USING POISSON AND EXPONENTIAL MIXTURES IN ESTIMATING AUTOMOBILE INSURANCE PREMIUMS.

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I56/69573/2013

A dissertation submitted to the school of graduate studies in partial fulfillment of the requirements for the award of masters in actuarial science.

July 2015

DECLARATION

This research project is my original work and has not been presented for the award of a degree in any other university or institution.

Signature ……………………………… Date ………………………………….

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This research project has been submitted for examination with approval of the undersigned as the university supervisor:

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Prof J.A.M Ottieno

ACKNOWLEDGEMENT

Special thanks to my supervisor Prof J.A.M Ottieno for his guidance and commitment to completion of this work. His valuable suggestions and review that made this work a success. To my friends and family thanks for your support. To my director Mr. Mackred Ochieng thanks for your encouragement and support.

Above all, thanks to our Almighty God for seeing me this far and enabling me accomplish this work

DEDICATION.

To entire Munai's family for their unconditional love and support throughout my studies.

God bless you all abundantly.

ABSTRACT

The main purpose of this project was to design an optimal bonus malus system that incorporates both the number of claims and the claim size. Majority of insurance companies charge premiums based on the number of accidents. This way a policyholder who had an accident with a small size of loss is penalized in the same way with a policyholder who had an accident with a big size of loss, thus the need to develop a model that incorporates both the frequency and the severity components. The frequency component was modelled using Poisson mixtures where the number of claims is Poisson distributed and the underlying risk for each policyholder or group of policyholders is the mixing distribution. We considered the mixing distribution to be gamma, exponential, Erlang and Lindely distribution. For the severity component we used exponential gamma mixture (Pareto distribution) where the claim amount is exponential distributed and the mean claim amount is inverse Gamma. Using the Bayes theory we obtain the posterior structure function for the frequency and the severity component. The premium was estimated as the mean of the posterior structure function for the frequency component if we compute premiums based on the number of claims only. The premium based on both frequency and severity components was estimated as the product of the mean of the posterior structure function of the frequency component and the mean of the posterior structure function of the severity component. We applied the data presented by Walhin and Paris (2000) with some adjustment of the claim amount data to fit the Pareto distribution. The study established that if we consider only the frequency component, the system was unfair to policyholders with small claim amounts. However optimal BMS based on frequency and severity component was found to be fair to all policyholder since policyholders with large claim amounts were charged higher malus due to the risk they pose to portfolio. Therefore we recommend a system that considers both frequency and severity components.

Keywords. BMS, Poisson mixtures, exponential mixtures, frequency component, severity component.

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CHAPTER 1: GENERAL INTRODUCTION

1.1 Background of study.

Bonus malus system hereafter referred to as BMS was established by insurance companies to reward good drivers and penalize the bad ones. A BMS usually is based on classes where premium paid in each class is based on the number of accidents irrespective of their size. Under these systems, if an insured makes a claim he moves to a class where he is required to pay a higher premium (malus) or remains at the highest premium class and if he does not make a claim he either stays in the same class or moves to a class where he is required to pay a lower premium (bonus). Bonus malus system is normally determined by three elements: the premium scale, the initial class, and the transition rules that determine the transfer from one class to another when the number of claims is known. An insured enters the system in the initial class when he applies for insurance, and throughout the entire driving lifetime, the transition rules are applied upon each renewal to determine the new class. The transition probabilities are determined by factors that can be broadly classified into two; that is, the priori and the posteriori classifications. The posteriori classification criteria considered the number and the severity of accidents that a policyholder made under the years of observation. The priori classification criteria considered variables whose values are known before the policyholder starts to drive such as age, horse power of the vehicle, and other characteristics of the driver and the automobile. However, there are other important or 'hidden' factors that cannot be taken into account by a priori classification. These include swiftness of reflexes, aggressiveness behind the wheel, or knowledge of Highway Code, all of which have bearing on the frequency and severity of motor insurance claims. The existence of these attitudinal factors renders a priori classification yet heterogeneous despite the use of many classification variables.

BMS in different countries.

The regulatory environment in the different countries are extremely diversified from total freedom to government imposed systems.

