UNIVERSITY OF NAIROBI

COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCES

SCHOOL OF MATHEMATICS

ON EXTENSIONS OF CLOSABLE OPERATORS FOR THE CASE OF A WHOLE PLANE, A HALF PLANE, A STRIP OR A LINE

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Declaration

Declaration by the student

I, the undersigned, declare that this project is my original work to the best of my knowledge and has not been used as a basis for any award of any degree in any university.

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Declaration by the supervisors

This project has been submitted for examination with our approval as the supervisors.

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Contents

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Dedication

I dedicate this research project to my parents, wife and daughter.

Abbreviations and symbols

The following are the lists of abbreviations and Symbols used in this paper.

Symbols

T :	An operator
$\mathbb{C}$:	Complex number field
N :	Normed linear space
$\lvert x \rvert$:	Absolute value of x
B :	Banach space
$\overline{0}$:	Zero vector
H :	Hilbert space
$\{x_n\}$:	A sequence of x
$D(T)$:	Domain of an operator T
$V \times V$:	Cross product of two sets

$W(T)$: Numerical range of T

$\langle , \rangle$: Inner product function

$\overline{W(T)}$: Closure of numerical range of T

x_n : An n^{th} term in a sequence

$\sigma(T)$: Spectrum of an operator

$\overline{\langle x, y\rangle}$: Complex conjugate of an inner product

$\widetilde{T}$: Extension of T

$\beta(H)$: Set of bounded operators on H

$\mathcal{I}$: Inner product space

$\|x\|$: Norm of x

$\widetilde{T}$: An extension of an operator

$x \perp y$: Orthogonal elements

$x^{\perp}$: Orthogonal complement of x

I : Identity operator

$R(T)$: Range of T

∞ : Infinity

$N(T)$: Kernel of T

$\bigoplus$: Direct sum symbol

$\exists$: Existence symbol

$\forall$: For all symbol

$\beta(W, V)$:	Set of bounded operators from W to V
$\overline{T}$:	Closure of operator T
T^* :	Adjoint of an operator
$\cup$:	Union
T^{-1} :	Inverse of T
$\cap$:	Intersection
$\emptyset$:	Empty set
$\rho(T)$:	Resolvent set of T
$P\sigma(T)$:	Point spectrum of T
$w(T)$:	Numerical radius
$R\sigma(T)$:	Residual spectrum of T
$S\|_{D(\overline{T})}$:	Restriction of S on the set $D(\overline{T})$
X' :	The dual space of X
$C\sigma(T)$:	Continuous spectrum of T
$x_n \xrightarrow{s} x$:	Strong convergence of $\{x_n\}$ to x
$\phi(x, x)$:	Sesquilinear functional
ψ :	Sesquilinear functional
$\phi(x)$:	Quadratic form associated with $\phi(x, x)$
D :	Direct sum symbol
$\varphi(x) = \widetilde{\phi}(x)$:	Extension of a sesquilinear functional

Abbreviations

sup : Supremum

max : Maximum

$dist$: Distance

Re : Real

Im : Imaginary

Abstract

It is well known that for a linear $T \in \beta(H)$ where H is a complex Hilbert space, the spectrum of T is contained in the closure of the numerical range of T. The result may not hold when the domain $D(T)$ of T is a proper subspace of H as is usually the case when T is an unbounded operator. Among unbounded operators, the closest to the theory of bounded operators are the so-called linear operators and it does happen that most unbounded operators met with in practice are closed.
In this study, our main interest is to extend the domain of those operators
$T : D(T) \to H$ for which the relation

$$\sigma(T) \subseteq W(T)$$

is not valid, that is, to find $\widetilde{T} : D(\widetilde{T}) \to H$ for which

$$\sigma(\widetilde{T}) \subseteq \overline{W(\widetilde{T})}$$

such that $\widetilde{T}$ extends T and investigate the conditions for such a T to have a closed extension.

We seek for answers when T is densely defined and $\overline{W(T)}$ is not the whole plane, a half plane, a strip or a line and examine the complementary problem, that is, when $\overline{W(T)}$ is one of these sets. Finally we deal, briefly, with the case when T is symmetric and seek for situations when such a T has a self-adjoint extension.

CHAPTER 1

INTRODUCTION

When a specific situation involving linear differential equations or linear integral equations with solutions in linear spaces with inner product structure arises, we may be able to formulate the situation in linear operator formalism and the problems involved can be reduced to solution of an operator equation involving a linear operator $T : D(T) \to H$ where $D(T)$ is a linear subspace of H, where H is a Hilbert space.

There are situations in which the domain $D(T)$ of the operator T is all of H and the operator T is bounded. In such situations, it turns out that the spectrum $\sigma(T)$ of the operator T is a subset of the closure of the numerical range, $W(T)$. However, if the domain $D(T)$ of the operator T is a proper linear subspace of H, the spectrum of T need not to be a subset of $\overline{W(T)}$. This deficiency occurs because the domain $D(T)$ is not large enough. In this study, we seek for situations where we can extend T to an operator $\widetilde{T}$ on a larger domain $D(\widetilde{T})$ of H so that $\sigma(\widetilde{T}) \subseteq \overline{W(\widetilde{T})}$, so that we overcome the above deficiency and resurrect the available relation as prevails between $\sigma(T)$ and $W(T)$ in the situation when T belongs to $\beta(H)$.

As the condition next best to boundedness of T when boundedness is not there, is closedness and many unbounded operators which arise in many practical situations do turn out to be closed, we seek answers to the objectives stated when T is a densely defined closed linear operator in H. We seek for answers in the various situations when $\overline{W(T)}$ is not the whole complex plane, half plane, a strip or a line, and these in turn guarantee sufficiently general situations met with in practice.

We also seek answers to the simpler question as to when will a linear operator $T : D(T) \to H$ as above merely have a closed extension?

We also seek an answer to the complementary problem of extension when $\overline{W(T)}$ happens to be a whole complex plane, a half plane, a strip or a line?

1.1 Introduction to operators

In this study, we are interested in four major mathematical structures of a vector space (sometimes referred to as a linear space) over a complex field in which our functions, functionals and operators will be defined on and the outputs will be described. The most important characteristic of these structures is a measure. They all have a sense of measure which is termed as a norm.

Definition 1.1.1. Norm

Let V be a vector space and F a field. A norm denoted $\|\cdot\|$ is a function $\|\cdot\| : V \to F$ such that

for $x, y \in V$ and $\alpha \in F$

1. $\|x\| \geq 0$, the property referred to as positiveness of a norm.

2. $\|x\| = 0$ if and only if $x = \overline{0}$, the zero vector.

3. $\|\alpha x\| = |\alpha|\|x\|$, where $|\alpha|$ is the absolute value of α property referred to as homogeneity of a norm.

4. $\|x + y\| \leq \|x\| + \|y\|$ a property referred to as the triangle inequality for norms, [1].

In out study we will take the field F to be a complex field $\mathbb{C}$.

Having defined a norm in a vector space, we now proceed to define a structure of a vector space having this characteristics, a norm. This structure is called a normed linear space.

Definition 1.1.2. Normed Linear Space

Let V be a vector space over a field F, the vector space together with the norm on V, written as an ordered pair $(V, \|\cdot\|)$ is called **a normed linear space**. We will denote it by N. When F is a complex field, then the N becomes a complex normed linear space,[1].

From the definition of the norm, in this structure, we are able to determine the norm, that is the magnitude, of vectors in this space.Thus the members of N satisfies the properties of the vector space and also that of a norm. In this study, by referring to a normed linear space, we will be implying a complex normed linear space.

Definition 1.1.3. Banach space

Let N be a normed linear space over $\mathbb{C}$. Then N is a Banach space if N is complete with respect to the norm in N. That is if $\{x_n\}$ be a convergent sequence of members of N, then the limit, say, x is a member of N. We will denote a Banach space by B.
That is

$$\|x_n - x\| \to 0 \text{ as } n \to \infty \text{ for } x, x_n \in N$$

then N is a Banach space, B,[1].

We now make the available vector structures richer by introducing the concept of angle and measure in vector spaces. To begin with, we define a function, the inner product function, that gives rise to such structures.

Definition 1.1.4. Inner product

Let V be a vector space over a complex field $\mathbb{C}$. An inner product over a field $\mathbb{C}$ is a mapping, denoted $\langle , \rangle$, from the product space $V \times V$ to the set of complex numbers, that is, $\langle , \rangle : V \times V \to \mathbb{C}$ such that for $x, y, z \in V$ and $\alpha \in \mathbb{C}$, the following are satisfied.

1. $\langle x, y \rangle \geq 0$ and $\langle x, y \rangle = 0$ if and only if $x = y$

   the property referred to as positive definiteness.

2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

   and $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$

   the property referred linearity in the first variable.

3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, [1]

   a property referred to as conjugate symmetry. We take conjugate symmetry because we are using a complex field.

Definition 1.1.5. Inner product space

Let V be a vector space and $\langle , \rangle$ an inner product, then a vector space V or a field F together with the inner product is an inner product space and is denoted by the ordered pair $(V, \langle , \rangle)$. In this study, we will consider a complex inner product space, that is when $\mathbb{C} = F$. We will denote it by $\mathcal{I}$,[1].

The inner product space induces a norm which is defined by

$$\|x\| = \sqrt{\langle x, x \rangle} \text{ for } x \in X \text{ where } V \text{ is a vector space}$$

When an inner product is complete, we get yet another structure that is mostly used in operator theory.

Definition 1.1.6. Hilbert space

Let $\mathcal{I}$ be an inner product space over $\mathbb{C}$. Then $\mathcal{I}$ is a Hilbert space if $\mathcal{I}$ is complete with respect to the induced norm on $\mathcal{I}$. That is if $\{x_n\}$ be a convergent sequence of members of $\mathcal{I}$, then the limit, say, x is contained in $\mathcal{I}$. We will denote a Hilbert space by H. That is

$$\|x_n - x\| \to 0 \text{ as } n \to \infty \text{ for } x, x_n \in \mathcal{I}$$

then $\mathcal{I}$ is a Banach space, H, [3].

Definition 1.1.7. Finite dimensional linear space

A linear space is finite dimensional if it is spanned by a countable number of vectors. That is, its basis must be countable, [3].

Definition 1.1.8. Infinite dimensional linear space

A linear space is infinite dimensional if each of its member can be expressed as a linear combination of some infinite sequence,[3].

Definition 1.1.9. Closed set

It is a set that has all its accumulation points. Similarly, a point set is closed if it is equal to its closure, [3].

Definition 1.1.10. Open set

It is a set whose all its points have neighbourhoods that are contained in the set, [3].

Definition 1.1.11. Convex set

Let N be a normed linear space over $\mathbb{C}$. A set $C \in N$ is convex if for $x, y \in C$ and $\alpha \in \mathbb{C}$

$$\alpha x + (1 - \alpha)y \in C, [3].$$

Definition 1.1.12. Cauchy Bunyakowki's Inequality

Let I be an inner product space and $x, y \in I$ then

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

is valid, [3].

Definition 1.1.13. Parallelogram Law

Let I be an inner product space and $x, y \in I$ then

$$\|x + y\| + \|x - y\| = 2\|x\| + 2\|y\|, [3]$$

Theorem 1.1.1. Orthogonal complement in H

Let H be a Hilbert space and if $x, y \in H$ are such that $\langle x, y\rangle = 0$ then x and y are orthogonal elements of H, denoted $x \perp y$. They are orthogonal complements of one another. If N and M are closed subspaces of H that are orthogonal,denoted $M \perp N$, that is

$$\langle x, y\rangle = 0 \text{ for } x \in N \text{ and } y \in M$$

Then N and M are orthogonal complements of and another. We will denote them as $N = M^{\perp}$ or $M = N^{\perp}$, [3].

Theorem 1.1.2. Pythagorean theorem

Let H be a Hilbert space and $x, y \in H$ such that $\langle x, y\rangle = 0$ then

$$\|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

Also

$$\|x\|^2 + \|y\|^2 = \|x - y\|^2, [3].$$

Theorem 1.1.3. *Let H be a Hilbert space and N a any closed subspace of H. There exists a closed linear subspace $N^{\perp} \in H$ such that $N^{\perp} + N$ closed and the two subspaces are disjoint,[3].*

Theorem 1.1.4. Orthogonal decomposition theorem

Let H be a Hilbert space and N a closed linear subspace of H Then

$$N^{\perp} \bigoplus N = H, [3].$$

Definition 1.1.14. An operator

It is a transformation that maps an element from one vector space to another element in the same or different vector space,[3].

Definition 1.1.15. A linear operator

Let N and M be a normed linear spaces over a complex field $\mathbb{C}$. An operator $T : N \to M$ is linear if for $x, y \in N$ and $\alpha, \beta \in \mathbb{C}$ we have

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \in M, [3].$$

Definition 1.1.16. A functional

Let X be a normed linear spaces over a complex field $\mathbb{C}$. A functional $f : X \to C$ is map such that its image a scalar. The set of all functionals on X is called **the dual of** X denoted X',[3].

Definition 1.1.17. The domain of an operator

Let X and Y are normed linear spaces and T an operator such that $T : X \to Y$. The domain of T is the set

$$D(T) = \{x \in X : Tx \in Y\}, [3].$$

Definition 1.1.18. The range of an operator

Let X and Y are normed linear spaces and T an operator such that $T : X \to Y$. The range of T is the set

$$R(T) = \{Tx \in Y : x \in X\}, [3].$$

Definition 1.1.19. The kernel of an operator

Let X and Y are normed linear spaces and T an operator such that $T : X \to Y$. The kernel of T is the set

$$N(T) = \{x \in X : Tx = 0 \in Y\}, [3].$$

Definition 1.1.20. The norm of an operator

Let X and Y are normed linear spaces and T an operator such that $T : X \to Y$. The norm of T is the number

$$\|T\| = \sup_{x\neq 0}\{\frac{\|Tx\|}{\|x\|}, x \in X, \|x\| = 1\}, [3].$$

Definition 1.1.21. A bounded operator

Let X be a normed linear spaces over $\mathbb{C}$ and T an operator defined on X. T is a bounded operator if

$$\exists k \in \mathbb{C} : \|Tx\| \leq k\|x\|, x \in X, [3].$$

Definition 1.1.22. Closed operator

Let X and Y are normed linear spaces. Let $\{x_n\} \in D(T) \subseteq X$ be such that

$$x_n \xrightarrow{s} x \in X \text{ and } Tx_n \xrightarrow{s} y$$

then $Tx = y$ and $x \in D(T)$,[3].

Definition 1.1.23. An unbounded operator

Let X and Y be a normed linear spaces over $\mathbb{C}$, an operator T is unbounded if its is defined on $D(T) \subseteq X$ such that

$$T : D(T) \to Y$$

and for a bounded $x \in X$, Tx is unbounded.

In most cases the closure of $D(T)$ is usually X. We refer to the operators having this property as densely defined operators,[3].

Definition 1.1.24. Adjoint of an operator

Let V and W be inner product spaces and $T \in \beta(V, W)$ be an operator such that $T : V \rightarrow W$. $S \in \beta(W, V)$ such that

$$S : W \rightarrow V \text{ and } \langle Tx, y \rangle = \langle x, Sy \rangle = \langle x, T^* y \rangle$$

is an adjoint of the operator T and is denote T^*, [3].

Definition 1.1.25. Extension of an operator

Let N be a normed linear space. An operator $A \in \beta(N)$ is an extension of an operator $T \in \beta(N)$ if

$$D(T) \subseteq D(A) \text{ and } Ax = Tx \text{ for } x \in D(T), [3].$$

Definition 1.1.26. Symmetric operator

Let H be a Hilbert space. An operator T is symmetric if for $x, y \in D(T)$

$$\langle Tx, y \rangle = \langle x, Ty \rangle \text{ and } D(T) \subseteq D(T^*), [3].$$

Definition 1.1.27. Self Adjoint Operator

Let H be a Hilbert space. An operator T is A self adjoint operator if for $x, y \in D(T)$

$$\langle Tx, y \rangle = \langle x, Ty \rangle \text{ and } D(T) = D(T^*) \text{ that is, } T = T^*, [3].$$

Definition 1.1.28. Positive Operator

Let H be a Hilbert space. An operator T is a positive operator if

$$\langle Tx, x \rangle \geq 0 \ \ \forall x \in D(T), [3].$$

Definition 1.1.29. Unitary Operator

Let H be a Hilbert space. An operator T is a Unitary operator if

$$T^*T = TT^* = I$$

The condition $T^*T = I$ implies that

$$\|Tx\| = \|x\|, \text{ for } x \in H, [3].$$

Definition 1.1.30. Isometric Operator

Let H be a Hilbert space. An operator T is an Isometry operator if

$$T^*T = I.$$

From the definition, it is true that an Isometric operator is unitary,[3].

Definition 1.1.31. Separable space

A space is referred to as a separable if it has a countable dense subset,[3].

Theorem 1.1.5. *Let M be a subspace of a normed linear space N and suppose $x_0 \in N$ such that*

$$d = d(x_0, M) = \inf_{x \in M} \|x_0 - x\| > 0.$$

Then there is a bounded linear functional F on N such that $\|F\| = 1, F(x_0) = d$ and $F(x) = 0$ for $x \in M$,[3].

