Standard Cosmology

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We present a simplified formalism for computing growth rates for perturbations within standard cosmology. We show that converting the perturbation evolution equations into ordinary differential equations in terms of D-functions, related to the growth functions, not only simplifies the equations, but also eliminates the commonly encountered complex problem of computing initial conditions in cosmological studies, avoiding making any assumptions - as is usually done - when setting up initial conditions for the perturbations.

1. Introduction

Structure formation in the Universe is well described by the cosmological linear perturbation theory, which predicts that the currently observed structures in the Universe are assumed to have formed from small perturbations in the baby universe that grew in size via gravitational attraction. The currently accepted theory of our universe is Einstein's theory of general relativity, which connects the geometry of the universe to the matter content and is summarized by the field equations (EFEs):

$$G_{\nu}^{\mu} = 8\pi G T_{\nu}^{\mu} \qquad (1)$$

Where, G_{ν}^{μ} is the Einstein tensor, G is Newton's gravitational constant and T_{ν}^{μ} is the energy momentum tensor.

In standard cosmological perturbation theory the formation of structure in the universe is described by the perturbed form of Eqn. (1), which becomes

$$\overline{G}_{\nu}^{\mu} + \delta G_{\nu}^{\mu} = 8\pi G(\overline{T}_{\nu}^{\mu} + \delta T_{\nu}^{\mu}) \qquad (2)$$

Here, bars on symbols stand for background quantities. Perturbations in Eqn. (2) are

$$\delta G_{\nu}^{\mu} = 8\pi G \delta T_{\nu}^{\mu} \qquad (3)$$

Since the perturbed Einstein tensor on the left-hand side of Eqn. (3) contains information about the geometry of the Universe, while the perturbed energy-momentum tensor on the right hand side carries information about the energy content of the Universe, therefore, in order to obtain the equations describing the perturbation dynamics, we require:

- The perturbed line element in a gauge of choice and
- A form for the perturbed energy-momentum tensor.

We define the perturbed line element in the widely used Newtonian gauge, which takes the form

$$ds^2 = -(1-2\Phi)dt^2 + (1+2\Phi)\delta_{ij}dx^i dx^j \qquad (4)$$

Where, Φ is the gravitational potential given by the relativistic Poisson equation.

We assume the form of energy-momentum tensor to be that of the commonly used perfect fluid, which is given as

$$T_{\nu}^{\mu} = (\rho + p)u^{\mu}u_{\nu} + p\delta_{\nu}^{\mu} \qquad (5)$$

Where, ρ and p are the perturbed density and pressure respectively given by

$$\rho = \overline{\rho} + \delta\rho \qquad (6)$$

$$p = \overline{p} + \delta p \qquad (7)$$

And, $\delta\rho$ and δp are the density and pressure perturbations.

The four-velocities are defined by [1,2,3] as

$$u^{\mu} = a^{-1}(1-\Phi, \partial^i \mathbf{v}) \qquad (8)$$

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$$u_v = a(-1-\Phi, \partial_i \mathbf{v}) \quad (9)$$

Where, $\mathbf{v}$ is the peculiar velocity.

From the perturbed EFE's (Eqn. (3)) and the perturbed metric, one obtains the evolution equation for the gravitational potential Φ as

$$\Phi' + \mathcal{H}\Phi = -\frac{3}{2}H^2\Omega_m \mathbf{v} \quad (10)$$

Where the 'prime' denotes the conformal time derivative and $\mathcal{H} = aH$, where H is the Hubble parameter.

The perturbed energy-momentum tensor conservation condition $\nabla_\mu T^{\mu\nu} = 0$ yields the energy and momentum conservation equations

$$\delta' + 3\mathcal{H}\Phi - k^2\mathbf{v} + \frac{9}{2}\mathcal{H}^2\Omega_m \mathbf{v} = 0 \quad (11)$$

$$\mathbf{v}' + \mathcal{H}\mathbf{v} + \Phi = 0 \quad (12)$$

Hence Eqns. (8) and (9) are the evolution equations describing the growth of cosmological perturbations.

Introducing the co-moving density contrast [1,2]

$$\Delta = \delta - 3\mathcal{H}\mathbf{v} \quad (13)$$

and using the definition of the conformal time $' = d/d\eta = a\mathcal{H} = a^2 H$ where H is the Hubble rate, Eqn. (9) reduces to the simple ordinary differential equation (ODE)

$$\frac{d\Delta}{da} = \frac{l^2 u}{a^2 E} \quad (14)$$

Where, $l = k/H_0$ is the normalized wave number, $u = H_0\mathbf{v}$ is the dimensionless peculiar velocity and $E = H/H_0$ is the dimensionless Hubble rate.