BMS in Belgium

In Belgium, third party automobile insurance was made compulsory in 1956. Only the characteristics of the automobile model such as horse power were used to differentiate premiums with a moderate deductible for young drivers. BMS was introduced in 1961 by a single middle sized company. The company gave the customers option to either adopt the old policy without experience rating or the BMS. The initial premium for the BMS was set at 20% higher, however vast majority of customers preferred this system. In 1971 the state enacted a BMS that had eighteen classes that had to be applied by all companies. The premium ranged from sixty for class one to two hundred for class eighteen. The entry point differed depending on whether the customer is private driver or business driver. The transition rules were for claim free years there was a reward of one class discount. The first claim in any given year led to a two class increase, any subsequent claim reported during the same year was penalized by three classes. Policies with four consecutive claim free years could not be in a class above class ten. To prevent switching of companies to evade any penalties imposed in the past, companies came up with systems to track customers where any move to another company required a certificate from the current company clearly stating the bonus malus level attained. There was an imbalance in the bonuses awarded and the maluses imposed on the policyholders since most of the drivers were in class one. This led to creation of a study group in 1983 whose mandate was to recommend a new tariff structure to the control authorities (Lemaire 1985). The new system applicable in 1992 which recommended the following changes among others.

Companies be allowed to use other variable such as age
Companies to communicate their rates to the authorities
Consideration for young (under 23 years of age) drivers was optional
All policies become one year renewable contracts.

The new system consisted of twenty three classes with premium ranging from fifty four to two hundred. For a claim free year a one class discount, penalty of five classes for the first claim and five classes' penalty for subsequent claims. Policyholders with four consecutive free claim years cannot be above level one hundred.

BMS in Brazil

The BMS was based on seven classes, premium ranging from 65 to 100. The starting premium was one hundred. For a claim free year a bonus of one class was awarded and for each claim a penalty of one class was imposed.

BMS in Denmark.

BMS here was based on 10 classes with premiums ranging from thirty to one fifty. The starting premium was one hundred. For a claim free year there was bonus of one class and for each claim a penalty of two classes.

BMS in Germany.

In Germany BMS the old system had eighteen classes with premiums ranging from forty to two hundred. The starting premium was set as 175 or 125 for drivers licensed for at least three years. The transition rule was, for a claim free year a bonus of one class and for each claim a penalty of one or two classes for highest levels and four to five years for the lowest levels.in the new system they had twenty two classes with premiums ranging from thirty to two hundred. The starting premium was 175 or 125 depending on the experience and other cars in the household. For a claim free year a bonus of one class was awarded and for each claim a penalty of one class for upper classes to nine for lowest class.

BMS in Kenya.

BMS in Kenya was based on seven classes with premiums ranging from forty for class one to one hundred for class seven. The transition rules were, for claim free year a bonus of one year was granted and for each claim all discounts were lost.

BMS in Korea

BMS was based on thirty seven classes with premium level from forty to two hundred and twenty. The entry premium was one hundred. For a claim free year the premium level decreases by ten. However moving down was only allowed after three claim free years. The malus was based on the level of severity of the accident. Property damage was penalized by 0.5 or 1 penalty point depending on cost. Depending on the type of injury, bodily injury claims were penalized 1 to 4 points depending on the type of injury. Serious injuries were assessed and imposed with penalties of supplementary points of up to three. The premium increased by ten levels per penalty point with a few exceptions.

BMS in Norway

The old BMS system was based on an infinite number of classes with the minimum premium level being thirty and increasing by ten in each class. The entry premium was one hundred. The transition rules were, for a claim free year a bonus of one class or a premium of 120 if more favorable. For the first claim a malus of two classes for highest levels and three classes for lowest levels was imposed. Any subsequent claim was penalized with two classes. A new system was introduced in 1987 by a leading company where several BMS coexist. The system had infinite number of classes with premium levels being all integers from 25 and above. The starting premium level was 80 for drivers aged at least 25 insuring privately owned vehicle and 100 for al, other customers. For a claim free year a bonus of 13% was awarded. For each claim, a fixed amount premium was imposed as penalty. The penalty however could not exceed 50% of the basic premium. The penalty was reduced by half for the drivers who have had between five and nine consecutive claim-free years at level 25, for their first claim. It is waived for drivers who have had at least ten consecutive years at the 25 level, for their first claim. An extra deductible is enforced if the claimant is at a higher level than 80, prior to the claim.

BMS in the United Kingdom

The system is made of seven classes with premium levels ranging from 33 to 100. The starting premium is seventy five. For acclaim free year, a one class bonus is awarded. For the first claim for a policy holder in class one a penalty of three classes is imposed, for class two and three a penalty of two classes and for the other classes a penalty of one class. As British insurers enjoy complete tariff structure freedom, many BMS coexist. Many insurers have recently introduced "protected discount schemes": policy- holders who have reached the maximum discount may elect to pay a surcharge, usually in the [10%-20%] range, to have their entitlement to discount preserved in case of a claim. More than two claims in five years result in disqualification from the protected discount scheme. Both the protected and unprotected forms are analyzed.

The study showed that all BMS were based on the priori component and the number of claims ignoring the size of claim with the exception of Korea level of severity was incorporated by classifying claims as either property damage or bodily injury.