Theorem 1.1.6. *If M is a closed convex subset of a normed linear space X and $x_0 \in X$ is not in M, then $\exists f \in X'$ such that*

$$Ref(x_0) \geq Ref(x), x \in X \text{ and } Ref(x_0) \neq Ref(x_1) \text{ for some } x_1 \in M, [3].$$

Theorem 1.1.7. Riesz Representation Theorem

For every bounded linear functional F on a Hilbert Space, H, there is a unique element $y \in H$ such that

$$F(x) = \langle x, y \rangle, \forall x \in H.$$

Moreover

$$\|y\| = \sup_{x \in H, x \neq 0} \frac{|F(x)|}{\|x\|}, [3].$$

1.2 Overview of some concepts in spectral theory

Definition 1.2.1. Invertible operator

Let N be a normed linear space and T an operator on N such that $T \in \beta(N)$. T is invertible if

$$\exists S : TS = ST = I$$

. Thus

$$S = T^{-1} \text{ and } T^{-1} \in \beta(T), [5].$$

Having known the definition of an invertible function, we proceed to find out the necessary and sufficient conditions for a bounded operator in a general Banach space can be invertible and also an onto transformation. When we select an element from the Banach space, the domain of T then find its image under the transformation T, our interest is to find out if the image is bounded below. If x chosen is arbitrary, then we have the required results. Furthermore, when the set

$$\{\overline{Tx} : \forall x \in B\} = B$$

then T is onto.

Theorem 1.2.1. *Let B be a Banach space and T be an operator such that $T \in \beta(T)$. Then T is invertible if and only if $\overline{R(T)} = B$. Moreover,*

$$\exists \alpha > 0 : \|Tx\| \geq \alpha \|x\|, \forall x \in B, [5].$$

The Theorem give an important condition that shows when an operator is onto and one to one hence invertible. The condition $\overline{R(T)} = B$ shows that the operator is onto while $\|Tx\| \geq \alpha\|x\|$ where the image of the operator is bounded below shows that the operator is one to one. This guarantees that the operator is invertible.

The spectral theory in infinite dimensional space involves the description the range, nullity and the inverse of operator $(T - I\lambda)$ which are derived from the bounded operator T and eigen value $\alpha \in N$. Based on these descriptions, we are able to define the resolvent set and the spectrum of an operator. We will therefore introduce the following definitions.

Definition 1.2.2. Resolvent set

The number λ is in the resolvent set, denoted $\rho(T)$ if $R(T - I\lambda)$ is dense in N and $(T - I\lambda)^{-1}$ exists and is bounded on the range, $R(T - I\lambda)$.Thus

$$\overline{R(T - I\lambda)} = N \text{ and } (T - I\lambda)\text{is one to one and } (T - I\lambda) \in \beta(N), [5].$$

Definition 1.2.3. Spectrum of a bounded operator

The number λ is in the spectrum of a bounded operator, denoted $\rho(T)$ if $(T - I\lambda)^{-1}$ does not exists. In a complex finite dimensional linear space, the set implies the eigen value of the operator T. Based on different characteristics of a member of a spectrum, we can have three different types of spectra are highlighted below,[5].

Definition 1.2.4. Point Spectrum

The number λ is in the point spectrum, denoted $P\sigma(T)$ if $(T - I\lambda)^{-1}$ does not exists implying that the operator $(T - I\lambda)$ is not one to one transformation,[5].

Definition 1.2.5. Residual Spectrum

The number λ is in the Residual Spectrum, denoted $R\sigma(T)$ if $R(T - I\lambda)$ is not dense in N and $(T - I\lambda)^{-1}$ exists but may be bounded on the range or not. This implies that $(T - I\lambda)$ is a one to one transformation that may be bounded or not,[5].

Definition 1.2.6. Continuous Spectrum

The number λ is in the Continuous Spectrum, denoted $C\sigma(T)$ if $R(T - I\lambda)$ is dense in N and $(T - I\lambda)^{-1}$ exists but not bounded on the range,[5].

The spectrum of an operator $T, \sigma(T)$, is therefore the union of the point spectrum,the residual spectrum and the continuous spectrum. Furthermore,the three subdivisions of a spectrum are disjoint.

$$\sigma(T) = P\sigma(T) \cup R\sigma(T) \cup C\sigma(T).$$

The spectrum is the complement of a the resolvent set. The union of a spectrum and the resolvent set form the whole of the complex plane.Thus, if N is a normed linear space on a complex number field, then

$$\sigma(T) \cap \rho(T) = \emptyset \text{ and } \sigma(T) \cup \rho(T) = N.$$

Definition 1.2.7. Spectral radius

Let $\{\lambda_i, i = 1, 2, 3...\}$ be a set of all eigenvalues of an operator. Then the number

$$\gamma = \max\{|\lambda_i|, i = 1, 2, 3...\}, [5].$$

The following are some of the relevant theorems in spectral theory.

Theorem 1.2.2. *Let B be a Banach space T an operator such that $T \in \beta(B)$, then $\rho(T)$ is an open set and $\sigma(T)$ is a closed set. Moreover, $\sigma(T)$ is bounded and contained in the disc $|\lambda| \leq \|T\|$,[5].*

1.3 Numerical range and the consequences

Definition 1.3.1. Numerical range and numerical radius

Let $T \in \beta(H)$. The set given by

$$W(T) = \{\langle Tx, x\rangle : x \in H \text{ such that } \|x\| = 1\}$$

is referred to as the numerical range of the operator T while

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$$

is referred to as the numerical radius of T.

The numerical range is a convex set, by the preceeding theorem hence can be enclosed in a circle of a radius which can be determined in the complex plane.

The numerical radius is the radius of the smallest circle in the complex plane which contain the numerical radius, [4].

Theorem 1.3.1. *Let $T \in \beta(H)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ then*

$$W(\alpha_1 I + \alpha_2 T) = \alpha_1 + \alpha_2 W(T), [4].$$

Theorem 1.3.2. *Let $T \in \beta(H)$, then*

$$\sigma(T) \subseteq \overline{W(T)}$$

and if $d = dist(\lambda, \overline{W(T)})$ then $\lambda I - T$ has an inverse and

$$\|(\lambda I - T)^{-1}\| < \frac{1}{d}, [4].$$

Theorem 1.3.3. Haudorff-Toeplitz Theorem

For any $T \in \beta(H), W(T)$ is a convex set,[4].

1.4 Symmetry in sesquilinear forms

Definition 1.4.1. Sesquilinear form

Let X be a normed linear space. The map $\phi : X \times X \to \mathbb{C}$ is a sesquilinear functional if for $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\phi(\alpha x + \beta y, z) = \alpha\phi(x, z) + \beta\phi(y, z)$$

$$\phi(x, \alpha y + \beta z) = \overline{\alpha}\phi(x, z) + \overline{\beta}\phi(y, z), [1].$$

A Sesquilinear form is also called a sesquiliear functional.

Definition 1.4.2. Quadratic form associated with a sesquilinear functional

Let ϕ be a sesquilinear functional, the map ϕ' defined as

$$\phi'(x) = \phi(x, x), \text{ for } x \in D(x)$$

is called a quadratic form associated with the sesquilinear functional. In this paper, we will use, $\phi(x)$ to imply $\phi'(x)$, [1].

Definition 1.4.3. Symmetric Sesquilinear form

Let $\phi(x, y)$ be a sesquilinear form; then it is symmetric if

$$\phi(x, y) = \phi(y, x) \text{ or } \phi(x, y) = \overline{\phi(x, y)}, [1].$$

Lemma 1.4.1. *Let $\phi(x, y)$ be a symmetric sesquilinear form such that $|\phi_1(x)| \leq \beta\phi_2(x), x \in D(\phi_1) \cap D(\phi_2)$ for a scalar β then*

$$|\phi_1(x, y)|^2 \leq \beta^2\phi_2(x)\phi_2(y), x \in D(\phi_1) \cap D(\phi_2), [1].$$

Corollary 1. *If* $\phi_2(x, y)$ *is symmetric,* $\phi_1(x, y)$ *is not and* $|\phi_1(x)| \leq \beta\phi_2(x), x \in D(\phi_1) \cap D(\phi_2)$ *then*

$$|\phi_1(x, y)|^2 \leq 4\beta^2\phi_2(x)\phi_2(y), x \in D(\phi_1) \cap D(\phi_2), \textit{ for a scalar } \beta, [1].$$

Definition 1.4.4. Associated operator of a sesquilinear functional

Let ϕ be a sesquilinear functional and and T an operator such that $D(\phi) = D(T)$. The operator T is an associated operator of a sesquilinear functional if

$$\phi(x, y) = \langle Tx, y\rangle \text{ for } x, y \in D(\phi), \ [1].$$

Theorem 1.4.2. *Let* $\phi(x, y)$ *be a densely defined sesquilinear form and* T *its associtade operator. If* $\lambda \notin \overline{W(\phi)}$, *then*

(i). $T - \lambda I$ *is injective and*

$$\|x\| \leq k\|(T - \lambda)x\|, x \in D(T)$$

(ii). T *is closed and the range of* $T - \lambda I$ *is closed in* H, *[1]. .*

Theorem 1.4.3. *The following statements are equivalent for sesquilinear forms;*

(i). $\phi(x, y)$ *is symmetric*

(ii). $Im\phi(x) = 0, x \in D(\phi)$

(iii). $Re\phi(x, y) = Re\phi(y, x), x, y \in D(\phi)$, *[1]. .*

CHAPTER 2

LITERATURE REVIEW

Basic material about numerical ranges of linear operators is given in F. Bonsall and J. Duncan's article [5] of 1980 and Hilbert Space Problem book of Halmos [13] of 1967 along with Functional Analysis by Bachman and Narici [3].

With reference to unbounded operators, the principal results connected with our work are discussed in the work of Kato T. [14] of 1952, M. Schecter [21] in 1967 and Friedrichs [10] in 1962.

The extension technique for sesquilinear forms was originally due to K.O. Friedrich's [9] in 1934 and Freudenthal[8] in 1936 but was refined later by Schecter in 1966 and Kato in 1967 to particular situations in which we are interested.

The generalization of the results, used in the study, on numerical ranges and the spectrum were provided in the work done by Winter, A [22] in 1929, Halmos P. [13] in 1967, Fillmore, P. A. [7] in 1970 and De Bara et al [4] in 1972.

A very technical account is provided in Kato's monograph [17] of 1976, and we have adopted a much simplified version of the account as would suit our modest requirements. We also used Aronszajns N. brief remarks of 1966 on extension of unbounded operators in a Hilbert space [2].

Applications of the extension technique discussed are due to Lax P. D. and Milgram [18] in 1954 and Philips [19] in 1959.

CHAPTER 3

EXTENSIONS OF CLOSABLE OPERATORS FOR THE CASE OF A WHOLE PLANE AND A HALF PLANE

3.1 Closable operators and sesquilinear functionals

It is not always the case that the spectrum of any operator, T, will be contained in the closure of its numerical range, that is $\sigma(T) \subseteq \overline{W(T)}$. Therefore, we investigate the conditions under which this property may be true. We investigate the operators that are densely defined so that we get the desired results. This chapter aims at determining the closed extensions of a densely defined operator which satisfies the condition above, $\sigma(\widetilde{T}) \subseteq \overline{W(\widetilde{T})}$. We also seek to determine the conditions when an extension of an operator can satisfy the condition

$$\sigma(\widetilde{T}) \subset \overline{W(\widetilde{T})} = \overline{W(T)}.$$

We begin by defining the terms that will be useful in this section.

Definition 3.1.1. A straight line

This refers to a set of points that are connected and extending infinitely in two opposite directions. We will use the term a line to refer to a straight line.

Remaks

An important remark about a straight line is that in a complex plane, when it is extended infinitely in both directions, it divides the plane into two half planes. In case such lines that are parallel are extended in both directions, they divide the complex plane into three portions, the portion in between them is one of our interest in this study.

Definition 3.1.2. A whole plane

This refers to the entire flat surface where any line entirely lies. In our case, we will consider a complex plane.

Definition 3.1.3. A half plane

This is an area on a plane that lies on one side of a line. It may include the line or not.

Definition 3.1.4. A strip

This is an area between two parallel lines. We will consider a plane strip only.

Definition 3.1.5. Densely defined operator

Let T be an operator and X and Y be normed linear spaces such that $T : X \to Y$. Then T is densely defined if $D(T) = X$.

Definition 3.1.6. Closure of an operator

The closure of an operator T is the smallest extension of T, denoted $\overline{T}$, such that if S is any closed extension of T then S extends $\overline{T}$, that is

$$D(\overline{T}) \subseteq D(S) \text{ and } S|_{D(\overline{T})} = \overline{T}.$$

Definition 3.1.7. Closable operator

Let T be a linear operator from a normed linear space X to a normed linear space Y. Then T is referred to as a closable operator if for

$$x_n \in D(T), \forall n \in N, x_n \xrightarrow{s} \bar{0}$$

$$Tx_n \xrightarrow{s} y \text{ implies } y = \bar{0} \in X$$

Thus every closed operator is closable.

We now discuss in general, the conditions for a closable operator to have a closed extension. This will give us a platform on which we can determine the conditions under which a closable operator will have a closed extension when the numerical range or its closure is predetermined, that is, when it is a half plane, a strip or a line.

Remarks

An operator is closed when its graph is closed. When T is a closed operator and, X and Y are normed linear spaces. Let $\{x_n\} \in D(T) \subseteq X$ be such that $x_n \xrightarrow{s} x \in X$ and $Tx_n \xrightarrow{s} y$, Then $Tx = y$ and $x \in D(T)$.

Taking $x = \overline{0}$, then $y = T\overline{0} = \overline{0}$ implying that a closed operator is closable.

Theorem 3.1.1. *A linear operator T has a closed extension if and only if it is closable.*

Proof. Suppose T has a closed extension $\widetilde{T}$ and let $\{x_n\}$ be a sequence of elements of $D(T)$ such that $x_n \xrightarrow{s} \bar{0} \in X$ while $Tx_n \xrightarrow{s} y \in Y$. We show that $y = \bar{0}$ to prove that T is closable.

Since $\widetilde{T}$ is an extension of T,$D(T) \subseteq D(\widetilde{T})$ hence $x_n \in D(T) \subseteq D(\widetilde{T})$. Thus $x_n \in D(\widetilde{T})$ for each $n \in \mathbf{N}$. Since $\tilde{T}$ is closed we have $\tilde{T}\bar{0} = y$, but $\tilde{T}\bar{0} = \bar{0}$ thus $y = \bar{0}$.

Conversely, let $\overline{T}$ be the closure of T. Then by definition, for $x \in X$ is in $D(\overline{T})$ if there is a sequence of $\{x_n\} \in D(T)$ such that

$$x_n \xrightarrow{s} x \text{ and } Tx_n \xrightarrow{s} y$$

for some element $y \in Y$ and $\overline{T}x = y$.

This definition does not depend upon the choice of the particular sequence $\{x_n\}$ for if $\{z_n\}$ is a sequence of elements in $D(T)$ and $z_n \xrightarrow{s} x$ and $Tz_n \xrightarrow{s} w \in Y$, then

$$x_n - z_n \xrightarrow{s} \bar{0}$$

. and

$$T(x_n - z_n) \xrightarrow{s} y - w$$

Since T is closable, $y - w = \bar{0}$, hence $y = w$.

$\overline{T}$ is a linear extension of T. To prove this, let $\{x_n\}, \{z_n\} \in D(T)$ such that

$$x_n \xrightarrow{s} x, z_n \xrightarrow{s} z \text{ and } Tx_n \xrightarrow{s} p, Tz_n \xrightarrow{s} q$$

By definition of closure of T, $\overline{T}x = p$ and $\overline{T}z = q$

Thus

$$Tx_n \xrightarrow{s} \overline{T}x \text{ and } Tz_n \xrightarrow{s} \overline{T}z$$

For $\alpha_1, \alpha_2 \in \mathbb{C}$ we have

$$\alpha_1 Tx_n \xrightarrow{s} \alpha_1 \overline{T}x \text{ and } \alpha_2 Tz_n \xrightarrow{s} \alpha_2 \overline{T}z$$

Adding the equations, we have

$$\alpha_1 Tx_n + \alpha_2 Tz_n \xrightarrow{s} \alpha_1 \overline{T}x + \alpha_2 \overline{T}z$$

Since T is linear, we have

$$\alpha_1 Tx_n + \alpha_2 Tz_n = T\alpha_1 x_n + T\alpha_2 z_n = T(\alpha_1 x_n + \alpha_2 z_n) \xrightarrow{s} \alpha_1 \overline{T}x + \alpha_2 \overline{T}z$$

Thus

$$T(\alpha_1 x_n + \alpha_2 z_n) \xrightarrow{s} \alpha_1 \overline{T}x + \alpha_2 \overline{T}z \tag{3.1}$$

But $(\alpha_1 x_n + \alpha_2 z_n) \in D(T)$ which it is a linear space, hence we must have

$$T(\alpha_1 x_n + \alpha_2 z_n) \xrightarrow{s} \overline{T}(\alpha_1 x_n + \alpha_2 z_n) \tag{3.2}$$

by definition of closure of T From equations (3.2) and (3.1), we have

$$\overline{T}(\alpha_1 x_n + \alpha_2 z_n) = \alpha_1 \overline{T}x + \alpha_2 \overline{T}z$$

Thus $\overline{T}$ is linear

We now verify that $\overline{T}$ is closed. Let $\{x_n\}$ be a sequence of elements of $D(\overline{T})$ and $x_n \xrightarrow{s} x$ and $Tx_n \xrightarrow{s} y$. But $D(\overline{T}) = \overline{D(T)}$ hence $D(\overline{T})$ is closed, thus,all points in it are limit points. Then for each $n \in \mathbb{N}$ there is a sequence $\{w_{n_k}\}_{k=1}^{\infty}$ of elements of $D(T)$ such that $\{w_{n_k}\} \xrightarrow{s} \{x_n\}$ and $Tw_{n_k} \xrightarrow{s} \overline{T}x_n$. In particular, one can find $z_n \in D(T)$ such that

$$\|z_n - x_n\| \leq \frac{1}{n}; \|Tz_n - \overline{T}x_n\| \leq \frac{1}{n}$$
$$\|z_n - x\| = \|z_n - x_n + x_n - x\| \leq \frac{1}{n}$$

Using triangle inequality for the norms, we have

$$\|z_n - x\| \leq \|z_n - x_n\| + \|x_n - x\| \to 0$$
$$\|Tz_n - y\| = \|Tz_n - \overline{T}x_n + \overline{T}x_n - y\| \leq \frac{1}{n}$$

Using triangle inequality for the norms, we have

$$\|Tz_n - y\| \leq \|Tz_n - \overline{T}x_n\| + \|\overline{T}x_n - y\| \to 0$$

This implies that $x \in D(T)$ and $\overline{T}x = y$. Hence $\overline{T}$ is closed. □

Remarks

Let S be any extension of T, then S extends $\overline{T}$. We prove this postulate. S exists since $\overline{T}$ is one of such extensions. Let $x \in D(\overline{T})$, then there is a sequence $\{x_n\}$ of elements of $D(T)$ such that;

$$x_n \xrightarrow{s} x \text{ and } Tx_n \xrightarrow{s} \overline{T}x$$

Now since S is an extension of $T, D(S) \supseteq D(T)$ and we have that $x_n \xrightarrow{s} x$ and $Sx_n \xrightarrow{s} \overline{T}x$ But S is closed, hence $x \in D(S)$ and $Sx = \overline{T}x$. This shows that $D(\overline{T}) \subseteq D(T)$ and $S|_{D(\overline{T})} = \overline{T}$.