Similarly, the ODE's for Φ and u become

$$\frac{du}{da} = -\frac{\Phi}{a^2 E} - \frac{u}{a} \quad (15)$$

and

$$\frac{d\Phi}{da} = -\frac{3}{2}E\Omega_m u - \frac{u}{a} \quad (16)$$

Structure formation in the Universe occurred after decoupling at red-shift $z = 1100$ or scale factor $a = 10^{-3}$. Thus in order to study the evolution of cosmic structures, one needs to integrate the perturbation equations using initial conditions set by decoupling i.e., u_d, Φ_d, Δ_d (where 'd' denotes quantities at decoupling). However, the problem of computing initial conditions for the perturbation is quite complex and usually require a number of assumptions given our poor knowledge about physics of the very early universe. In this paper, we develop a formalism that fixes the problem of initial conditions and provides a simplified technique to compute structural growth rates within the standard cosmological model.

2. The Growth Rates

If we define the the D-functions:

$$\tilde{D}_\Phi(k,a) = \frac{\Phi(k,a)}{\Phi_d(k)} \quad (17)$$

$$\tilde{D}_\Delta(k,a) = \frac{\Delta(k,a)}{\Delta_d(k)} \quad (18)$$

$$\tilde{D}_u(k,a) = \frac{u(k,a)}{u_d(k)} \quad (19)$$

Where it is easy to show that [1]

$$\Delta_d = -\frac{21^2\Phi_d}{a_d^2\Omega_d E_d^2} \quad (20)$$

$$u_d = -\frac{2\Phi}{3a_d\Omega_d E_d^2} \quad (21)$$

$$\Phi_d = \frac{9}{10}T(k)\Phi_p(k) \quad (22)$$

Here $T(k)$ is the transfer function for the perturbations from the radiation dominated era into the matter domination epoch and $\Phi_p(k)$ is the primordial scalar potential from inflation [5]. In the derivation presented here, we used the well known transfer function from [5] while $\Phi_p(k)$ is given by the equation

$$\Phi_p(k) = \frac{\Omega_0}{D_\Phi(k, a=1)}\left(\frac{k}{H_0}\right)^{(n-4)/2} \quad (23)$$

Where, n is the scalar spectroscopic index and Ω_0

is the value of the matter density in the Universe today.

Using the definitions (17)-(19), Eqns. (14)-(16) yield the D-functions ODE's:

$$\frac{d\tilde{D}_\Delta}{da} = \frac{l^2}{a^2 E} I_{u\Delta} \tilde{D}_u \tag{24}$$

$$\frac{d\tilde{D}_u}{da} = -\frac{l1}{a^2 E} I_{\Phi u} \tilde{D}_\Phi - \frac{\tilde{D}_u}{a} \tag{25}$$

$$\frac{d\tilde{D}_\Phi}{da} = -\frac{3}{2} E\Omega I_{u\Phi} \tilde{D}_u - \frac{\tilde{D}_\Phi}{a} \tag{26}$$

Where, we have introduced the definitions

$$I_{u\Delta} = \frac{u_d}{\Delta_d}$$
$$I_{\Phi_u} = \frac{\Phi_d}{u_d} = I_{\Phi u}^{-1} \tag{27}$$

We note that from Eqns. (17)-(19) the initial conditions required to solve the ODE's (Eqns. (20)-(22)) simplify to

$$\tilde{D}_{\Delta d} = \tilde{D}_{ud} = \tilde{D}_{\Phi d} = 1 \tag{28}$$

Thus using Eqns. (24), the ODE's (Eqns. (20)-(22)) can be integrated numerically to yield the functions $D_A (A = \Delta, u, \Phi)$.

The growth rates for the perturbations, Δ , u and Φ can be related to the D-functions D_A via the equations

$$D_\Delta(k,a) = a_d \tilde{D}_\Delta(k,a) \tag{29}$$

$$D_\Phi(k,a) = a\tilde{D}_\Phi(k,a) \tag{30}$$

$$D_u(k,a) = \alpha_u \tilde{D}_u(k,a) \tag{31}$$

The above can be evaluated directly once the functions D_A are known. The power spectrum of matter in the Universe can then be evaluated straightforwardly as

$$P = |D_A(k,a)|^2 \tag{32}$$

3. Summary and Conclusion

We have presented a simple powerful technique for computing the growth rates within the standard paradigm. By expressing the perturbation variables as ratios of their values at decoupling, we obtained simple ODE's in terms of D-functions and shown that these can be solved using very simple initial conditions that avoids the usual assumptions made in computing cosmological perturbations. Our formalism can be extended to higher order and even more complex equations that can be simplified greatly. This technique has been applied in recent studies on interacting dark sector models (see e.g. [1,2]) to simplify the initial conditions as well as the calculation.

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