BMS in Nigeria

The Nigerian BMS recognizes three categories of motor vehicles, private motor cars, commercial vehicles on schedule 1 to 5 and commercial vehicles on schedule 6.

For the private motor cars if the policyholder reported no accident during the previous insurance year, he would be given a 20% bonus in the current period. Where no accident is reported during the second year, the bonus will be increased to 25%. For the third, fourth and fifth claim-free insurance years, the premium discount is 33.3%, 40% and 50% respectively. The premium discount, however, cannot exceed 50%, as no discounts are allowed after the fifth claim-free year. The initial premium is 100. In case of a claim all the discount gained is lost and the policyholder starts from 100 all over again. If an insured changes the insurance company, he will go direct to the discount level achieved in the new insurance company if the policyholder can document the discount level attained with the previous insurance company. For the commercial vehicles on schedule 1 to 5 a discount of 15% in premium is allowed where no claim is made or pending during the preceding year or years of insurance. While as for the commercial vehicles in schedule 6 a discount of 10% is allowed irrespective of the number of claim free years. (Ibiwoye, Adeleke & Aduloju 2011). However the study argued that the system was not optimal since it did not take into consideration factors such as claim severity and depreciation of the motor vehicle, the transfer of information between insurance companies was inefficient and the loss of all discount attained in case of a claim.

Optimal BMS based on the posteriori information.

There has been great effort to model an optimal bonus malus system. Frangos, and Vrontos, (2001) defined an optimal BMS as one that is financially balanced for the insurer that is the total amount of bonuses is equal to the total amount of maluses and fair for the policyholder that is each policyholder pays a premium proportional to the risk that he imposes to the pool. In this effort

Lemaire (1995) developed a BMS quadratic error loss function, the expected value premium calculation principle and the Negative Binomial as the claim frequency distribution. Similarly, Tremblay (1992) designed an optimal BMS using the quadratic error loss function, the Poisson Inverse Gaussian as the claim frequency distribution and the zero-utility premium calculation principle. However all this studies did not consider the claim severity component but considered the frequency component only. This system was unfair since there is no difference between the policyholder having an accident with a small size of loss and a big size of loss. That is the policyholder with a small claim size is penalized highly compared to a policyholder with a big claim size. This lead to policyholders with small claim amounts not to report the claims due to the fear of paying higher premium in future because of the malus imposed. This could go to the extent of the policyholder paying the third party than to report the claim. Lemaire (1977) referred to this as the hunger for bonus. Therefore there was need to incorporate the claim amount in the bonus malus system. A BMS which incorporates both the claim frequency and the claim amount is said to be optimal. Here a policyholders pays premium proportional to the risk he imposes to the pool. Motivated by this Frangos, and Vrontos (2001) designed an optimal BMS based on both the claim frequency component and the claim severity using negative binomial distribution to model the claim frequency and Pareto distribution to model the claim severity. Premium was computed using the net premium principle. similarly Ibiwoye, Adeleke & Aduloju (2011) considered the design of optimal BMS based on both frequency and severity components using Poisson exponential mixture (Geometric distribution) and Poisson Gamma mixture (negative binomial) for the frequency component and Pareto for the severity component. Also Mert and Saykan (2005) considered both frequency and severity in the design of an optimal BMS system taking claim frequency to be Geometric distributed and claim severity to be to be Pareto distributed.

Optimal BMS based on both posteriori and priori information.

All the models mentioned above are function of time and of past number of accidents and do not take into consideration the characteristics of each individual. However there was need to design an optimal BMS based on both the posteriori and priori classification. Motivated by this Dionne and Vanasse (1989), stated that the premiums do not vary simultaneously with other variables that affect the claim frequency distribution. The BMS was derived as a function of the years that the policyholder is in the portfolio, the number of accidents and the individual characteristics which are significant for the number of accidents. Similarly Picech (1994) and Sigalotti (1994) derived a BMS that incorporates the a posteriori and the a priori classification criteria, with the engine power as the single a priori rating variable. Frangos, and Vrontos, (2001) suggested a generalized optimal BMS. The extended the work of Dionne and Vanasse (1989, 1992) by introducing the severity component. The study proposed a generalized BMS that integrates a priori and a posteriori information on an individual basis based both on the frequency and the severity component. This generalized BMS was derived as a function of the years that the policyholder is in the portfolio, the number of accidents, the exact size of loss that each one of these accidents incurred, and the significant individual characteristics for the number of accidents and for the severity of the accidents. Some of the a priori rating variables that could were used include the age, the sex and the place of residence of the policyholder, the age, the type and the cubic capacity of the car, etc.

1.2 Problem statement

Usually we consider claim frequency in Bonus Malus System without taking into consideration the size of the claim. This system is unfair since policyholders with large claim amounts are penalized the same way with policyholders with large claim amounts (Frangos, and Vrontos, (2001). The study further proposed a generalized BMS that incorporates both the posteriori and priori information.