Lemma 3.1.2. *A Convex set in the plane which is not the whole plane is contained in a half plane*

Proof. Let V be a convex set in the complex Hilbert space, H, which is nor the whole plane. Then its closure $\overline{V}$ is not the whole plane either. Let the plane have a scalar product defined as $\langle z_1, z_2 \rangle = z_1\overline{z_2}$.

Let $x_0 \notin \overline{V}$. Then by theorem 1.1.6, there is a bounded linear functional $F \neq 0$ such that

$$ReF(x) \leq ReF(x_0) \text{ for } x \in \overline{V}$$

. By Riesz Representation theorem, there is a complex number $y \neq 0$ such that

$$F(x) = x\overline{y} \text{ and } F(x_0) = x_0\overline{y}, x_0 \notin \overline{V}, x \in \overline{V}$$

Upon substitution, we have $Rex\overline{y} \leq Rex_0\overline{y}$

Let $Rex\overline{y} = a$ then $Rex_0\overline{y} \geq a$ is a plane to the right of $x = a$ including the vertical line $x = a$. Also let $Rex_0\overline{y} = b$, then $Rex\overline{y} \leq b$ is the plane to the left of $x = b$ including the vertical line $x = b$. Taking $a = b$, we have $Rex\overline{y}$ being of the left of $x = a$ and $Rex_0\overline{y}$ is on the right. Hence for all $x \in \overline{V}$, $Rex\overline{y} \leq a$. Since any straight line divides a complex plane into two halves, $x = a$ does thus the set $\{x : x \in \overline{V}\}$ is a half plane. □

Definition 3.1.8. Numerical range of a sesquilinear functional

Let ϕ be a sesquilinear functional. The numerical range of a Sesquilinear functional is

$$\lambda = W(\phi) = \phi(x) = \phi(x, x) : \|x\| = 1 \text{ and } x \in D(\phi)$$

The relation between the numerical range of a sesquilinear functional and that of its associated operator is that

$$\{\lambda = W(\phi) = \phi(x) = \phi(x, x) = \langle Tx, x\rangle : \|x\| = 1 \text{ and } x \in D(\phi) = D(T)\}$$

Thus

$$\phi(x) = \langle Tx, x\rangle \text{ for } x \in D(\phi) = D(T)$$

Corollary 2. *If ϕ is a sesquilinear functional such that $\overline{W(\phi)}$ is not the whole plane, then there are constants r, k_0 with $|r| = 1$ such that*

$$Re[r\phi(x) + k_0\|x\|^2] \geq 0, \forall x \in D(\phi) \tag{3.3}$$

Proof. $W(\phi)$ is convex, be lemma 3.1.2, it must be contained in a half plane.
By the fact that a plane can be divided into two when taking the negative or positive sides of the an axis, we have every half plane being of the form

$$Re[rz + k_0] \geq 0, |r| = 1$$

Where z is any point in the complex plane. Taking $z = \phi(x)$ for $x \in D(\phi)$ such that $\|x\| = 1$ we have

$$Re[r\phi(x) + k_0] \geq 0, x \in D(\phi), \|x\| = 1$$

But $\|x\| = 1$ implying that $\|x\|^2 = 1$. Multiplying by k_0, we have

$$Re[r\phi(x) + k_0\|x\|^2] \geq 0, x \in D(\phi), \|x\| = 1 \text{ as required .}$$

□

Theorem 3.1.3. *The numerical range of a sesquilinear functional is a convex set in the plane*

Proof. Let T be the associated operator of the sesquilinear functional ϕ, then for $x \in D(\phi)$ then

$$\phi(x, x) = W(\phi) = W(T) = \langle Tx, x\rangle \text{ for } x \in D(\phi) = D(T) \text{ and } \|x\| = 1$$

Since $W(\phi) = W(T)$
From the Hausdorff - Toeplitz theorem, the numerical range of an associated operator of a sesquilinear functional is convex. Thus $W(\phi)$ is convex. □

Definition 3.1.9. An extension of a Sesquilinear functional

Let ϕ and Φ be sesquilinear functionals. Then Φ is an extension of a sesquilinear functional, ϕ, if

$$D(\phi) \subset D(\Phi) \text{ and } \phi(x, y) = \Phi(x, y) \text{ for } x, y \in D(\phi)$$

Definition 3.1.10. Closed Sesquilinear functional

Let $\phi(x, y)$ be a sesquilinear functional. It is closed if a sequence $\{x_n\}$ which is a members of $D(\phi)$ is such that $x_n \to x \in H, \phi(x_n - x_m) \to 0$ as $m, n \to \infty$ implies that

$$x \in D(\phi) \text{ and } \phi(x_n - x) \to 0$$

Definition 3.1.11. A dense domain of a Sesquilinear functional

Let $D(\phi)$ be the domain of a sesquilinear functional (ϕ). A set $X \subset D(\phi)$ is dense in $D(\phi)$ if for each $y \in D(\phi)$ and each real $\epsilon > 0$ there is an $x \in X$ such that

$$\phi(y - x) < \epsilon \text{ and } \|y - x\| < \epsilon.$$

Having set that a densely defined closable operator has a closed linear extension, we can extend the notion to bilinear forms too, the sesquilinear functionals. This is possible by introducing similar conditions of closability of an operator to a sesquilinear functional. Furthermore, we also need to define the numerical range in a way that it is not a whole plane to get the desired results. The result will then set a basis for determining when the numerical range of the associated operator can contain its spectrum. We first have some theorems that will lead to the above results.

Theorem 3.1.4. *Let $\phi(x, y)$ be sesquilinear functional such that $W(\phi)$ is not the whole plane, a half plane, a strip or a line. There is a symmetric sesquilinear functional ϕ_2 with $D(\phi_1) = D(\phi_2)$ such that*

$$\beta^{-1}|\phi_1(x)| \leq \phi_2(x) \leq |\phi_1(x)| + \beta\|x\|^2, x \in D(\phi_1) \tag{3.4}$$

In particular, if

$\{x_n\} \subset D(\phi_1), x \in D(\phi_1), x_n \to x$ *and* $\phi_1(x_n - x) \to 0$

Then

$$\phi_1(x_k, y) \to \phi_1(x, y), y \in D(\phi_1)$$

Proof. From Corollary 2, there are constants r, k_0 such that

$Re[r\phi(x) + k_0\|x\|^2] \geq 0, \forall x \in D(\phi)$. Since the value is more than zero, we can get a constant k and a sesquilinear functional $\phi(x), x \in D(\phi)$ such that

$\|\phi(x)\| \leq k[Re[r\phi(x) + k_0\|x\|^2]] \geq 0, \forall x \in D(\phi)$.

If we take $\phi_*(x,y)$ to be the real part of $r\phi(x,y)$ and set

$$\phi_2(x,y) = \phi_*(x,y) + k_0\langle x,y\rangle$$

so that

$$\phi_2(x) = \phi_*(x) + k_0\langle x,x\rangle = \phi_*(x) + k_0\|x\|^2.$$

By substitution, we get that

$$|\phi(x)| \leq k\phi_2(x), x \in D(\phi)$$

or

$$|\phi(x)| \leq k\phi_2(x) = \phi_*(x) + k_0\|x\|^2 = Rer\phi(x,y) + k_0\|x\|^2 \leq |\phi(x)| + k_0\|x\|^2, x \in D(\phi).$$

Hence

$$|\phi(x)| \leq k\phi_2(x) \leq |\phi(x)| + k_0\|x\|^2, x \in D(\phi)$$

or

$$k^{-1}|\phi(x)| \leq \phi_2(x) \leq |\phi(x)| + k_0\|x\|^2, x \in D(\phi).$$

Which is equivalent to inequality 3.4.

We now prove that $\phi_1(x_k,y) \to \phi_1(x,y), y \in D(\phi_1)$.

$|\phi_1(x_k,y) - \phi_1(x,y)| = |\phi_1(x_k - x, y)|$ (by properties of sesquilinear functionals)

and

$|\phi_1(x_k,y) - \phi_1(x,y)|^2 = |\phi_1(x_k - x, y)|^2$.

Using corollary 1, we have

$|\phi_1(x_k - x, y)|^2 \leq 4\beta^2\phi_2(x_k - x)\phi_2(y)$.

Thus

$$|\phi_1(x_k, y) - \phi_1(x, y)|^2 \leq 4\beta^2\phi_2(x_k - x)\phi_2(y)$$
$$\leq 4\beta^2\phi_2(y)[\phi_1(x_k - x) + \beta\|x_k - x\|].$$

Since $\leq 4\beta^2\phi_2(y)[\phi_1(x_k - x) + \beta\|x_k - x\|$ approaches 0 as $k \to \infty$ we have that $|\phi_1(x_k, y) - \phi_1(x, y)|^2 \to 0$ or $|\phi_1(x_k, y) - \phi_1(x, y)| \to 0$ if an only if $\phi_1(x_k, y) - \phi_1(x, y) \to 0$ if an only if $6\phi_1(x_k, y) \to \phi_1(x, y)$. □

Theorem 3.1.5. *Let $\phi(x, y)$ be a closed sesquilinear form such that $W(\phi)$ is not a half plane, trip or a line and such that $0 \notin W(\phi)$. Then for each linear functional Fy on $D(\phi)$ satisfying*

$$|Fy|^2 \leq C[\phi(y)], y \in D(\phi).$$

There are unique elements $w, x \in D(\phi)$ such that

(i). $Fy = \phi(y, w), y \in D(\phi)$

(ii). $Fy = \overline{\phi(x, y)}, y \in D(\phi)$.

Proof. Since $\phi(x, y)$ satisfies the hypothesis of theorem 3.1.4., there is a symmetric sesquilinear functional $\phi(x, y)_1$ such that $D(\phi_1) = D(\phi_2)$ and

$$\beta^{-1}|\phi(x)| \leq \phi_1(x) \leq |\phi(x)| + \beta\|x\|^2, x \in D(\phi_1).$$

Since $0 \notin W(\phi) = \phi(x), \|x\| = 1$ and $x \in D(\phi)$ there is a $\delta > 0$ such that

$$|\phi(x)| \geq \delta > 0\|x\| = 1 \text{ and } x \in D(\phi).$$

Hence

$$|\phi(x)| \geq \delta\|x\|^2 x \in D(\phi).$$

Also

$$|\|x\|^2 \leq \delta^{-1}|\phi(x)|, x \in D(\phi).$$

Taking $\delta^{-1} = M_2$ and using 3.4 there is an a constant say M_3 such that

$$|\|x\|^2 \leq M_2|\phi(x)| \leq M_3\phi_1(x), x \in D(\phi).$$

Without loss of generality, let there be another constant M_1 such that

$$\|x\|^2 \leq M_1\phi_1(x) \leq M_2|\phi(x)| \leq M_3\phi_1(x), x \in D(\phi), \tag{3.5}$$

which is true due to the inequality $\|x\|^2 \leq \delta^{-1}|\phi(x)|$. Let X be a vector space equipped with the scalar product $\phi_1(x, y)$. We show that X is a Hilbert space.

Since $\phi_1(x, y)$ is a sesquilinear functional, it satisfies all the properties of an inner product function hence we only need to show that X is complete given the scalar product $\phi_1(x, y)$.

Let $\{x_n\}$ be a Cauchy sequence in X , that is, $\phi_1(x_n - x_m) \to 0$ as $n, m \to \infty$. By 3.5 $\|x\|^2 \leq M_1\phi_1(x)$ implies $\{x_n\}$ is a Cauchy sequence hence in H hence $\phi(x_n - x_m) \to 0$ as $n, m \to \infty$. Since H is complete, there is an element $x \in H$ such that $x_n \to x \in H$ as $n \to \infty$.

Since $\phi(x, y)$ is a closed sesquilinear form, by implication, we have $\phi(x_n - x) \to 0$ as $n \to \infty$. Hence by 3.5 , $M_1\phi_1(x) \leq M_2|\phi(x)|$, $\phi(x_n - x) \to 0$ as $n \to \infty$ implies $\phi_1(x_n - x) \to 0$ as $n \to \infty$ showing that X is complete.

Now for each $z \in X$ we have

$$Gy = \phi(y, z), y \in X$$

is a linear functional on X. This is true when applying the Riesz Representation theorem to any linear functional G and using associated operator of ϕ. By 3.5 $\phi(y, w) = Gy$ is bounded above (even below). Hence by Reisz Representation theorem, there is an element $Tz \in X$ such that

$$Gy = \phi_1(y, Tz), y \in X.$$

Since $z \in X$ and $Tz \in X$, T is a map of X onto itself. Moreover, it is also linear.
We show this. Let $u, v \in X$ and $\alpha, \beta \in \mathbb{C}$. Since X is a Hilbert space hence complete and closed under addition and scalar multiplication, $\alpha u, \beta v \in X$ and $\alpha u + \beta v \in X$.
By definition of T; $Tu, Tv \in X$ also $T(\alpha u + \beta v) \in X$.
Also, since $Tu, Tv \in X$; $\alpha Tu, \beta Tv \in X$ hence $\alpha Tu + \beta Tv \in X$.
Therefore $T(\alpha u + \beta v) \in X$ and $\alpha Tu + \beta Tv \in X$ shows that T is linear.
Also T is bounded. We show this.
We already seen that $Gy = \phi_1(y, Tz) = \phi(y, z)$. Hence
$\phi_1(Tz)^2 = \phi(Tz, z)^2 \leq 4M_3^2\phi_1(Tz)\phi_1(z)$ by corollary 1. Thus canceling the common factors, we get

$$\phi_1(Tz) \leq 4M_3^2\phi_1(z).$$

T is also one to one and has a closed range. We prove this. Applying 3.5 on the above inequality, we get

$$M_1^2\phi_1(z)^2 \leq M_2^2|\phi(z)| = M_2^2|\phi_1(z, Tz)| \leq M_2^2|\phi_1(z)\phi_1(Tz)|, x \in D(\phi). \qquad (3.6)$$

Dividng through $\phi_1(z)M_1^2$ we get

$$\phi_1(w) \leq \frac{M_2^2\phi_1(Tz)}{M_1^2}.$$

Showing that T is bounded below hence one to one. We claim that $R(T) = X$. Indeed if we take any element $h \in X$ orthogonal to $R(T)$ then

$$\phi_1(h, Tz) = 0, z \in X.$$

By $Gy = \phi_1(y, Tz)$ and $Gy = \phi(y, z)$ we have $\phi_1(h, Tz) = 0$ implying $\phi(h, z) = 0$ for $z \in X$. But this is possible only if $h = z$ implying that $\phi(h, z) = \phi(h, h) = \phi(h) = 0$. By a similar arguement as the one above $\phi(h) = 0$ implies $\phi_1(h) = 0$. Thus $h = 0$.
This implies that R(T) is dense and equal to X since the only vector orthogonal to a complete space is the zero vector.
Now let F be any linear functional in $D(\phi)$ such that $|Fy|^2 \leq \beta|\phi(y)|, y \in D(\phi)$, then F is bounded above, hence a bounded linear functional on X. Hence there is an element $f_1, f_2 \in X$ such that

$$Fy = \phi_1(y, f_1), \text{ and } Fy = \overline{\phi_1(f_2, y)}, y \in X.$$

Since T is one to one and onto (onto because $R(T) = X$) there is a $w \in X$ and an $x \in X$ such that $Tw = f_1$ and $Tx = f_2$. Hence

$$Fy = \phi_1(y, f_1) = \phi_1(y, Tw) = \phi(y, w), y \in X$$

and

$$Fy = \overline{\phi_1(f_2, y)} = \overline{\phi_1(Tx, y)} = \phi_1(y, Tw) = \overline{\phi(x, y)}, y \in X.$$

□

Theorem 3.1.6. *Let $\phi(x, y)$ be a densely defined sesquilinear functional such that $\overline{W(\phi)}$ is not the whole plane, a half plane, a strip or a line. If T is its associated operator, then T is closed and*

$$\sigma(T) \subset \overline{W(\phi)} = \overline{W(T)}$$

Proof. We first proof that T is closed.
Let $\{x_n\} \in D(T)$ such that $x_i \to x$ and $Tx_i \to h \in H$.
Then $\phi(x_i - x_n) = \phi(x_i - x_n, x_i - x_n) = \langle T(x_i - x_n), x_i - x_n\rangle = \langle Tx_i - Tx_n, x_i - x_n\rangle$.
Taking absolute values, we get

$|\phi(x_i - x_n)| = |\langle Tx_i - Tx_n, x_i - x_n\rangle|$. By Cauchy Bunyakowski's inequality, we get
$|\phi(x_i - x_n)| \leq \|Tx_i - Tx_n\|\|x_i - x_n\|$.
Since $x_i \to x$, $\|x_i - x_n\| \to 0$ as $i, n \to \infty$ so is $|\phi(x_i - x_n)|$.
Thus $\phi(x_i - x_n) \to 0$ $x_i \to x$ and $\phi(x_i - x_n) \to 0$ shows that $\phi(x, y)$ is closed. From the definition of a closed linear functional, we get the implication that $x \in D(\phi)$ and $\phi(x_i - x) \to 0$ as $n \to \infty$.
By theorem 3.1.4 $\{x_n\} \subset D(\phi), x \in D(\phi), x_n \to x$ and $\phi(x_n - x) \to 0$ implies

$$\phi(x_i, y) \to \phi(x, y) \text{ as } i \to \infty, y \in D(\phi).$$

From the definition of associated operator, we have that $\phi(x_i, y) = \langle Tx_i, y\rangle, y \in D(\phi)$
From the limit computation, we got $Tx_i \to h \in H$ and $\phi(x_i, y) \to \phi(x, y)$ implying that $\phi(x, y) = \langle Tx_i, y\rangle, y \in D(\phi)$ implying that $x \in D(T)$.Hence T is closed.
Then show that $\sigma(T) \subset \overline{W(\phi)}$.
Let $\lambda \notin \overline{W(\phi)}$. Then by theorem 1.4.2 $T - \lambda I$ is injective and there is a $\delta > 0$ such that

$$|\phi(x) - \lambda| \geq \delta, \|x\| = 1, x \in D(T).$$

Since $\|x\| = 1$, $\|x\|^2 = 1$ and we have

$$|\phi(x) - \lambda\|x\|^2| \geq \delta\|x\|^2, \|x\| = 1.x \in D(T). \tag{3.7}$$

Let us define a number dependent of λ as $\phi_\lambda(x, y) = \phi_(x, y) - \lambda\langle x, y\rangle$ by 3.5.
Thus ϕ_λ satisfies hypothesis of theorem 3.1.5 and if $h \in H$ be arbitrary then $\langle x, h\rangle$ is a linear functional on $D(\phi_\lambda) = D(\phi)$ and $|\langle x, h\rangle|^2 \leq \|x\|^2\|h\|^2$ by Cauchy Bunyakowki's inequality and $|\langle x, h\rangle|^2 \leq \|x\|^2\|y\|^2 \leq C|\phi_\lambda(y)|$.
Hence by theorem 1.4.3 there is an $x \in H$ such that $\phi_\lambda(x, y) = \langle h, y\rangle, y \in D(\phi)$ showing that $x \in D(T)$ and $(T - \lambda I)x = h$. Since h was arbitrarily taken from H, we find that $R(A - \lambda) = H$ showing that λ is in the resolvent set of T.

We now show that $\overline{W(\phi)} = \overline{W(T)}$. It is enough to prove that $W(T) \subset W(\phi) \subset \overline{W(T)}$.

Let $x \in D(T)$, then $Tx, x = W(T)$, for $\|x\| = 1$ and $Tx, x = \phi(x) = \phi(x, x) = W(\phi)$.

Thus $W(T) \subset W(\phi)$.

Then we show that $W(\phi) \subset \overline{W(T)}$. We will achieve this by showing that for each $x \in D(\phi)$, there is a sequence $\{x_n\} \subset D(T)$ such that $\phi(x_k) \to \phi(x)$.

Let $\phi_1(x, y)$ be a symmetric sesquilinear functional such that

$$\beta^{-1}|\phi(x)| \leq \phi_1(x) \leq |\phi(x)| + \beta\|x\|^2, x \in D(\phi_1).$$

Then

$$\begin{aligned}|\phi(y) - \phi(x)| &= |\phi(y) - \phi(x) + \phi(x, y) - \phi(x, y)| \\ &= |\phi(y, y) - \phi(x, x) + \phi(x, y) - \phi(x, y)| \\ &= |\phi(y - x, y) + \phi(x, y - x)| \text{ by triangle inequality we have} \\ &\leq |\phi(y - x, y)| + |\phi(x, y - x)|.\end{aligned}$$

By applying corollary 1 to each term of $\leq |\phi(y - x, y)| + |\phi(x, y - x)|$ we have

$$\begin{aligned}|\phi(y) - \phi(x)| \leq |\phi(y - x, y)| + |\phi(x, y - x)| &\leq 2\beta\phi_1(y - x)^{\frac{1}{2}}\phi_1(y)^{\frac{1}{2}} + 2\beta\phi_1(x)^{\frac{1}{2}}\phi_1(y - x)^{\frac{1}{2}} \\ &= 2\beta\phi_1(y - x)^{\frac{1}{2}}[\phi_1(y)^{\frac{1}{2}} + \phi_1(x)^{\frac{1}{2}}].\end{aligned}$$

Thus it suffices to show that for $x \in D(\phi)$, there is a sequence $\{x_k\} \subset D(T)$ such that $\phi_1(x_k - x) \to 0$.

Consider $D(\phi)$ as a vector space with scalar product $\phi_1(x, y) + \langle x, y\rangle$, then it is a normed vector space, say X, we the norm $[\phi_1(x) + \|x\|^2]^{\frac{1}{2}}$. We show that $D(\phi)$ is dense in X. If were not dense, we would have a $z \in X$ such that $\min |x - z| > 0, \forall x \in D(T)$.

By theorem 1.1.5, there is a bounded linear functional $F \neq 0$ on X such that $F(x) = 0$ $\forall x \in D(T)$. Let λ be any scalar such that $\lambda \notin \overline{(W(\phi))}$ and $\phi_\lambda(x, y) = \phi_(x, y) - \lambda\langle x, y\rangle$

as above. Then ϕ_λ satisfies the hypothesis of theorem theorem 3.1.5. thus by 3.4 and 3.5

$$|Fy|^2 \leq K[\phi_1(y) + \|y\|^2] \leq K'|\phi_\lambda(y)|, y \in X.$$

Thus by theorem 3.1.5. there is $z \in X$ such that

$Fy = \phi_\lambda(y, z), y \in X$. Since F annihilates $D(T)$, we have $Fy = \phi_\lambda(y, z) = 0$, or

$\phi_\lambda(y, z) = 0, y \in D(T)$.

This is equivalent to $((T - \lambda)y, z) = 0, y \in D(T)$ implying that $R(T - \lambda) = H$ so that there is a $y \in D(T)$ such that $(T - \lambda)y = z$. Thus $z = 0$ implying that $F = 0$. But $F \neq 0$ and $F = 0$ is a contradiction, thus, we must have $D(T)$ being dense in X. Thus $W(\phi) \subset \overline{W(T)}$ and $\overline{W(\phi)} = \overline{W(T)}$ As required. □

Theorem 3.1.7. *Let ϕ be a densely defined sesquilinear functional such that $\overline{W(\phi)}$ is not the whole plane, a half plane, a strip or a line. Suppose that there are sequences $\{x_n\}$ in the domain of $D(\phi)$ where $x_n \to 0, \phi(x_n - x_m)$ imply $\phi(x_n) \to 0, n, m \in \mathbb{Z}$. Then $\phi(x, y), x, y \in D(\phi)$ has a closed extension denoted $\widetilde{\phi}(x, y)$ such that $D(\phi)$ is dense in $D(\widetilde{\phi})$ and*

$$W(\phi) \subset W(\widetilde{\phi}) \subset \overline{W(a)}.$$

Proof. Let us define the extension $\widetilde{\phi}(x, y)$ as follows:

$x \in D(\widetilde{\phi})$ if there is a $\{x_n\} \in D(\phi)$ such that $\phi(x_n - x_m) \to 0$ and $x_n \to x \in H$.If $\{y_n\}$ is such a sequence for y, the define

$$\widetilde{\phi}(x, y) = \lim_{n \to \infty} \phi(x_n, y_n). \tag{3.8}$$

We proof that this limit exists.

$$\begin{aligned}\phi(x_n, y_n) - \phi(x_m, y_m) &= \phi(x_n, y_n) - \phi(x_m, y_m) + \phi(x_n, y_m) - \phi(x_n, y_m)\\ &= \phi(x_n, y_n) - \phi(x_n, y_m) + \phi(x_n, y_m) - \phi(x_m, y_m)\\ &= \phi(x_n, y_n - y_m) + \phi(x_n - x_m, y_m).\end{aligned}$$

Taking absolute values, we get

$|\phi(x_n, y_n) - \phi(x_m, y_m)| = |\phi(x_n, y_n - y_m) + \phi(x_n - x_m, y_m)|$.

By triangle inequality, we have

$|\phi(x_n, y_n) - \phi(x_m, y_m)| \leq |\phi(x_n, y_n - y_m)| + |\phi(x_n - x_m, y_m)|$.

By theorem 3.1.4, there is a symmetric sesquilinear form $\phi_1(x, y)$ such that $\beta^{-1}|\phi(x)| \leq \phi_1(x) \leq |\phi(x)| + \beta\|x\|^2, x \in D(\phi)$. Thus inequality satisfies the hypothesis of corollary 1, $|\phi(x)| \leq \beta\phi_1(x)$ hence applying the corollary on the right hand side, we get

$|\phi(x_n, y_n) - \phi(x_m, y_m)| \leq 2\beta\phi_1(x_n)^{\frac{1}{2}}\phi_1(y_n - y_m)^{\frac{1}{2}} + 2\beta\phi_1(y_m)^{\frac{1}{2}}\phi_1(x_n - x_m)^{\frac{1}{2}}$.

Upon factorization of common factors, we get

$|\phi(x_n, y_n) - \phi(x_m, y_m)| \leq 2\beta(\phi_1(x_n)^{\frac{1}{2}}\phi_1(y_n - y_m)^{\frac{1}{2}} + \phi_1(y_m)^{\frac{1}{2}}\phi_1(x_n - x_m)^{\frac{1}{2}})$.

By the equation $\beta^{-1}|\phi(x)| \leq \phi_1(x) \leq |\phi(x)| + \beta\|x\|^2, x \in D(\phi)$, $\phi_1(x, y)$ is sandwiched between two sequences of $\phi(x, y)$ which converges by the hypothesis. Thus $|\phi(x_n, y_n) - \phi(x_m, y_m)|$ converges to zero. Moreover the limit 3.8 is unique, that is, it does not depend on the sequence chosen. We proof this claim.

Let $\{x'_n\}$ and $\{y'_n\}$ be other sequences for x and y respectively. Set

$x''_n = x'_n - x_n, y''_n = y'_n - y_n$ then by theorem 3.1.4 we have

$$\phi(x''_n - x''_m)^{\frac{1}{2}} \leq \phi(x'_n - x'_m)^{\frac{1}{2}} + \phi(x_n - x_m)^{\frac{1}{2}} \to 0 \text{ as } m, n \to \infty.$$

Thus $\phi(x'_n - x'_m) \to 0$ and similarly $\phi(y''_n - y''_m) \to 0$. Since $x''_n \to 0, y''_n \to 0$ in H, we conclude by hypothesis that

$$\phi(x_n'') \to 0, \phi(y_n'') \to 0 \text{ as } n \to \infty,$$

which implies that

$$\phi_1(x_n'') \to 0, \phi_1(y_n'') \to 0 \text{ as } n \to \infty.$$

Hence,

$$|\phi(x_n', y_n') - \phi(x_n, y_n)| \leq |\phi(x_n', y_n'')| + |\phi(x_n'', y_n)| \leq 2k[\phi_1(x_n')^{\frac{1}{2}}\phi_1(x_n'')^{\frac{1}{2}} + \phi_1(x_n'')^{\frac{1}{2}}\phi_1(x_n)^{\frac{1}{2}}] \to 0.$$

From the way $\widetilde{\phi}(x, y)$ was defined, we arrive at the conclusion

$$W(\phi) \subset W(\widetilde{\phi}) \subset \overline{W(a)}$$

We now show that $D(\phi)$ is dense in $D(\overline{\phi})$.

Let $x \in D(\overline{(\phi)}$, then there is a sequence $\{x_n\} \in D(\phi)$ such that $\phi(x_n - x_m) \to 0$ while $x_n \to x \in H$ and $\phi(x_n) \to \widetilde{\phi}(x)$. In particular for each n,

$\phi(x_n - x_m) \to \widetilde{\phi}(x_n - x)$ as $m \to \infty$. Now let $\epsilon > 0$ be given, and take N so large enough that

$$|\phi(x_n - x_m)| < \epsilon, m, n > N.$$

Letting $m \to \infty$, we obtain

$$|\widetilde{\phi}(x_n - x)| \leq \epsilon, n > N$$

implying that

$$\widetilde{\phi}(x_n - x) \to 0 \text{ as } n \to \infty,$$

showing that $D(\phi)$ is dense in $D(\widetilde{\phi})$.

Finally, we show that $\widetilde{\phi}$ is closed.

$\widetilde{\phi}$ is a sesquilinear functional and since we are given that that $W(\phi)$ is not a whole plane, half plane, strip or a line, we take $W(\widetilde{\phi})$ to satisfies the same conditions too. Then by theorem 3.1.4 there exists a symmetric sesquiliear functional $\widetilde{\phi}_1$ with $D(\widetilde{\phi}_1) = D(\widetilde{\phi})$ such that

$$\beta^{-1}|\widetilde{\phi}(x)| \leq \widetilde{\phi}_1(x) \leq |\widetilde{\phi}(x)| + \beta\|x\|^2, x \in D(\widetilde{\phi}).$$

Let $x_n \in D(\widetilde{\phi}), \widetilde{\phi}(x_n - x_m) \to 0$ and $x_n \to x \in H$.
By 3.4 since the equivalent of $\widetilde{\phi}_1$ is sandwiched by the equivalents of $\widetilde{\phi}$ and $\widetilde{\phi} + \beta\|x\|$ which converges to 0 by the hypothesis we find that $\widetilde{\phi}_1(x_n - x_m) \to 0$ as $n \to \infty$. Since $D(\phi)$ is dense in $D(\widetilde{\phi})$ we have that for each n there is a $y_n \in D(\phi)$ such that

$$\widetilde{\phi}(x_n - y_n) \leq \tfrac{1}{n^2}, \|x_n - y_n\| < \tfrac{1}{n}.$$

By 3.4

$$\widetilde{\phi}_1(x_n - y_n) \leq |\widetilde{\phi}(x_n - y_n)| + \beta\|x_n - y_n\|.$$

Upon substitution, we have

$$\widetilde{\phi}_1(x_n - y_n) < \tfrac{1}{n^2} + \tfrac{\beta}{n}.$$

Without loss of generality and for simplification purposes, we can also have

$$\widetilde{\phi}_1(x_n - y_n) < \tfrac{1}{n^2} + \tfrac{\beta}{n^2} = \tfrac{1+\beta}{n^2}$$

which is true. Since $\widetilde{\phi}_1$ is a sesquilinear form we have

$$\begin{aligned}\widetilde{\phi}_1(y_n - y_m)^{\frac{1}{2}} &= \widetilde{\phi}_1(y_n - x_n + x_n - y_m + y_m - y_m)^{\frac{1}{2}} \\ &\leq \widetilde{\phi}_1(y_n - x_n)^{\frac{1}{2}} + \widetilde{\phi}_1(x_n - y_m)^{\frac{1}{2}} + \widetilde{\phi}_1(y_m - y_m)^{\frac{1}{2}}.\end{aligned}$$

The right hand side approaches 0 as $n, m \to \infty$.

Hence

$\widetilde{\phi}_1(y_n - y_m)^{\frac{1}{2}} \to 0$ as $n, m \to \infty$ or $\widetilde{\phi}_1(y_n - y_m) \to 0$ as $n, m \to \infty$.