In literature only Poisson Gamma (Negative Binomial) distribution, Poisson inverse Gaussian and Poisson exponential (Geometric) distribution has been used as the Poisson mixtures in modelling the frequency component, while only the exponential Gamma (Pareto) distribution has been used to model the severity component. Different claim frequency distributions and different claim severity distributions would give different strictness in terms of bonuses awarded to good drivers and malus imposed on bad drivers. This will intern affect the competitiveness of the insurance company in the market. Lemaire (1998), stated that in Belgium when the BMS was introduced customers were given option on whether to take the traditional policies or the BMS. Most of the customers preferred the BMS though it was expensive. Therefore the researcher sought to investigate and compare the level of strictness on application of different frequency distributions on the optimal BMS holding the severity distribution to be Pareto distribution.

1.3 Objectives

General objectives

The main objective of the study was to calculate automobile premiums taking into account both claim frequency and claim severity components.

Specific objectives

The following were the specific objectives:

i. To estimate frequency component using Poisson mixture.
ii. To estimate severity component using exponential mixture.
iii. To use the claim frequency component mean and claim severity component mean to estimate automobile insurance premium.
iv. To compare the premium charged and the level of strictness under different frequency distributions.

1.4 Significance of study

The finding of the study will play a great role in comparing the level of strictness of different claim frequency distributions. The level of strictness in turn determines the competitiveness level of an insurance company in the market. The study further opens up areas of study such as investigation and comparing the claim severity distributions in terms of their strictness in design of optimal BMS. This can be done much easily by use of a link between Poisson mixtures and exponential mixtures.

CHAPTER 2: LITERATURE REVIEW

2.1 introduction

In this section we review the work that has been done on the bonus malus system. We consider the bonus malus system based on the posteriori components. First we review studies on BMS based on the frequency component then BMS based on both frequency and severity components. We will then summarize our finding and state the gaps we have identified in our review some of which this paper will be based on.

2.2 BMS based on frequency component

In this case the number of claims a policyholder makes determines the premium he/she is charged. The claim frequencies under insurance policies show a considerable heterogeneity, especially in the early years. Therefore it's not possible to model frequency as homogeneous sub-groups. Hence most of the work done takes the frequency component as a distribution

Lemaire (1995) considered the design of an optimal BMS based on the number of claims of each policyholder. The optimal estimate of the policyholder's claim frequency is the one that minimizes the loss incurred. Lemaire (1995) considered, among other BMS, the optimal BMS obtained using the quadratic error loss function, the expected value premium calculation principle and the Negative Binomial as the claim frequency distribution.

Tremblay (1992) considered the design of an optimal BMS based on the Poisson Inverse Gaussian as the claim frequency distribution. He took the frequency of claims to be Poisson distributed assuming that the frequency of claims vary with portfolio. He further assumed that the portfolio risk in any particular portfolio has a Poisson distribution with mean Λ, where Λ is itself a random variable with distribution representing the expected risks inherent in the given portfolio. He took Λ to be inverse Gaussian arguing that it has thick tails and has a closed form expression of the moment generating functions. The mixed Poisson provided a better fit from the insurer's point since its variance is greater than its mean as compared to the Poisson distribution where the variance is equal to the mean. He used the quadratic error loss function to estimate the parameter that minimizes loss and using the Bayesian theory he estimated the posterior distribution for the portfolio inherent risk given the claim frequency in the past n years. Premium was computed using the zero-utility principle.

Walhin and Paris (1999) extended the work of Lemaire (1995) and Tremblay (1992) who used the Poisson Gamma (Negative Binomial) distribution and the Poisson inverse Gaussian distribution as the claim frequency distributions respectively by using the Hofmann's distribution which is a three parameter distribution that encompasses the Poisson, Negative Binomial and the Inverse Gaussian Distributions. For comparison purpose, Walhin and Paris worked with a portfolio published by Buhlmann (1970) and used by Lemaire (1985) and Tremblay (1992). They showed that the Hofmann's distribution gives a better fit to the claim frequency data.

Dionne and Vanasse (1989, 1992) presented a BMS that integrates a priori and a posteriori information on an individual basis. This BMS is derived as a function of the years that the policyholder is in the portfolio, of the number of accidents and of the individual characteristics which are significant for the number of accidents.

2.3 BMS based on frequency and severity components.

In the models described above only the number of accidents is considered in design of the BMS ignoring the size of the claim.in this way policyholders with the same number of claims are penalized the same. This is unfair to policyholders with small amount of claims (Frangos, N. E., and Vrontos, S. D. 2001)

Lemaire (1995) pointed out that all BMS in the world with the exception of Korea consider the number of claims in BMS ignoring the claim size. In Korea claim severity was subdivided into two, those with bodily damage and those with property damage. Policyholders with bodily injuries were to pay higher maluses depending on the severity of the accident.