Using 3.4 that $|\widetilde{\phi}(x)| \leq \beta\widetilde{\phi}_1(x)$ we get that $\widetilde{\phi}(y_n - y_m) \to 0$ too. Since $v_n \in D(\phi)$ and $\widetilde{\phi}$ is an extension of ϕ, by restriction fo the domain, we have that
$\phi(y_n - y_m) = \widetilde{\phi}(y_n - y_m) \to 0$.
But $\|y_n - x\| = \|y_n - x_n + x_n - x\| \leq \|y_n - x_n\| + \|x_n - x\|$ by triangle inequality for norms and we know that $x_n \to x$ and
$\|y_n - x_n\| = \| - 1(x_n - y_n)\| = | - 1|\|(x_n - y_n)\| = \|x_n - y_n\| < \frac{1}{n} \to 0$ as $n \to \infty$, we have $\|y_n - x\| = \|y_n - x_n + x_n - x\| \leq \|y_n - x_n\| + \|x_n - x\| \to 0$ as $n \to \infty$. Since $\|y_n - x\| \to 0$ as $n \to \infty, y_n \in D(\phi)$ we see that $x \in D(\widetilde{\phi})$ by definition of an extension. Since $x_n \to x$, $y_n \to x$ and $\widetilde{\phi}(x_n \to x) \to 0$ as $n \to \infty$, we have that

$$\widetilde{\phi}(y_n \to x) \text{ and } \widetilde{\phi}_1(y_n \to x) \to 0 \text{ as } n \to \infty.$$

The inequality $\widetilde{\phi}_1(x_n - y_n) < \frac{1+\beta}{n^2} \to 0$ as $n \to \infty$ together with
$\widetilde{\phi}_1(y_n \to x) \to 0$ as $n \to \infty$ implies that $\widetilde{\phi}_1(x_n \to x) \to 0$ as $n \to \infty$ which in turn by 3.4 imply that $\widetilde{\phi}(x_n \to x) \to 0$ as $n \to \infty$. Thus, by definition of a closed sesquilinear functional, we have that $\widetilde{\phi}$ is closed. □

Theorem 3.1.8. *Let T be a densely defined linear operator on H such that $\overline{W(T)}$ is not the whole plane, a half plane, a strip or a line. Then T has a closed extension $\widetilde{T}$ such that*

$$\sigma(\widetilde{T}) \subset \overline{W(T)} = \overline{W(\widetilde{T})}.$$

Proof. Let $\phi(x, y)$ be a sesquilinear functional and T be its associated operator, then $\phi(x, y) =\rangle Tx, y\rangle x, y \in D(T) = D(\phi)$ and $W(\phi) = W(T)$.
By theorem 3.1.4 there is a symmetric sesquilinear form $\phi_1(x, y)$ such that 3.4 holds
Suppose $\{x_n\} \subset D(\phi), x_m \to 0$ and $\phi(x_n - x_m) \to 0$

$$\begin{aligned}\phi(x_n) &= \phi(x_n) + \phi(x_n, x_m) - \phi(x_n, x_m)\\ &= \phi(x_n, x_n) - \phi(x_n, x_m) + \phi(x_n, x_m), \text{ by properties of sesquilinear forms, we have}\\ &= \phi(x_n, x_n - x_m) + \phi(x_n, x_m), \text{ by properties of associated operator, we have}\\ &= \phi(x_n, x_n - x_m) + \langle Tx_n, x_m\rangle, \text{ taking absolute values, we get}\end{aligned}$$

$|\phi(x_n)| = |\phi(x_n, x_n - x_m) + \langle Tx_n, x_m\rangle|$, by triangle inequality for absolute values , we get

$|\phi(x_n)| \leq |\phi(x_n, x_n - x_m)| + |\langle Tx_n, x_m\rangle|$.

From 3.4 we have $|\phi_1(x)| \leq \beta\phi_2(x)$ thus satisfying the hypothesis of corollary 1.

Applying the Corollary 1, we get

$|\phi(x_n)| \leq 2\beta\phi_1(x_n)^{\frac{1}{2}}\phi_1(x_n - x_m)^{\frac{1}{2}} + |\langle Tx_n, x_m\rangle|$ By Cauchy Bunyakowki's inequality, we have

$$|\phi(x_n)| \leq 2\beta\phi_1(x_n)^{\frac{1}{2}}\phi_1(x_n - x_m)^{\frac{1}{2}} + \|Tx_n\|\|x_m\|. \tag{3.9}$$

For $x_n, x_m \in D(\phi) \cap D(\phi_1) \subset D(\phi)$

hence $x_n - x_m \in D(\phi) \cap D(\phi_1) \subset D(\phi)$ hence by 3.4 we have

$\beta^{-1}\phi(x_n - x_m) \leq \phi_1(x_n - x_m) \leq \phi(x_n - x_m) + \beta\|x_n - x_m\|^2$.

Taking $\phi_1(x_n - x_m) \leq \phi(x_n - x_m) + \beta\|x_n - x_m\|^2$. Since $\phi(x_n - x_m) \to 0$ and $\|x_n - x_m\| \to 0$ implies $\phi_1(x_n - x_m) \to 0$ as $n, m \to \infty$. This implies that $\phi_1(x_n)$ is bounded hence there is a constant τ such that $\phi_1(x_n) \leq \tau^2, \forall n$. Let $\epsilon > 0$, then we can choose a number N larger enough such that

$$\phi_1(x - n - x_m) < \frac{\epsilon^2}{4\beta^2\tau^2} \text{ for } n, m > N$$

or

$$\phi_1(x - n - x_m)^{\frac{1}{2}} < \frac{\epsilon}{2\beta\tau} \text{ for } n, m > N.$$

Substituting into 3.9, we get

$|\phi(x_n)| \leq \epsilon + \|Tx_n\|\|x_m\|$ for $n, m > N$.

Taking $m \to 0, \|x_m\| \to 0$ we have $|\phi(x_n)| < \epsilon, n > N$ Thus $\phi(x_n) \to 0$ as $n \to \infty$ satisfying the hypothesis of theorem 3.1.7. Hence $\phi(x, y)$ has a closed extension $\widetilde{\phi}(x, y)$ with $D(\phi)$ dense in $\widetilde{\phi}(x, y)$. Let $\widetilde{T}$ be the associated operator for $\widetilde{\phi}(x, y)$. Then by 3.1.6 $\sigma(\widetilde{T}) \subset \overline{W(\widetilde{\phi})} = \overline{W(\widetilde{T})}$.

But

$\overline{W(\widetilde{\phi})} = \overline{W(\phi)} = \overline{W(\widetilde{T})}$

hence

$$\sigma(\widetilde{T}) \subset \overline{W(\widetilde{\phi})} = \overline{W(\widetilde{T})}.$$

We now show that $\widetilde{T}$ is an extension of T.

Let $x \in D(T)$. This implies that $\phi(x, y) = \langle x, y \rangle, y \in D(\phi) = D(T)$.

Now $\widetilde{\phi}(x, y) = \phi(x, y) = \langle Tx, y \rangle$ since $\widetilde{T}$ is an extension, we have that $\widetilde{\phi}(x, y) = \langle Tx, y \rangle, y \in D(\phi)$. We claim that this holds for all $y \in D(\widetilde{\phi})$.

This follows since $D(\phi) = D(\widetilde{\phi})$.

Let $y \in D(\widetilde{\phi})$ hence there is a sequence $\{y_n\} \in D(\phi)$ such that

$$\widetilde{\phi}(y_n - y) \to 0 \text{ and } \|y_n - y\| = 0.$$

Since $\overline{W(\widetilde{\phi})}$ is not the whole plane, half plane, strip or a line, we apply theorem 3.1.4 so that $\widetilde{\phi}(x, y_n) \to \widetilde{\phi}(x, y)$. Since $\widetilde{\phi}(x, y_n) = \langle Tx, y_n \rangle$ we have that $\widetilde{\phi}(x, y) = \langle Tx, y \rangle \forall y \in D(\phi)$.

Thus $x \in D(\widetilde{T})$ and $\widetilde{T}x = Tx$. Thus $\widetilde{T}$ is an extension of T. □

Theorem 3.1.9. *Let ϕ be a densely defined sesquilinear functional such that $\overline{W(\phi)}$ is not the whole plane. Let T be the operator associated with ϕ. If $D(T)$ is dense in H, then T is closable.*

Proof. Since $\overline{W(\phi)}$ is not the whole plane and it is convex, it must be contained in a half plane, hence applying the above corollary, there are constants r, k_0 such that for $\|r\| = 1$

$$Re[r\phi(x) + k_0\|x\|^2] \geq 0, \forall x \in D(\phi)$$

which is equivalent to

$$Re[r\phi(x,x) + k_0\langle x, x\rangle] \geq 0, \forall x \in D(\phi).$$

For some $y \in D(\phi)$, it is true that

$$Re[r\phi(x,y) + k_0\langle x, y\rangle] \geq 0, \forall x, y \in D(\phi).$$

But $\phi(x,y) \in \mathbb{C}$ since ϕ is a functional and $r, k_0 \in \mathbb{C}$ hence $r\phi(x,y) + k_0\langle x, y\rangle \in \mathbb{C}$. Thus, we can get a sesquilinear functional whose image is $r\phi(x,y) + k_0\langle x, y\rangle$. Let ψ be the said functional, thus

$$\psi(x,y) = r\psi(x,y) + k_0\langle x, y\rangle.$$

This is compatible to

$$S = rT + k_0 I,$$

Since T is an operator associated with ϕ and I is an identity operator, S is the operator associated with ψ, moreover, T is closable if and only if S is. If S is closable then it has a closed extension by theorem 3.1.1 and

$$\sigma(\widetilde{S}) \subseteq \overline{W(\widetilde{S})} = \overline{W(S)}$$

by theorem 3.1.8. Hence it suffices to show that S is closable. Let $\{x_n\}$ be a sequence of elements of $D(T)$ such that

$$x_n \xrightarrow{s} \overline{0} \text{ and } Sx_n \xrightarrow{s} z.$$

Then for $\alpha > 0$ and $w \in D(T)$ then $x_n - \alpha w \in D(T)$ since $D(T)$ is a vector space and

$$\psi(x_n - \alpha w) = \psi(x_n - \alpha w, x_n - \alpha w)$$

$$\psi(x_n - \alpha w) = \psi(x_n, x_n) - \psi(x_n, \alpha w) - \psi(\alpha w, x_n) + \psi(\alpha w, \alpha w)$$

$$\psi(x_n - \alpha w) = \psi(x_n, x_n) - \alpha\psi(x_n, w) - \alpha\psi(w, x_n) + \alpha^2\psi(w, w)$$

(by linearity in the first and second variable)

$$\psi(x_n - \alpha w) = \psi(x_n) - \alpha\psi(x_n, w) - \alpha\psi(w, x_n) + \alpha^2\psi(w).$$

But S is associate operator of ψ hence replacing the latter appropriately by the former, we have

$$\psi(x_n) - \alpha\psi(x_n, w) - \psi(w, x_n) + \alpha^2\psi(w) = \langle Sx_n, x_n\rangle - \alpha\langle Sx_n, w\rangle - \alpha\langle Sw, x_n\rangle + \alpha^2\psi(w).$$

We have that

$$x_n \xrightarrow{s} \overline{0} \text{ and } Sx_n \xrightarrow{s} z \text{ as } n \to \infty.$$

Substituting these into the above equation, we have

$$\langle Sx_n, x_n\rangle - \alpha\langle Sx_n, w\rangle - \alpha\langle Sw, x_n\rangle + \alpha^2\psi(w) \to -\alpha\langle z, w\rangle + \alpha^2\psi(w).$$

Hence

$$Re[-\alpha\langle z, w\rangle + \alpha^2\psi(w)] \geq 0, \alpha > 0, w \in D(T).$$

Dividing through by α we get

$$Re[-\langle z, w\rangle + \alpha\psi(w)] \geq 0, \alpha > 0, w \in D(T).$$

Letting $\alpha \to 0$ we find that

$$Re[-\langle z, w\rangle \geq 0, \alpha > 0, w \in D(T).$$

Dividing through by negative one, we have

$$Re[\langle z, w\rangle \leq 0 \text{ for all } w \in D(T)]. \tag{3.10}$$

Since $D(T)$ is dense in H, there is a sequence $\{y_n\}$ of elements of $D(T)$ such that

$$y_n \xrightarrow{s} z \in H.$$

Since $\{y_n\} \in D(T)$ we use $\{y_n\}$ in place of w to have $Re\langle z, y_n\rangle \leq 0$.
Taking the limits, we have $Re\langle z, y_n\rangle \leq 0 \to Re\langle z, z\rangle \leq 0$, which implies have in the limit $Re\|z\|^2 = \|z\|^2 \leq 0$ (since $\|z\|^2$ is real) which shows that $z = \overline{0}$. Hence S is closable.
This implies that T is also closable. □

We now show the results quoted in the beginning of this section

Theorem 3.1.10. *Let T be a densely defined linear operator in H such that $W(T)$ is not the whole complex plane. Then T has a closed extension.*

Proof. From theorem 3.1.1, an operator has a closed extension if it is closable, thus it suffices to show that a densely defined operator T in H is closable if $W(T)$ is not the whole plane.
Set $\phi(x, y) = \langle Tx, y\rangle$ for all $x, y \in D(T)$, then T is the associated operator of a sesquilinear functional,ϕ. By virtue of the definition of T, we have that $D(\phi) = D(T)$ and $W(\phi) = W(T)$
Since ϕ satisfies all the hypothesis of theorem 3.1.9. Hence we conclude that T is closable and by theorem 3.1.1, T has a closed extension. □

If $\overline{W(T)}$ is the whole plane then we have

$$\sigma(\widetilde{T}) \subseteq \overline{W(T)} = \overline{W(\widetilde{T})}$$

is vacuously true for all extensions of T.

On the other hand, the existence of a closed extension of T depends on whether or not T is closable as provided in theorem 3.1.1. Hence this case is not interesting.

In all other cases $W(T)$ is contained in a half plane. Thus there are constants r, k with $|r| = 1$ such that

$$Re[r\langle Tx, x\rangle - k\|x\|^2] \leq 0 \text{ for all } x \in D(T).$$

From this equation, we can form an operator S from T and I such that

$$S = rT - kI. \tag{3.11}$$

Then for all $x \in D(T)$ we have $Sx = rTx - kIx$ Thus

$$D(S) = D(T) \text{ and } Re\langle Sx, x\rangle \leq 0 \text{ for all } x \in D(S). \tag{3.12}$$

3.2 Dissipative operators

The study of extensions of densely defined operators has led to operators whose image have a real part that is contained wholly in the left hand side of the imaginary line of the complex plane. This was achieved when we came to a conclusion that when the numerical range of a sesquilinear operator , ϕ is not the whole plane, then it is contained in the half plane and is given by

$$Re[r\phi(x) + k_0\|x\|^2] \geq 0, \forall x \in D(\phi)$$

The previous discussions let to the function whose image is given be $r\phi(x) + k_0\|x\|^2] \geq 0$ leading to sesquilinear function whose associated operator is defined by equation (3.11), an operator whose real part is less than or equal to zero. This operator is called **dissipative operator**. We now proceed to give the formal definition of a dissipative operator which will ensure that our study onwards remains in one half plane, the negative real part of the complex plane.

Definition 3.2.1. Dissipative operator

Let S be a linear operator. S is called a dissipative operator if

$$Re\langle Sx, x\rangle \leq 0 \text{ for all } x \in D(S).$$

Our interest now shifts to determining if there can be an extension for our new operator given that its numerical range is a half plane.

Theorem 3.2.1. *Let S be a dissipative operator in H with $D(S)$ dense in H. Then S has a closed dissipative extension $\widetilde{S}$ such that $\sigma(S)$ is contained in the half plane $Re\lambda \leq 0$*

Proof. S is dissipative, hence

$$Re\langle Sx, x\rangle \leq 0 \text{ for all } x \in D(S).$$

Multiplying the equation by negative, we have

$$Re(-\langle Sx, x\rangle) \geq 0 \text{ for all } x \in D(S). \tag{3.13}$$

Let x be such that $\|x\| = 1$, then $\|x\|^2 = 1$. Adding $\|x\|^2 = 1$ on both sides of inequality (3.13) we have

$$\|x\|^2 + Re(-\langle Sx, x\rangle) \geq \|x\|^2 \text{ for all } x \in D(S).$$

By definition of a norm, $\|x\|$ is real so is $\|x\|^2$ hence $\|x\|^2 = Re(\|x\|^2)$. Substituting into the above inequality, we get

$$Re\|x\|^2 + Re(-\langle Sx, x\rangle) \geq \|x\|^2 \text{ for all } x \in D(S)$$

or

$$Re(\|x\|^2 - \langle Sx, x\rangle) \geq \|x\|^2 \text{ for all } x \in D(S)$$

Since $\|x\|^2 = \langle x, x\rangle = \langle Ix, x\rangle$ we have

$$Re(\langle Ix, x\rangle - \langle Sx, x\rangle) \geq \|x\|^2 \text{ for all } x \in D(S).$$

By definition of inner product, the inequality reduces to

$$Re(\langle Ix - Sx, x\rangle) \geq \|x\|^2 \text{ for all } x \in D(S),$$

then

$$Re\langle (I - S)x, x\rangle \geq \|x\|^2.$$

This inequality shows that the operator $(I - S)$ is bounded below, hence by theorem 1.2.1 $(I - S)$ is one to one. Since injectiveness implies the existence of an inverse, $(I - S)$ has an inverse, $(I - S)^{-1}$, defined on $R(I - S)$, that $D(I - S)^{-1}) = R(I - S)$.