Pinquet (1997) considered the designed an optimal BMS which makes allowance for the severity of the claims first starting from a rating model based on the analysis of number of claims and of costs of claims, then heterogeneity components are added. This represent unobserved factors that are relevant for the explanation of the severity variables. The costs of claims follow Gamma or lognormal distribution. The rating factors, as well as the heterogeneity components are included in the scale parameter of the distribution. Considering that the heterogeneity also follows a Gamma or lognormal distribution, a credibility expression is obtained which provides a predictor for the average cost of claim for the following period

Mert and Saykan (2005) considered both frequency and severity in the design of an optimal BMS system taking claim frequency to be Geometric distributed and claim severity to be to be Pareto distributed. They used the quadratic loss function to estimate parameters and computed premium based on the net premium method as a product of the mean of the posterior claim frequency component and the mean of the posterior severity component.

Frangos, and Vrontos, (2001) designed an optimal BMS based on both number of claims (frequency) and claim amount (severity) using negative binomial distribution for the frequency component and Pareto distribution for the severity component. The number of claims were assumed to be Poisson distributed with mean λ. Where λ is the underlying risk of each policyholder which varies from one policyholder hence a random variable. The underlying risk was assumed to be Gamma distributed thus the mixed Poisson Gamma (Negative binomial). For the severity component, the amount of claims were assumed to be exponential distributed with mean claim size y which varies with policyholder hence a random variable. The mean was assumed to be Inverse Gamma distributed. Thus the exponential inverse Gamma mixture (Pareto distribution). Using the Bayesian theory, they obtained the posterior structure functions of the frequency component and for the severity component for the number of years the policyholder has been under observation. The premium estimate was based on the net premium principle as a product of the mean of the posterior structure function of the frequency component and the posterior structure function of the severity component.

Ibiwoye, Adeleke & Aduloju (2011) considered the design of optimal BMS based on both frequency and severity components using Poisson exponential mixture (Geometric distribution) and Poisson Gamma mixture (negative binomial) for the frequency component. The number of claims were assumed to be Poisson distributed while the underlying risk of the group of policyholder was taken first to be Exponential giving the Geometric distribution then Gamma giving rise to the negative Binomial distribution. They modelled the claim size to be an exponential inverse Gamma mixture (Pareto distribution) where the claim size for the k^{th} claim was assumed to be exponential and the mean claim amount to be inverse Gamma distributed. The expected value of the parameters was estimated using the quadratic loss method. The risk premium was estimated as the product of the mean claim frequency and claim severity components.

Promislow (2006) made an analysis on how to choose the frequency and the severity distributions comparing Binomial, Poisson and Negative binomial distributions for the frequency component and Normal, Gamma and Pareto for the severity component.

2.4 Summary

In most of the work reviewed the frequency component is modelled as a Poisson mixture where the number of claims is Poisson distributed and the underlying risk distribution is the mixing distribution. The mixing distributions used include:

a. Gamma distribution.
b. Exponential distribution.
c. Inverse Gaussian distribution.

For the severity component, an exponential mixture has been applied to model the frequency component. The claim size is taken to be exponential distributed while the mean claim size distribution is the mixing distribution. The mixing distribution considered include:

a. Gamma distribution
b. Inverse Gamma distribution
c. Lognormal distribution.

The Bayesian theorem is used to obtain the posterior structure functions for the frequency and the severity components. The mean of this functions is used to estimate the premiums to be charged to a policyholder who have been under observation.

2.5 Research gaps.

Only a few Poisson and exponential mixtures have been used to model the frequency and the severity components respectively. This can be extended by considering among others the following mixing distributions:

i. Erlang distribution
ii. Lindely distribution

iii. Normal distribution

There is need to come up with a link between Poisson and exponential mixture that will simplify the comparison of the various mixing distributions in design of an optimal BMS.

CHAPTER 3: METHODOLOGY

3.1 Introduction

In this chapter we design the optimal BMS based on both frequency and severity components. First we will consider the frequency component and fit several distributions in modelling the frequency component. Secondly we model the frequency component using a Poisson mixture. Finally we will estimate the premiums charged to a policyholder based on the frequency and severity component.

The severity and frequency components will be assumed to be independent.

3.2 Frequency component

In automobile insurance, when the portfolio is considered to be heterogeneous, all policyholders will have a constant but unequal underlying risk of having an accident. That is, the expected number of claims differs from policyholder to policyholder. As the mixed Poisson distributions have thicker tails than the Poisson distribution, it is seen that the mixed Poisson distributions provide a good fit to claim frequency data when the portfolio is heterogeneous. We will use the following Poisson mixture distributions to model the frequency component.

i. Poisson Gamma distribution
ii. Poisson exponential distribution
iii. Poisson Erlang distribution
iv. Poisson Lindley distribution

3.2.1 Poisson Gamma distribution.

Consider the number of claims k, given the parameter λ is distributed according to Poisson (λ)

$$p\left(k/\lambda\right)=\frac{e^{-\lambda}\lambda^{k}}{k!}$$

$k=0,1,2,3,\ldots$ and $\lambda>0$

λ denotes the different underlying risk of each policyholder to have an accident.

We assume that λ follows gamma (α, τ) distribution, with pdf of the form:

$$u(\lambda)=\frac{\lambda^{\alpha-1}\tau^{\alpha}e^{-\tau\lambda}}{\Gamma\alpha} \qquad \alpha>0,\ \lambda>0,\ \tau>0$$

With mean E(Λ)=α/τ and variance var(Λ)=α/τ^2

The unconditional distribution of the number of claims k will be:

$$p(k) = \int_0^\infty p(k/\lambda)\, u(\lambda)\, d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!} \frac{\lambda^{\alpha-1}\tau^\alpha e^{-\tau\lambda}}{\Gamma\alpha} d\lambda$$

$$= \int_0^\infty \frac{\lambda^{k+\alpha-1}\tau^\alpha e^{-\lambda(1+\tau)}}{k!\,\Gamma\alpha} d\lambda$$

$$= \int_0^\infty \frac{\Gamma(\alpha+k)(1+\tau)^{\alpha+k}\lambda^{k+\alpha-1}\tau^\alpha e^{-\lambda(1+\tau)}}{\Gamma(\alpha+k)(1+\tau)^{\alpha+k} k!\,\Gamma\alpha} d\lambda$$

$$= \frac{\tau^\alpha\Gamma(\alpha+k)}{(1+\tau)^{\alpha+k} k!\,\Gamma\alpha} \int_0^\infty \frac{(1+\tau)^{\alpha+k}\lambda^{k+\alpha-1} e^{-\lambda(1+\tau)}}{\Gamma(\alpha+k)} d\lambda$$

$$= \frac{\tau^\alpha(\alpha+k-1)!}{(1+\tau)^{\alpha+k} k!\,(\alpha-1)!}$$

$$= \frac{(\alpha+k-1)!}{k!\,(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^\alpha \left(\frac{1}{1+\tau}\right)^k$$

$$= \binom{\alpha+k-1}{k}\left(\frac{\tau}{1+\tau}\right)^\alpha \left(\frac{1}{1+\tau}\right)^k$$

which is probability density function of Negative binomial (α,τ)

$$E(K) = \sum_{k=0} k \binom{k+\alpha-1}{k}\left(\frac{\tau}{1+\tau}\right)^\alpha \left(\frac{1}{1+\tau}\right)^k$$

$$= \sum_{k=0} k\, \frac{(k+\alpha-1)!}{k!(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^\alpha \left(\frac{1}{1+\tau}\right)^k$$

$$= \sum_{k=0} \frac{(k+\alpha-1)!}{(k-1)!(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^\alpha \left(\frac{1}{1+\tau}\right)^k$$

$$=\frac{(1+\tau)}{\tau}*\frac{1}{(1+\tau)}\alpha\sum_{k=0}\binom{k+\alpha-1}{k}\left(\frac{\tau}{1+\tau}\right)^{\alpha+1}\left(\frac{1}{1+\tau}\right)^{k-1}$$

$$=\frac{\alpha}{\tau}$$

$$\operatorname{var}(K)=E(K(K-1))+E(K)-[E(K)]^2$$

$$E(K(K-1))=\sum_k k(k-1)\binom{k+\alpha-1}{k}\left(\frac{\tau}{1+\tau}\right)^{\alpha}\left(\frac{1}{1+\tau}\right)^{k}$$

$$=\sum_k \frac{k(k-1)(k+\alpha-1)!}{k!(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^{\alpha}\left(\frac{1}{1+\tau}\right)^{k}$$

$$=\sum_k \frac{(k+\alpha-1)!}{(k-2)!(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^{\alpha}\left(\frac{1}{1+\tau}\right)^{k}$$

$$=\sum_k \alpha(\alpha+1)\frac{(k+\alpha-1)!}{(k-2)!(\alpha+1)!}\left(\frac{\tau}{1+\tau}\right)^{\alpha}\left(\frac{1}{1+\tau}\right)^{k}$$

$$=\frac{\alpha(\alpha+1)}{\tau^2}\sum_k\binom{k+\alpha-1}{k-2}\left(\frac{\tau}{1+\tau}\right)^{\alpha+2}\left(\frac{1}{1+\tau}\right)^{k-2}$$

$$=\frac{\alpha(\alpha+1)}{\tau^2}$$

$$\operatorname{var}(K)=\frac{\alpha(\alpha+1)}{\tau^2}+\frac{\alpha}{\tau}-\left(\frac{\alpha}{\tau}\right)^2$$

$$=\frac{\alpha(1+\tau)}{\tau^2}$$

$$=\frac{\alpha}{\tau}\left(1+\frac{1}{\tau}\right)$$

The variance of the negative binomial exceeds its mean, this will help us to deal with over dispersion.