Let T be an operator such that

$$T = (I + S)(I - S)^{-1}$$

Let $x \in D(I - S)^{-1}$, then $x \in D(I - S)$. Then for $x \in D(I - S)^{-1}$ there is a $y \in R(1 - s)^{-1}$ such that

$$y = (I - S)^{-1}x \tag{3.14}$$

For x to be acted upon by $(I - S)^{-1}$ we must have it being acted upon by T too. Thus

$$Tx = (I + S)(I - S)^{-1}x \text{ for all } x \in D(I - S)^{-1}. \tag{3.15}$$

This implies that the $D(T) = D(I - S)^{-1}) = R(I - S)$. Thus

$$D(T) = R(I - S).$$

Substituting y with $(I - S)^{-1}x$ in equation (3.9) we have

$$Tx = (I + S)y. \tag{3.16}$$

This implies that y is in the domain of $(I + S)$ which is also in the range of $(I - S)^{-1}x$ for all $y \in (I - S)$. Since this is true for all $x \in D(I - S)^{-1}$ all $y \in R(I - S)^{-1}$, we have that

$$D(I + S) = R(1 - S)^{-1}.$$

We now determine if T is bounded. From equation (3.16), $Tx = (I + S)y$, we have

$$\|Tx\| = \|(Iy + Sy)\| = \|(y + Sy)\|$$

$$\|Tx\|^2 = \|(y + Sy)\|^2.$$

By the definition of a norm induced by inner product, we have

$$\|Tx\|^2 = \|(y + Sy)\|^2 = \langle y + Sy, y + Sy\rangle.$$

On expansion, we get

$$\langle y + Sy, y + Sy\rangle = \langle y, y\rangle + \langle y, Sy\rangle + \langle Sy, y\rangle + \langle Sy, Sy\rangle.$$

But $\langle y, y\rangle = \|y\|^2$ Thus on simplification, we have

$$\langle y, y\rangle + \langle y, Sy\rangle + \langle Sy, y\rangle + \langle Sy, Sy\rangle = \|y\|^2 + \langle y, Sy\rangle + \langle Sy, y\rangle + \|Sy\|^2.$$

Since the sum of two conjugates is twice their real part, we have

$$\|y\|^2 + \langle y, Sy\rangle + \langle Sy, y\rangle + \|Sy\|^2 = \|y\|^2 + 2Re\langle Sy, y\rangle + \|Sy\|^2. \tag{3.17}$$

Since S is dissipative , $Re\langle Sy, y\rangle \leq 0$ consequently, $2Re\langle Sy, y\rangle \leq 0$.

Thus by multiplying the inequality by negative one, we have $-2Re\langle Sy, y\rangle \geq 0$.

Substituting the inequality in right hand side of equation (3.17) we have

$$\|y\|^2 + 2Re\langle Sy, y\rangle + \|Sy\|^2 \leq \|y\|^2 - 2Re\langle Sy, y\rangle + \|Sy\|^2$$

On expanding the middle term, we have

$$= \|y\|^2 - \langle y, Sy\rangle - \langle Sy, y\rangle + \|Sy\|^2$$
$$= \langle y, y\rangle - \langle y, Sy\rangle - \langle Sy, y\rangle + \langle Sy, Sy\rangle$$
$$= \langle y - Sy, y - Sy\rangle$$
$$= \|y - Sy\|^2$$
$$= \|(I - S)y\|^2.$$

Finally we have

$$\|Tx\|^2 \leq \|(I - S)y\|^2.$$

From equation (3.14), $y = (I - S)^{-1}x$, hence $x = (I - S)y$ and $\|x\| = \|(I - S)y\|$ and $\|x\|^2 = \|(I - S)y\|^2$.

Thus

$$\|Tx\|^2 \leq \|(I - S)y\|^2 = \|x\|^2.$$

Hence

$$\|Tx\|^2 \leq \|x\|^2$$

and

$$\|Tx\| \leq \|x\|$$

$$\|Tx\| \leq \|x\| \text{ for all } x \in D(T). \tag{3.18}$$

We now find the extension of T and determine if its is bounded.

Thus, extend T to $\overline{D(T)}$. That is, if $x \in \overline{D(T)}$ then it is an accumulation point. That is there is a sequence $\{x_n\}$ of elements of $D(T)$ such that $x_n \xrightarrow{s} x$.

By (3.18), Tx_n is a Cauchy sequence in H , since such sequences are convergent, it has a limit say z. Define to be $\overline{T}x = z$. We now check that this definition is independent of the sequence chosen and that $\overline{T}$ is an extension of T to $\overline{D(T)}$

Let x be any element of H, then by Projection theorem $x = w + y$, where $w \in \overline{D(T)}$ and y is orthogonal to $D(T)$. Define $\widetilde{T}x = \overline{T}w$ then

$$\|\widetilde{T}x\| = \|\overline{T}w\| \leq \|w\| \leq \|x\|.$$

Thus $\widetilde{T} \in B(H)$ and since

$$\|\widetilde{T}x\| \leq \|x\|$$

implies

$$\|\widetilde{T}\| \leq 1. \tag{3.19}$$

Now by (3.14) and (3.16)

$$x = y - Sy, Tx = y + Sy. \tag{3.20}$$

Adding these two we have $2y = x + Tx$ and subtracting the two, we have $2Sy = Tx - x$. Thus, we have the system

$$2y = x + Tx, 2Sy = Tx - x. \tag{3.21}$$

From $2y = x + Tx$, we have by linearity if $(I + T)$ that $2y = (I + T)x$, we therefore substitute for y using equation (3.14),$y = (I - S)^{-1}x$ to get $2(I - S)^{-1}x = (I + T)x$, thus

$$2(I - S)^{-1} = (I + T) \tag{3.22}$$

implies that $(I + T)$ has an inverse, thus invertible, consequently, one to one operator. Also, from the relation above, we get that

$R(I + T) = R((1 - S)^{-1}) = D(1 - S) = D(S)$.

Therefore, $R(I + T) = D(S)$.

Now $D((I + T)^{-1}) = R(I + T) = D(S)$.

Likewise, from equation (3.21), $2Sy = Tx - x$ can be written as $2Sy = (T - I)x$.

Substituting for y using equation (3.14),$y = (I - S)^{-1}x$ to get $2S(I - S)^{-1}x = (T - I)x$, thus

$$2S(I - S)^{-1} = (T - I). \tag{3.23}$$

Dividing equation (3.23) by (3.22), we get

$$\frac{2S(I - S)^{-1}}{2(I - S)^{-}1} = \frac{(T - I)}{(I + T)}.$$

Hence

$$S = \frac{(T - I)}{(I + T)}$$

Since $(T+I)$ is invertible, we have the following

$$S = (T - I)(I + T)^{-1}. \tag{3.24}$$

A candidate for the extension is $\widetilde{S}$ is

$$\widetilde{S} = (\widetilde{T} - I)(I + \widetilde{T})^{-1}. \tag{3.25}$$

With $D(\widetilde{S}) = R(I + \widetilde{T})$.

Just like in equation (3.18) where we had to validate it after confirming that $(I+T)$ is one to one hence is invertible, we cannot assume that (3.25) holds without determining the invertibility of $(I + \widetilde{T})$.

Let $x \in H$ such that

$$(I + \widetilde{T})x = \overline{0}. \tag{3.26}$$

Let y be any element of H;

set $z = (I + \widetilde{T})y$, then $z = y + \widetilde{T}y$ and $z - y = \widetilde{T}y$.

Applying the norm function, we get $\|z - y\| = \|\widetilde{T}y\|$ By triangle inequality for norm, we have

$\|z - y\| = \|\widetilde{T}y\| \leq \|\widetilde{T}\|\|y\|$. Due to boundedness of $\widetilde{T}$ in (3.19) we have

$$\|z - y\| = \|\widetilde{T}y\| \leq \|\widetilde{T}\|\|y\| \leq \|y\|.$$

Hence $\|z - y\| \leq \|y\|$.

For some α a positive real number $\|z - y + \alpha x\| \leq \|y - \alpha x\|$.

Factorizing -1 on the left hand side ans squaring both sides, we get

$$\|z - y + \alpha x\|^2 \leq \|y - \alpha x\|^2 \tag{3.27}$$

$\|z - (y - \alpha x)\|^2 \leq \|y - \alpha x\|^2$.

We the expand the inequality on both sides by using definition of the norm generated by an inner product to get

$$\langle z-(y-\alpha x), z-(y-\alpha x)\rangle \leq \|y-\alpha x\|^2.$$

On expansion, we get

$$\langle z, z\rangle - \langle z, y-\alpha x\rangle - \langle y-\alpha x, z\rangle + \langle y-\alpha x, y-\alpha x\rangle \leq \|y-\alpha x\|^2.$$

Which reduces to $\|z\|^2 - 2Re\langle z, y-\alpha x\rangle + \|y-\alpha x\|^2 \leq \|y-\alpha x\|^2$

$\|z\|^2 - 2Re\langle z, y-\alpha x\rangle \leq 0$

$\|z\|^2 - 2Re\langle z, y\rangle + 2Re\langle z, \alpha x\rangle \leq 0$

$\|z\|^2 - 2Re\langle z, y\rangle + 2\alpha Re\langle z, x\rangle \leq 0.$

Dividing through by alpha gives

$$\frac{\|z\|^2}{\alpha} - \frac{2Re\langle z, y\rangle}{\alpha} + 2Re\langle z, x\rangle \leq 0$$

letting $\alpha \to \infty$ this gives

$2Re\langle z, x\rangle \leq 0$ which reduces to

$$Re\langle z, x\rangle \leq 0 \quad z \in R(I+\widetilde{T}). \tag{3.28}$$

Since $R(I+\widetilde{T}) \supseteq R(I+T) = D(T)$

We see that $R(I+T)$ is dense in H

Hence there is a sequence $\{z_n\}$ of elements in $R(I+\widetilde{T})$ such that $z_n \xrightarrow{s} x$ in H

Thus (3.28) implies $Re\|x\|^2 \leq 0$ which shows that $x = \overline{0}$ Thus the operator $\widetilde{S}$ given by (3.24) is well-defined and it is clearly an extension of S. We show that $\widetilde{S}$ is closed.

Suppose $\{x_n\}$ is a sequence of elements $D(\widetilde{S}) = R(I+\widetilde{T})$ such that

$$x_n = (I+\widetilde{T})w_n$$

and

$$w_n = (I + \widetilde{T})^{-1} x_n.$$

By (3.24)

$$\widetilde{S} x_n = (\widetilde{T} - I) w_n.$$

Adding $x_n = w_n + \widetilde{T} w_n$ to $-Sx_n = \widetilde{T} w_n - w_n$ yields $x_n - Sx_n = 2w_n$.
Hence

$$2w_n = x_n - \widetilde{S} x_n \xrightarrow{s} x - h \text{ as } n \to \infty$$

which is equivalent to

$$2w_n = (I - \widetilde{S}) x_n \xrightarrow{s} x - h \text{ as } n \to \infty.$$

Without loss of generality $w_n \xrightarrow{s} x - h$ as $n \to \infty$
From the above equation we have $w_n = (I + \widetilde{T})^{-1} x_n$ we manipulate it to be $x_n = (I + \widetilde{T}) w_n$. Multiplying $x_n = (I + \widetilde{T}) w_n$ and $\widetilde{S} x_n = (\widetilde{T} - I) w_n$ by 2, we get $2x_n = 2(I + \widetilde{T}) w_n$ and $2\widetilde{S} x_n = 2(\widetilde{T} - I) w_n$
Since $\widetilde{T} \in B(H)$, this implies

$$2x_n = 2(I + \widetilde{T}) w_n \xrightarrow{s} (I + \widetilde{T})(x - h)$$

$$2\widetilde{S} x_n = 2(\widetilde{T} - I) w_n \xrightarrow{s} (\widetilde{T} - I)(x - h)$$

from which we conclude

$$2x = (I + \widetilde{T})(x - h), 2h = (\widetilde{T} - I)(x - h) \text{ since } x_n \xrightarrow{s} x \text{ and } \widetilde{S} x_n \xrightarrow{s} h$$

In particular, we see that

$$x \in R(I + \widetilde{T}) = D(\widetilde{S}),$$

and

$$\widetilde{S}x = (\widetilde{T} - I)(I + \widetilde{T})^{-1}x = \frac{1}{2}(\widetilde{T} - I)(x - h) = h$$

hence $\widetilde{S}$ is a closed operator.

We also show that $\widetilde{S}$ is a dissipative operator.

For $w = (I + \widetilde{T})^{-1}x, \quad x = (I + \widetilde{T})w$

$\widetilde{S}x = (\widetilde{T} - I)(I + \widetilde{T})^{-1}x = (\widetilde{T} - I)w.$

Therefore,

$$\begin{aligned}\langle \widetilde{S}x, x\rangle &= \langle (\widetilde{T} - I)w, (I + \widetilde{T})w\rangle \\ &= \|\widetilde{T}w\|^2 - \langle w, \widetilde{T}w\rangle + \langle \widetilde{T}w, w\rangle - \|w\|^2.\end{aligned}$$

Since a norm is aways real and range of an inner product is in a complex field (because we are working with complex inner product space)

$Re\langle \widetilde{S}x, x\rangle = \|\widetilde{T}w\|^2 - \|w\|^2 \leq \|\widetilde{T}\|^2\|w\|^2 - \|w\|^2 = (\|\widetilde{T}\|^2 - 1)\|w\|^2.$

By (3.13) $(\|\widetilde{T}\|^2 - 1) <= 0$ and since $\|w\|^2 \leq 0$

$$Re\langle \widetilde{S}x, x\rangle = \|\widetilde{T}w\|^2 - \|w\|^2 \leq 0. \tag{3.29}$$

Finally, we must verify that $\lambda \in \rho(\widetilde{S})$ for $Re\lambda > 0$.

That is, the resolvent set is the positive half plane excluding the vertical imaginary axis. If that is the case, then $\lambda \in \sigma(\widetilde{S})$ will be contained in the negative half plane including the vertical axis. This is because the two sets are disjoint and one is the complement of the other and also due to the fact that $\lambda \in \sigma(\widetilde{S})$ is closed and $\lambda \in \rho(\widetilde{S})$ is open.

$$\begin{aligned}Re\langle (\widetilde{S} - \lambda I)x, x\rangle &= Re\langle (\widetilde{S}x - \lambda x, x\rangle = Re\langle \widetilde{S}x, x\rangle - Re\langle \lambda x, x\rangle = Re\langle \widetilde{S}x, x\rangle - Re(\lambda\langle x, x\rangle) \\ &= Re\langle \widetilde{S}x, x\rangle - Re\lambda\|x\|^2.\end{aligned}$$

Since $\widetilde{S}$ is dissipative we have

$$Re\langle(\widetilde{S}-\lambda I)x,x\rangle \leq -(Re\lambda)\|x\|^2.$$

Multiplying through by -1, we get

$$Re\lambda\|x\|^2 \leq -Re\langle(\widetilde{S}-\lambda I)x,x\rangle \leq \|(\widetilde{S}-\lambda I)x\|\|x\|.$$

Hence $\|(\widetilde{S}-\lambda I)x\|x\| \geq Re\lambda\|x\|^2$.

This implies that $(\widetilde{S}-\lambda I)$ is bounded below hence one to one for $Re\lambda > 0$.

Thus all we need to show is that $R(\widetilde{S}-\lambda I) = H$ for $Re\lambda > 0$.

Now $(\widetilde{S}-\lambda I) = [(I-\lambda)\widetilde{T}-(I+\lambda)](I+\widetilde{T})^{-1}$.

It is possible to solve

$$(\widetilde{S}-\lambda I)x = z. \tag{3.30}$$

If and only if one can solve $[(I-\lambda)\widetilde{T}-(I+\lambda)](I+\widetilde{T})^{-1}x$ which is equivalent to

$$[(I-\lambda)\widetilde{T}-(I+\lambda)]w = z \tag{3.31}$$

where$w = (I+\widetilde{T})^{-1}x$.

Equation (3.31) can be solved $\forall z \in H$ when $Re\lambda > 0$.

Dividing (3.31) by $I-\lambda$ we get $[\widetilde{T}-\frac{I+\lambda}{I-\lambda}]w = \frac{z}{I-\lambda}$.

This is obvious for all $\lambda = 1$

If $\lambda \neq 1$, all we need to note is that for $Re\lambda > 0$, $|\frac{1+\lambda}{1-\lambda}| > 1$.

Since $\|\widetilde{T}\|$, is bounded , that is,$\|\widetilde{T}\| \leq 1$, $\frac{1+\lambda}{1-\lambda}$ is in $\rho(\widetilde{T})$ hence (3.31) can be solved for all $z \in H$. □

If $\overline{W(T)}$ is a half-plane then theorem 3.2.1 gives a closed extension $\widetilde{T}$ of T satisfying

$$\sigma(\widetilde{T}) \subseteq \overline{W(T)} \subseteq \overline{W(\widetilde{T})}.$$

In fact, all we need to do is to define S by 3.11 for appropriate r, k then extend S to $\widetilde{S}$ by as $S = rT - kI$.
The extension $\widetilde{T}$ define by $\widetilde{T} = \frac{\widetilde{S}+kI}{r}$
has all the desired properties. Hence theorem 3.2.1 implies the following results.

Theorem 3.2.2. *Let T be a densely defined linear operator on H such that $\overline{W(T)}$ is a half - plane. Then T has a closed extension $\widetilde{T}$ satisfying*

$$\sigma(\widetilde{T}) \subseteq \overline{W(T)} \subseteq \overline{W(\widetilde{T})}.$$

Proof. If T be a densely defined linear operator on H such that $\overline{W(T)}$ is a half - plane, by theorem 3.2.1, T has a closed extension such that the resolvent set is in a half plane which is a complement of the numerical range. This implies that the spectrum of this operator is in the other half plane,$Re\lambda \leq 0$ hence contained in the $\overline{W(T)} \subseteq \overline{W(\widetilde{T})}$. □

CHAPTER 4

EXTENSIONS OF THE CLOSABLE OPERATORS FOR THE CASE OF A STRIP OR A LINE

4.1 A case of a strip or a line

Having found that a densely defined dissipative operator can be extended when its numerical range is a half plane, to be precise, when its numerical range is the negative half plane containing the vertical axis, such that the extension is closed and its numerical range is contained in the negative half plane, in this chapter we proceed by considering a situation where the numerical range is a strip or a line in that particular half plane. We achieve this by defining a suitable constant that we will ensure that our strip or a line is entirely in the half plane.