Consider a policyholder or group of policyholders who have been under observation for the last t years.

Let $K=\sum_{i=1}^{t}k_i$ be the number of claims the policyholder had in the t years, where k_i is the number of claims that the policyholder had in year i=1,2,…,t.

Using the Bayes theorem we can obtain the posterior structure function for λ for a policyholder with claim history $(k_1,...,k_t)$, $u(\lambda/k_1,...,k_t)$

$$u(\lambda/k_1,...,k_t)\propto p(k_1,...,k_t)u(\lambda)$$

$$p(k_1,...,k_t/\lambda)=\prod_{i=1}^{t}\frac{e^{-\lambda}\lambda^{k}}{k!}$$

$$=\frac{e^{-\lambda t}\lambda^{K}}{\prod_{i=1}^{t}k_i!}$$

$$u(\lambda/k_1,...k_t)\propto e^{-\lambda t}\lambda^{K}\lambda^{\alpha-1}e^{-\lambda\tau}$$

$$\propto e^{-\lambda(t+\tau)}\lambda^{K+\alpha-1}$$

$$\int_0^\infty u(\lambda/k_1,...k_t)d\lambda\propto\int_0^\infty e^{-\lambda(t+\tau)}\lambda^{K+\alpha-1}d\lambda$$

$$\int_0^\infty Ae^{-\lambda(t+\tau)}\lambda^{K+\alpha-1}d\lambda=1$$

where A= $\frac{(t+\tau)^{K+\alpha}}{\Gamma(\alpha+K)}$

hence

$$u(\lambda/k_1,...k_t)=\frac{(t+\tau)^{K+\alpha}e^{-\lambda(t+\tau)}\lambda^{K+\alpha-1}}{\Gamma(\alpha+K)}$$

Which is the pdf for gamma $(\alpha+k,\ t+\tau)$

The optimal choice of λ_{t+1} for a policyholder with claim history $k_1,...k_t$ will be the mean of the posterior structure function, that is

$$\hat{\lambda}_{t+1} = \int_0^{\hbar\infty} \frac{(t+\tau)^{K+\alpha} \lambda e^{-\lambda(t+\tau)} \lambda^{K+\alpha-1}}{\Gamma(\alpha+K)} d\lambda$$

$$= \int_0^{\hbar\infty} \frac{(t+\tau)^{K+\alpha} \lambda^{K+\alpha} e^{-\lambda(t+\tau)}}{\Gamma(\alpha+K)} d\lambda$$

$$= \frac{\alpha+K}{t+\tau} \int_0^{\hbar\infty} \frac{(t+\tau)^{K+\alpha+1} \lambda^{K+\alpha} e^{-\lambda(t+\tau)}}{\Gamma(\alpha+K+1)} d\lambda$$

$$\hat{\lambda}_{t+1} = \frac{\alpha+K}{t+\tau} = \bar{\lambda}\left(\frac{\alpha+K}{\alpha+t\bar{\lambda}}\right) \text{ where } \bar{\lambda} = \frac{\alpha}{\tau} \qquad (3.2.1.1)$$

The occurrence of K accidents in t years necessitates an update of the parameters of gamma from α and τ to α+K and t+τ respectively.

3.2.2 Poisson exponential distribution.

Assume that the number of claims k is distributed according to Poisson with a given Parameter λ.

$$p(k/\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Let us assume that λ is distributed according to the Exponential distribution with parameter θ (that is, the structure function of λ is assumed to be an Exponential distribution). The probability density function of λ is as follows:

$$u(\lambda) = \theta e^{-\lambda\theta} \quad , \quad \lambda > 0$$

Then the unconditional distribution of k claims is as follows

$$p(k) = \int_0^\infty p(k/\lambda).u(\lambda) d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!} \theta e^{-\lambda\theta} d\lambda$$

$$= \int_0^\infty \theta \frac{e^{-\lambda(1+\theta)}\lambda^k}{k!} d\lambda$$

$$= \frac{\theta}{k!(1+\theta)^k} \int_0^\infty e^{-\lambda(1+\theta)} \left(\lambda(1+\theta)\right)^k d\lambda$$

using integration by substitution we have

$$p(k) = \frac{\theta}{k!(1+\theta)^{k+1}} \Gamma(k+1)$$

$$= \frac{\theta}{(1+\theta)^{k+1}}$$

$$= \left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)^k$$

which is Geometric distribution with parameter θ.