We will consider a line as a strip of thickness 0 and a strip with a thickness more than 0. Suppose $\overline{W(T)}$ is a strip of thickness $a-1$, where $a \geq 1$.

Just like in the previous chapter we can find a dissipative operator S of the form

$$S = rT - kI \tag{4.1}$$

such that $\overline{W(S)}$ is the strip $1 - a \leq Rez \leq 0$ where z is arbitrarily taken in the whole complex plane.

From this definition of $\overline{W(S)}$, It is clear that Rez is in the negative half plane and $1 - a$ is always a negative number when $a > 1$ making $\overline{W(S)}$ a strip and zero when $a = 1$ making $\overline{W(S)}$ a line, precisely, the vertical axis.

Since z is a complex number in the plane, there is an $x \in D(S)$and $\|x\| = 1$ such that $\langle Sx, x\rangle = z$.

Thus

$$1 - a \leq Re\langle Sx, x\rangle \leq 0, x \in D(S), \|x\|^2. \tag{4.2}$$

It is now clear that S is dissipative.

Suppose S has a closed extension $\widetilde{S}$ such that $\overline{W(\widetilde{S})}$ is the same strip and that $\rho(\widetilde{S})$ contains two complementary half planes.

Let

$$\widetilde{T} = (aI + \widetilde{S})(aI - \widetilde{S})^{-1}. \tag{4.3}$$

Let x be any element of Hsuch that

$$y = (aI - \widetilde{S})^{-1}x. \tag{4.4}$$

Then

$$x = (aI - \widetilde{S})y \tag{4.5}$$

and $\widetilde{T}x = (aI + \widetilde{S})(aI - \widetilde{S})^{-1}x$ can be reduced to

$$\widetilde{T}x = (aI + \widetilde{S})y. \tag{4.6}$$

Taking the norms of Tx and x and squaring them gives us $\|x\|^2 = \|ay - \widetilde{S}y\|^2$ and $\|Tx\|^2 = \|(ay + \widetilde{S}y)\|^2$.

Expanding the left hand side of the two equations gives us

$\|x\|^2 = \langle ay - \widetilde{S}y, ay - \widetilde{S}y\rangle$ and $\|Tx\|^2 = \langle ay + \widetilde{S}y, ay + \widetilde{S}y\rangle$ which is equivalent to

$\|x\|^2 = \|ay\|^2 - 2Re\langle \widetilde{S}y, ay\rangle + \|\widetilde{S}y\|^2$ and $\|Tx\|^2 = \|x\|^2 = \|ay\|^2 + 2Re\langle \widetilde{S}y, ay\rangle + \|\widetilde{S}y\|^2$

This is equivalent to

$\|x\|^2 = \|ay\|^2 - 2aRe\langle \widetilde{S}y, y\rangle + \|\widetilde{S}y\|^2$ and $\|Tx\|^2 = \|ay\|^2 + 2aRe\langle \widetilde{S}y, y\rangle + \|\widetilde{S}y\|^2$

Since a is real hence $|a| = a$

Subtracting the first from the second equation yields

$$\|\widetilde{T}x\|^2 - \|x\|^2 = 4aRe\langle \widetilde{S}y, y\rangle \tag{4.7}$$

From $x = (aI - \widetilde{S})y = ay - \widetilde{S}y$, it clear that $y \in D(\widetilde{S})$, hence by theorem 3.2.1 $\widetilde{S}$ is dissipative leading to

$$\|\widetilde{T}x\|^2 - \|x\|^2 = 4aRe\langle \widetilde{S}y, y\rangle \leq 0$$

We know that $\langle Sy, y\rangle \leq \|Sy\|\|y\| \leq \|S\|\|y\|\|y\| = \|S\|\|y\|^2$. Since $\|S\| \leq 1$,

$\langle Sy, y\rangle \leq \|y\|^2$ and $Re\langle Sy, y\rangle \leq \|y\|^2$. Since $1 - a < 0$ and $4a > 0$

$(1 - a)Re\langle Sy, y\rangle \geq (1 - a)\|y\|^2$ and $4a(1 - a)Re\langle Sy, y\rangle \geq 4a(1 - a)\|y\|^2$.

Hence

$$4a(1 - a)\|y\|^2 \leq 4a(1 - a)Re\langle Sy, y\rangle$$

Also from $\|\widetilde{T}x\|^2 - \|x\|^2 = 4aRe\langle \widetilde{S}y, y\rangle \leq 0$ it true that

$$\|\widetilde{T}x\|^2 - \|x\|^2 \geq 4a(1 - a)Re\langle \widetilde{S}y, y\rangle$$

since $(1-a)<0$

Thus by transitivity in inequalities, we get that

$$4a(1-a)\|y\|^2 \leq \|\widetilde{T}x\|^2 - \|x\|^2 \leq 0 \tag{4.8}$$

Subtracting (4.5) from (4.6) and adding the two equations, we get

$$(\widetilde{T}+I)x = 2ay, (\widetilde{T}-I)x = 2\widetilde{S}y \tag{4.9}$$

From (4.9), we have $y = \frac{(\widetilde{T}+I)x}{2a}$ and

$$\|y\| = \|\frac{(\widetilde{T}+I)x}{2a}\| = \frac{\|(\widetilde{T}+I)x\|}{2a}$$

Squaring both sides gives

$$\|y\|^2 = \|\frac{(\widetilde{T}+I)x}{2a}\| = \frac{\|(\widetilde{T}+I)x\|^2}{4a^2}$$

Substituting for $\|y\|^2$ in equation (4.8) we get

$$\frac{1-a}{a}\|(\widetilde{T}+I)x\|^2 \leq \|\widetilde{T}x\|^2 - \|x\|^2 \leq 0 \tag{4.10}$$

or

$$(1-a)\|\widetilde{T}x + x\|^2 \leq a\|\widetilde{T}x\|^2 - a\|x\|^2 \leq 0 \tag{4.11}$$

$$(1-a)\langle\widetilde{T}x + x, \widetilde{T}x + x\rangle \leq a\|\widetilde{T}x\|^2 - a\|x\|^2 \leq 0 \tag{4.12}$$

Upon expansion of the inner product function, we get

$$(1-a)(\|\widetilde{T}x\|^2 + 2Re\langle\widetilde{T}x, x\rangle + \|x\|^2) \leq a\|\widetilde{T}x\|^2 - a\|x\|^2 \leq 0 \tag{4.13}$$

$\|\widetilde{T}x\|^2 + 2Re\langle\widetilde{T}x, x\rangle + \|x\|^2 - a\|\widetilde{T}x\|^2 - 2aRe\langle\widetilde{T}x, x\rangle - a\|x\|^2 \leq a\|\widetilde{T}x\|^2 - a\|x\|^2 \leq 0$

$\|\widetilde{T}x\|^2 \leq a\|\widetilde{T}x\|^2 - a\|x\|^2 - 2Re\langle\widetilde{T}x, x\rangle - \|x\|^2 + a\|\widetilde{T}x\|^2 + 2aRe\langle\widetilde{T}x, x\rangle + a\|x\|^2 \leq 0$

$\|\widetilde{T}x\|^2 \leq 2a\|\widetilde{T}x\|^2 - \|x\|^2 + 2aRe\langle\widetilde{T}x, x\rangle - 2Re\langle\widetilde{T}x, x\rangle \leq 0$

$\|\widetilde{T}x\|^2 \leq 2a\|\widetilde{T}x\|^2 - \|x\|^2 + 2(a-1)Re\langle\widetilde{T}x, x\rangle \leq 0$

$\|x\|^2 \leq 2a\|\widetilde{T}x\|^2 - \|\widetilde{T}x\|^2 + 2(a-1)Re\langle\widetilde{T}x, x\rangle \leq 0$

$$\|x\|^2 \leq 2a(-1)\|\widetilde{T}x\|^2 + 2(a-1)Re\langle\widetilde{T}x, x\rangle \leq 0 \tag{4.14}$$

From (4.10), it is clear that $\|\widetilde{T}x\|^2 \leq \|x\|^2$, using this inequality together with (4.14), we get or

$$\|\widetilde{T}x\|^2 \leq \|x\|^2 \leq (2a-1)\|\widetilde{T}x\|^2 + 2(a-1)Re\langle\widetilde{T}x, x\rangle \tag{4.15}$$

Thus $\widetilde{T}$ is an extension of

$$T = (aI + S)(aI - S)^{-1}, D(T) = R(aI - S) \tag{4.16}$$

which satisfies (4.15) (thus bounded) and is such that

$$R(T) = H$$

Conversely, if we can find an extension $\widetilde{T}$ of T, we can show that

$$\widetilde{S} = a(\widetilde{T} - I)(\widetilde{T} + I)^{-1} \tag{4.17}$$

is a closed extension of S with $\widetilde{\sigma(S)} \subseteq \overline{W(S)} = \overline{W(\widetilde{S})}$.

Using the reasoning of theorem (3.2.1), $\widetilde{S}$ is closed and dissipative while $\rho(\widetilde{S})$ contains the half-plane $Re\lambda > 0$.

Moreover, (4.7), (4.10) and (4.15) implies that $\overline{W(\widetilde{S})}$ is the strip

$$1 - a \leq Rez \leq 0$$

.

In fact, from (4.15) it is clear that $\widetilde{T}x\|^2 - \|x\|^2 \leq 0$ since $\widetilde{T} \leq 1$ making it coincide with Rez. The section coinciding with $1-a$ is $\frac{1-a}{a}\|(\widetilde{T}+I)x\|^2$ since the factor $1-a$ is less than Rez and also less than zero.

However,our goal to have a strip contain the spectrum of the extension operator $\widetilde{S}$, thus, we define another half plane (remember a half plane is any section of a plane divided by any straight line) so that between that plane and the plane $Re\lambda > 0$ we have a strip. In that case, the new plane would be entirely a resolvent set so that the spectrum is in the strip only.

If we define the new half plane as $Re\lambda < 1-a$ and the earlier plane $Re\lambda > 0$, the strip would be the set

$$\{z|(Re\lambda < 1-a) \leq z \leq (Re\lambda > 0)\}$$

or simply

$$\{1-a \leq Rez \leq 0)\}.$$

The thickness of this strip would be $|a-1|$. Thus $\rho(\widetilde{S}) \in (Re\lambda < 1-a)$.

We know that λ to be in the resolvent set of the extension $\widetilde{S}$, $(\widetilde{S}-\lambda I)^{-}1$ exists and it is bounded and $R(\widetilde{S}-\lambda I)$ is dense in N. This implies that $(\widetilde{S}-\lambda I)$ is injective hence $N(\widetilde{S}-\lambda I) = \{\overline{0}\}$ and that $R(\widetilde{S}-\lambda I)$ is closed.

To achieve this, we only need to show that the half-plane $Re\lambda < 1-a$ contains one point in $\rho(\widetilde{S})$.

After having such point with the above properties, we can then talk of a boundary point in the resolvent set. Thus we can apply

Theorem 4.1.1. *Let T be a closed linear operator in a Banach space X. If λ is a boundary point of $\rho(T)$, the either $ker(\widetilde{T}-\lambda I) \neq \{\overline{0}\}$ or $R(\widetilde{T}-\lambda I)$ is not closed in X.*

The proof of this result is provided later, but we continue with the earlier argument.

If the half-plane $Re\lambda < 1-a$ contains a point of $\rho(\widetilde{S})$, then the entire half -plane must be in $\rho(\widetilde{S})$ because the point is arbitrary, otherwise, it would contain a boundary point

λ of $\rho(\widetilde{S})$. By theorem 4.1.1 this would imply that $N(\widetilde{S}-\lambda I) \neq \{\overline{0}\}$ or $R(\widetilde{S}-\lambda I)$ is not closed contradicting the conclusion reached above this would force the point to be in the strip having the spectrum of $\widetilde{S}$.

To complete the argument, let us show that the point $\lambda = -a$ is indeed in $\rho(\widetilde{S})$.

From (4.3), we have

$$\widetilde{T} = (aI + \widetilde{S})(aI - \widetilde{S})^{-1},$$

implying that

$$aI + \widetilde{S} = \widetilde{T}(aI - \widetilde{S}), \tag{4.18}$$

and since $R(T) = R(aI - \widetilde{S}) = H$, it follows that $R(aI + \widetilde{S}) = R(aI - \widetilde{S}) = H$. Since H is complete or simply closed, we have that the $R(aI - \widetilde{S}) = R(\widetilde{S} - \lambda I)$ for $\lambda = -a$.

This together with the fact that $N(aI + \widetilde{S}) = N(\widetilde{S} - \lambda I) =\neq \{\overline{0}\}$ for $\lambda = -a$ shows that $-a \in \rho(\widetilde{S})$.

Hence, we must try to find an extension $\widetilde{T}$ of T satisfying (4.10) or (4.15) and such that $R(\widetilde{T}) = H$.

Let us consider first the case of a line, that is, $a = 1$. Now by (4.10)

$$\frac{1-1}{1}\|(\widetilde{T}+I)x\|^2 \leq \|\widetilde{T}x\|^2 - \|x\|^2 \leq 0,$$

$$0 \leq \|\widetilde{T}x\|^2 - \|x\|^2 \leq 0.$$

This is only possible if

$$\|\widetilde{T}x\| = \|x\|, \forall x \in H \tag{4.19}$$

(4.19) shows that $\widetilde{T}$ is an isometry which also applies that it is a unitary operator since its range is its domain from the relation (4.19).

$$T = (aI + S)(aI - S)^{-1}, D(T) = R(aI - S) \tag{4.20}$$

Therefore, for $\lambda = a$ it all about having a unitary extension $\widetilde{T}$.

By (4.16),

$$T = (aI + S)(aI - S)^{-1}, D(T) = R(aI - S)$$

Since $T = (aI + S)(aI - S)^{-1}$ implies $(aI - S)T = (aI + S)$ hence

$R(aI - S) = R(aI + S)$.

Using a similar argument, the operator T is an Isometry of $R(I - S)$ onto $R(I + S)$.

By continuity we can extend it to be an Isometry $\widetilde{T}$ of $\overline{R(I - S)}$ onto $\overline{R(I + S)}$.

Thus to determine $\widetilde{T}$, we need only define it on $R(I - S)^{\perp}$. This follow from the general property of Isometries in Hilbert space.

$$\langle \widetilde{T}x, \widetilde{T}w \rangle = \langle x, w \rangle. \tag{4.21}$$

Moreover, $\widetilde{T}$ must map onto $R(I + S)^{\perp}$ for otherwise we could not have $R(\widetilde{T}) = H$.

From the discussion above we have the following results.

Theorem 4.1.2. *Let S be a densely defined linear operator on H such that $W(S)$ is the line $Re\lambda = 0$. Then a necessary and sufficient condition for S to have a closed extension $\widetilde{S}$ such that*

$$\sigma(\widetilde{S}) \subseteq \overline{W(\widetilde{S})} = \overline{W(S)} \tag{4.22}$$

is that there exists an Isometry from $R(I - S)^{\perp}$ onto $R(I + S)^{\perp}$.

In particular, this is true if they both have the same finite dimension or if they are both separable and infinite dimensional.

The last statement follows from the fact that $R(I + S)^{\perp}$ and $R(I - S)^{\perp}$ have complete orthogonal sequences (ϕ_k) and (ψ_k) respectively.

Moreover,these sequences are either both infinite or have the same finite number of elements. In either case, we define $\widetilde{T}$ by

$$\widetilde{T}\phi_k = \psi_k, k = 1, 2, 3, ... \tag{4.23}$$

In proving the theorem, we make use of the following lemma

Lemma 4.1.3. *Let T be a closed linear operator in a Banach space X. If λ is a boundary point of $\rho(T)$ and λ_n is a sequence of points in $\rho(T)$ converging to λ, then*

$$\|(T - \lambda_n I)^{-1}\| \to +\infty.$$

Proof. If this lemma were not true, there would be a sequence $\{\lambda_n\} \in \rho(T)$ such that $\{\lambda_n\} \to \lambda$ as $n \to \infty$

$$\|(T - \lambda_n I)^{-1}\| \leq C \tag{4.24}$$

where C is a positive constant.

Since

$$(T - \lambda_n I)^{-1} - (T - \lambda_m I)^{-1} = (T - \lambda_n I)^{-1}(\lambda_n - \lambda_m)(T - \lambda_m I)^{-1}$$

taking norms on both sides, we have

$$\begin{aligned}\|(T - \lambda_n I)^{-1} - (T - \lambda_m I)^{-1}\| &= \|(T - \lambda_n I)^{-1}(\lambda_n - \lambda_m)(T - \lambda_m I)^{-1}\| \\ &\leq \|(T - \lambda_n I)^{-1}\| \|(\lambda_n - \lambda_m)\| \|(T - \lambda_m I)^{-1}\|.\end{aligned}$$

Using inequality (4.24) we get

$$\|(T - \lambda_n I)^{-1} - (T - \lambda_m I)^{-1}\| \leq C^2 |\lambda_m - \lambda_n|.$$

As $n, m \to \infty$, we have

$$\|(T - \lambda_n I)^{-1} - (T - \lambda_m I)^{-1}\| \leq C^2 |\lambda_m - \lambda_n| \to 0.$$

Thus, $(T-\lambda_n I)^{-1}$ converges to an operator $U \in B(X)$ as $n \to \infty$. Moreover, if x is any element in X, then

$$y_n = (T-\lambda_n I)^{-1}x \xrightarrow{s} Ux \text{ as } n \to \infty.$$

This implies

$$(T-\lambda_n I)y_n = x.$$

But

$$Ty_n = Ty_n - \lambda_n y_n + \lambda_n y_n$$

or $Ty_n = (T-\lambda_n)y_n + \lambda_n y_n$. Substituting for $(T-\lambda_n)y_n$ we have $Ty_n = x + \lambda_n y_n$. which is equal to $Ty_n = x + \lambda_n((T-\lambda_n I)^{-1}x)$.

Taking limits as $n \to \infty$ we have

$$Ty_n = x + \lambda_n((T-\lambda_n I)^{-1}x) \xrightarrow{s} x + \lambda Ux.$$

Since T is closed $Ux \in D(T)$ and $TUx = x + \lambda Ux$ whence $TUx - \lambda Ux = x$ and

$$(T-\lambda I)Ux = x, \forall x \in X. \tag{4.25}$$

Similarly, if $x \in D(T)$ then

$$(T-\lambda_n I)^{-1}(T-\lambda_n I)x \xrightarrow{s} U(T-\lambda_n I)x \text{ as } n \to \infty.$$

But

$$(T-\lambda_n I)^{-1}(T-\lambda_n I)x = x - (\lambda-\lambda_n)(T-\lambda_n I)^{-1}x \xrightarrow{s} x \text{ as } n \to \infty$$

hence

$$U(T-\lambda_n I)x = x \forall x \in D(T) \tag{4.26}$$

This shows that $\lambda \in \rho(T)$, contrary to the assumption hence we must have
$\|(T-\lambda_n I)^{-1}\| \to +\infty$. □

Proof. **Proof of theorem 4.1.1.**

If the theorem were not true, then by theorem 1.4.2 there would be a constant k such that

$$\|x\| \leq k\|(T - \lambda I)x\|, \forall x \in D(T) \tag{4.27}$$

X, Y are Banach spaces and let T be a one to one closed linear operator from X to Y. Then a necessary and sufficient condition that $R(T)$ is closed in Y is that T is bounded from below. Since λ is a boundary point of $\rho(T)$, there is a sequence $\{\lambda_n\}$ of points of $\rho(T)$ converging to λ. Set

$$S_n = \frac{(T - \lambda_n I)^{-1}}{\|(T - \lambda_n)^{-1}\|},$$

then $\|S_n\| = 1$, in particular, for each $n \in N$, there is an element $x_n \in X$ such that

$$\|x_n\| = 1, \|S_n x_n\| > \frac{1}{2}. \tag{4.28}$$

Now

$$\begin{aligned}
(T - \lambda I)S_n = TS_n - \lambda IS_n &= TS_n - \lambda IS_n - \lambda_n IS_n + \lambda_n IS_n \\
&= TS_n - \lambda_n IS_n + \lambda_n IS_n - \lambda IS_n \\
&= (T - \lambda_n I)S_n + (\lambda_n - \lambda)S_n.
\end{aligned}$$

Taking norms on both sides, we have $\|(T - \lambda I)S_n\| = \|(T - \lambda_n I)S_n + (\lambda_n - \lambda)S_n\|$. Using triangle inequality for norms, we have

$$\begin{aligned}
\|(T - \lambda I)S_n\| &\leq \|(T - \lambda_n I)S_n\| + \|(\lambda_n - \lambda)S_n \\
&\leq \|(T - \lambda_n I)\|\|S_n\| + |(\lambda_n - \lambda)|\|S_n\| \\
&\leq \|(T - \lambda_n I)\| + |(\lambda_n - \lambda)| \text{ since } \|S_n\| = 1.
\end{aligned}$$

Hence

$$\|(T - \lambda I)S_n\| \leq \|(T - \lambda_n I)^{-1}\|^{-1} + |(\lambda_n - \lambda)|.$$

By lemma 4.1.3, this tends to 0 as $n \to \infty$. In particular, the norm of $(T - \lambda I)S_n$ can be made less that $\frac{1}{3}k$ for n sufficiently large But by (4.27) we have

$$\frac{1}{2} < \|S_n x_n\| \leq k\|(T - \lambda I)S_n x_n\| < \frac{1}{3}$$

for large n. This contradiction shows that (4.27) does not hold hence result of the theorem is true. □

If $a \neq 1$ (the case of a strip , $-\infty > a > 1$) it is necessary for $\widetilde{T}$ to map $R(aI - S)$ onto a closed subspace M such that

$$H = \overline{R(aI + S)} = M$$

as in such a way that (4.15) holds.

We shall just give a sufficient condition

Theorem 4.1.4. *Let S be a densely defined linear operator on H such that $\overline{W(S)}$ is a strip $1 - a \leq Rez \leq 0, a > 1$. If $\overline{R(aI - S)} = \overline{R(aI + S)}$, then S has a closed extension $\widetilde{S}$ satisfying*

$$\sigma(\widetilde{S}) \subseteq \overline{W(\widetilde{S})} = \overline{W(S)}. \tag{4.29}$$

Proof. On $R(aI - S)^{\perp} = R(aI + S)^{\perp}$ we defined $\widetilde{T}$ to be $-I$. Then $\widetilde{T}$ is Isometric on the set. Thus (4.10) and (4.15) holds for $x \in \overline{R(aI - S)}$ and for $x \in R(aI - S)^{\perp}$. For any $x \in H, x = x_1 + x_2$ where $x_1 \in R(aI - S)$ and $x_2 \in R(aI - S)^{\perp}$. Thus

$$\frac{1-a}{a}\|(\widetilde{T} + I)x\|^2 = \frac{1-a}{1}\|(\overline{T} + I)x_1\|^2 \leq \|\overline{T}x_1\|^2 - \|x_1\|^2 \leq 0.$$

But

$$\|\overline{T}x_1\|^2 - \|x_1\|^2 = \|\overline{T}x - x_2\|^2 - \|x_2 + x_2\|^2 = \|\widetilde{T}x\|^2 - \|x\|^2$$

hence (4.10) holds.

To prove (4.21), we expand both sides of

$$\|\widetilde{T}(x+y)\|^2 = \|x+y\|^2$$

or

$$\|\widetilde{T}x + \widetilde{T}y)\|^2 = \|x+y\|^2.$$

We get the equivalent

$\langle \widetilde{T}x + \widetilde{T}y, \widetilde{T}x + \widetilde{T}y\rangle = \langle x+y, x+y\rangle$.

Upon expansion, we get

$\|\widetilde{T}x\|^2 + \langle \widetilde{T}x, \widetilde{T}y\rangle + \langle \widetilde{T}y, \widetilde{T}x\rangle + \|\widetilde{T}y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$

which reduces to

$\|\widetilde{T}x\|^2 + 2Re\langle \widetilde{T}x, \widetilde{T}y\rangle + \|\widetilde{T}y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$.

Since $\widetilde{T}$ is an Isometric for $x, y \in D(\widetilde{T})$ we have $\|\widetilde{T}x\|^2 = \|x\|^2$ and $\|\widetilde{T}y\|^2 = \|y\|^2$. Using these properties and diving through by 2, our equation reduces to

$$Re\langle \widetilde{T}x, \widetilde{T}y\rangle = Re\langle x, y\rangle.$$

Now substitute ix in place of x we get

$Re\langle \widetilde{T}(ix), \widetilde{T}y\rangle = Re\langle ix, y\rangle$ which is equivalent to $Rei\langle \widetilde{T}x, \widetilde{T}y\rangle = Rei\langle x, y\rangle$. But

$$Re\langle ix, y\rangle = Rei\langle x, y\rangle = Im\langle x, y\rangle \text{ and } Re\langle \widetilde{T}ix, \widetilde{T}y\rangle = Rei\langle \widetilde{T}x, \widetilde{T}y\rangle = Im\langle \widetilde{T}x, \widetilde{T}y\rangle.$$

Hence $Im\langle \widetilde{T}x, \widetilde{T}y\rangle = Im\langle x, y\rangle$. Adding this to $Re\langle \widetilde{T}x, \widetilde{T}y\rangle = Re\langle x, y\rangle$, we get

$Re\langle \widetilde{T}x, \widetilde{T}y\rangle + iIm\langle \widetilde{T}x, \widetilde{T}y\rangle = Re\langle x, y\rangle + iIm\langle x, y\rangle$.

Hence we get $\langle \widetilde{T}x, \widetilde{T}y\rangle = \langle x, y\rangle$. □

4.2 Self-adjoint extensions

With self-ajoint operator, T, we have $T = T^*$ and $D(T)$ being dense for T^* to be defined. Moreover, T must be closed since T^* is.

For $x \in D(T)$

$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle T^*x, x\rangle = \langle Tx, x\rangle$.

Thus $\overline{\langle Tx, x\rangle} = \langle Tx, x\rangle$ which is only possible if $\langle Tx, x\rangle$ is a real number. Thus, by definition of a numerical range, $\langle Tx, x\rangle = W(T)$ is a real axis; thus it can be the whole of the real axis, a half axis or an interval.

We claim that

$$\sigma(T) \subset \overline{W(T)} \tag{4.30}$$

when T is self-adjoint the prove that it is true.

The statement $\lambda \notin \overline{W(T)}$ and $\lambda \notin \sigma(T)$ implies that $\sigma(T) \subset \overline{W(T)}$.

Let $\lambda \notin \overline{W(T)}$, by theorem 1.4.2, $N(T - \lambda I) = 0$ and $R(T - \lambda I)$ is closed. That is if $y \perp R(T - \lambda I)$ then

$$\langle y, (T - \lambda I)x = 0, x \in D(T)$$

showing that $y \in D(T^*)$. Since T is self-adjoint $y \in D(T^*) = D(T)$

and

$\langle y, (T - \lambda I)x\rangle = \langle (T - \lambda I)^* y, x\rangle = \langle (T^* - \overline{\lambda} I)y, x\rangle = \langle (T - \overline{\lambda} I)y, x\rangle = 0, x \in D(T)$ since $T = T^*$.

Since $D(T)$ is dense, anything orthogonal to it is a zero vector hence $(T - \overline{\lambda} I)y = 0$.

Now if λ is not real, so is $\overline{\lambda}$ its conjugate hence $\lambda \notin \overline{W(T)}$ implies $\overline{\lambda} \notin \overline{W(T)}$ since $\overline{W(T)}$ is real and $\overline{\lambda}$ and λ are complex.

If λ is real, then $\lambda = \overline{\lambda}$ hence $(T - \lambda I)y = 0$. Since $T - \lambda I \neq 0$ the only possibility is that $y = \overline{0}$. But $y \perp R(T - \lambda I)$ and $y = \overline{0}$ implies that $R(T - \lambda I)$ is complete, thus dense and closed since the only vector that can be zero and orthogonal to $R(T - \lambda I)$ is the zero vector.

From the definition of resolvent set $R(T - \lambda I)$ is dense of $\lambda \in \rho(T - \lambda I)$. Thus

$\lambda \notin \sigma(T-\lambda I)$ and consequently $\rho(T-\lambda I)$. Since $\rho(T)^c = \sigma(T)$, we have that $\lambda \notin \sigma(T)$ .Thus $\sigma(T) \subset \overline{W(T)}$.

Suppose T is densely defined linear operator on H.Then $W(T)$ is a subset of the real axis if and only if T is an isommetry, that is $\langle Tx, y\rangle = \langle x, Ty\rangle$.

Let $W(T)$ be a subset of real axis, then by theorem 1.4.3 $W(T)$ is a subset of the real axis if and only if T is symmetric.

Thus, $Re\langle Tx, y\rangle = \langle Tx, y\rangle$, $Re\langle x, Ty\rangle = \langle x, Ty\rangle$ and by the theorem $\langle Tx, y\rangle = \langle x, Ty\rangle, x, y \in D(\phi)$.

If T is symmetric and $\sigma(T) \subset W(T)$, then we show that T is self adjoint.

Suppose $y \in D(T^*)$ and $y \notin \mathbb{R}$ then $y \notin W(T)$. $y \notin \mathbb{R}$ implies $\overline{y} \notin \mathbb{R}$ thus $\overline{y} \notin W(T)$ either. Therefore $y \notin \sigma(T)$. Since $\sigma(T)$ and $\rho(T)$ are complements of each other, we get that $y \in \rho(T)$. Hence there is a $z \in D(T)$ such that

$$(T-\overline{\lambda})z = (T^* - \overline{\lambda})y. \tag{4.31}$$

Thus

$$\begin{aligned}
\langle y, (T-\lambda)x\rangle = \langle (T-\lambda)^* y, x\rangle &= \langle (T^* - \overline{\lambda})y, x\rangle. \\
&= \langle (T-\overline{\lambda})z, x\rangle \text{ By substitution using 4.31} \\
&= \langle Tz, x\rangle - \langle \overline{\lambda} z, x\rangle \\
&= \langle z, Tx\rangle - \overline{\lambda}\langle z, x\rangle \text{ since } T \text{ is symmetric} \\
&= \langle z, Tx\rangle - \langle z, \lambda x\rangle \\
&= \langle z, Tx - \lambda x\rangle \\
&= \langle z, (T-\lambda)x\rangle.
\end{aligned}$$

Therefore $\langle y, (T-\lambda)x\rangle = \langle z, (T-\lambda)x\rangle$ which is equivalent to $\langle y, (T-\lambda)x\rangle - \langle z, (T-\lambda)x\rangle = 0$ or $\langle y - z, (T-\lambda)x\rangle = 0, x \in D(T)$.

We have that $\lambda \in \rho(T)$ by definition implying that $R(T-\lambda)$ is dense and closed. Since $x \in D(T), (T-\lambda)x \in R(T-\lambda) \forall x \in D(T)$. Hence the vector orthogonal to $(T-\lambda)x$ is only the zero vector. Thus $y - z = 0$ or $y = z$ implying that $y \in D(T)$.

Due to the equality of the two vectors, equation 4.31 is transformed to

$$(T - \overline{\lambda})y = (T^* - \overline{\lambda})y.$$

which is equivalent to $Ty - \overline{\lambda}y = T^*y - \overline{\lambda}y$. By equality in addition of algebra, we have $Ty = T^*y, y \in D(T)$. Hence T is self adjoint.

In particular, since every bounded operator satisfies 4.30, we see that every bounded symmetric operator defined everywhere is self-adjoint. Now we find out if a densely defined operator on a Hilbert space has a self-adjoint extension.

First suppose $W(T)$ is not a whole real axis, then by theorem 3.1.8 T has a closed extension $\widetilde{T}$such that

$$\sigma(\widetilde{T}) \subset \overline{W(\widetilde{T})} = \overline{W(T)}. \tag{4.32}$$

In particular $\widetilde{T}$ is symmetric and satisfies 4.30. Hence $\widetilde{T}$ is self-adjoint.

Second, if $W(T)$ is the whole real axis, then T has a closed extension $\widetilde{T}$ such that $\sigma(\widetilde{T}) \subset \overline{W(\widetilde{T})} = \overline{W(T)}$ if an only if there is an Isometry of $R(i+T)^{\perp}$ onto $R(i+T)^{\perp}$ which is a condition to similar to one in theorem 4.1.4 when S is replaced by iT. Since in this case, an extension violating condition 4.32 could not be self-adjoint, the condition is both necessary and sufficient for T to have a self-adjoint extension.

CONCLUSION

In this study, we were faced with a problem finding a way in which the spectrum of an operator can be contained in the closure of the numerical range when the domain of the operator is a subset of the space of which it is defined.

We approached it by considering densely defined operators and determining if their extensions (which are closed) can satisfy the condition in question. To achieve this, we set the rules under which the condition,

$$\sigma(\widetilde{T}) \subset \overline{W(\widetilde{T})} = \overline{W(T)}.$$

can hold. This was achieved while fixing the numerical range of the operator to be a whole plane, a half plane, a strip or a line.

For the case of a half plane, we found that a suitable closed extension will exist if the operator is dissipative and densely defined.

For the case of a strip being the closure of numerical range of the dissipative operator S, we found that a suitable closed extension will exist if $\overline{R(aI - S)} = \overline{R(aI + S)}$.

For the case of a line being the numerical range of the dissipative operator S, we found that a suitable closed extension will exist if the there is an isometry from $R(I - S)^{\perp}$

onto $R(I+S)^{\perp}$ where our operator T is given by $T = (aI+S)(aI-S)^{-1}, a > 1$ and S is dissipative.

For the case of self-adjoint operator; if the numerical range is the whole of the real axis, then the suitable closed extension exists if there is an isometry of $R(i+T)^{\perp}$ onto $R(i+T)^{\perp}$.

RECOMMENDATIONS

Recommendations for further studies

For further studies, one may

(i). Find out the nature of the extensions described in the three cases (the case of a half plane, a strip or a line). That is, if they are the closure or containing the closure of the operator. One may also advance to find out if there can me more than one extension, if so, their relation may be developed.

(ii). Apply the concept to a case where a line, a strip or a half plane is parallel to the real axis then combine the result to the current one and talk of more cases being a square, a rectangle or a point (origin).

(iii). Approach the idea using a 3D view by taking half plane, a strip or a line as projections of some cuboid or cube that is equal to a half 3D plane, a cylinder or a straight line in space respectively. This is possible when defining a suitable projection operator to map corresponding pair of sets.

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