The conditional distribution of the total number of claims in t years, $K = \sum_{i=1}^{t} k_i$ given λ will be

$$p(k_1, ..., k_t / \lambda) = \prod_0^t \frac{e^{-\lambda} \lambda^k}{k!}$$

$$= \frac{e^{-\lambda t} \lambda^K}{\prod_{i=1}^{t} k_i!}$$

By applying the Bayesian, the posterior structure function for a group of policyholders with a claim history $k_1, ..., k_t$ can be obtained as follows

$$u(\lambda / k_1, ..., k_t) \propto p(k_1, ..., k / \lambda) u(\lambda)$$

$$\propto \mathrm{e}^{-\lambda t} \lambda^K e^{-\lambda\theta}$$

$$\propto e^{-\lambda(t+\theta)} \lambda^K$$

$$\int_0^\infty u(\lambda / k_1, ..., k_t) d\lambda \propto \int_0^\infty e^{-\lambda(t+\theta)} \lambda^K d\lambda$$

$$\int_0^\infty u(\lambda / k_1, ..., k_t) d\lambda = \int_0^\infty A e^{-\lambda(t+\theta)} \lambda^K d\lambda = 1$$

Therefore

$$\int_0^\infty A e^{-\lambda(t+\theta)} \lambda^K d\lambda = \frac{A}{(t+\theta)^K} \int_0^\infty e^{-\lambda(t+\theta)} \left(\lambda(t+\theta)\right)^K d\lambda$$

using integration by sustitution we have

$$\frac{A}{(t+\theta)^{K+1}}\Gamma(K+1)=1$$

$$A=\frac{(t+\theta)^{K+1}}{\Gamma(K+1)}$$

Hence

$$u(\lambda/k_1,\ldots,k_t)=\frac{(t+\theta)^{K+1}e^{-\lambda(t+\theta)}\left(\lambda(t+\theta)\right)^{K}}{\Gamma(K+1)}\ ,\qquad \lambda>0$$

which is pdf of Gamma (K+1, t+θ)

The optimal choice of λ_{t+1} for a policyholder with claim history $k_1,\ldots,k_t$ will be the mean of the posterior structure function

$$\lambda_{t+1}(k_1,\ldots,k_t)=\int_0^\infty \lambda\frac{(t+\theta)^{k+1}}{\Gamma(k+1)}e^{-\lambda(t+\theta)}\lambda^k d\lambda$$

$$=\frac{(t+\theta)^{k+1}}{\Gamma(k+1)}\int_0^\infty e^{-\lambda(t+\theta)}\lambda^{k+1}d\lambda$$

$$=\frac{(t+\theta)^{k+1}}{\Gamma(k+1)}\frac{\Gamma(k+2)}{(t+\theta)^{k+2}}\int_0^\infty\frac{(t+\theta)^{k+2}\lambda^{k+1}e^{-\lambda(t+\theta)}}{\Gamma(k+2)}d\lambda$$

$$=\frac{k+1}{t+\theta}$$

$$\hat{\lambda}_{t+1}=\frac{k+1}{t+\theta}$$

$$\hat{\lambda}_{t+1}=\frac{K+1}{t+\theta}=\bar{\lambda}\left(\frac{K+1}{\frac{t}{\theta}+1}\right)\ \text{where}\ \bar{\lambda}=\frac{1}{\theta}\qquad (3.2.2.1)$$

3.2.3 Poisson – Erlang distribution

The conditional distribution of the number of claims k given the underlying risk λ is assumed to be Poisson distributed with pdf

$$p\left(k/\lambda\right)=\frac{e^{-\lambda}\lambda^{k}}{k!}$$
$$k=0,1,2,3,\ldots \quad \text{and } \lambda>0$$

Let λ be Erlang distributed with parameter α

$$u(\lambda)=\alpha^{2}\lambda e^{-\lambda\alpha} \qquad \lambda>0$$

The unconditional distribution of k would therefore be

$$p(k)=\int_{0}^{\infty} p(k/\lambda).u(\lambda)d\lambda$$
$$=\int_{0}^{\infty}\tfrac{e^{-\lambda}\lambda^{k}}{k!}\alpha^{2}\lambda e^{-\lambda\alpha}d\lambda$$
$$=\frac{\alpha^{2}}{k!}\int_{0}^{\infty}\lambda^{k+1}e^{-\lambda(1+\alpha)}d\lambda$$
$$=\frac{\alpha^{2}\Gamma(k+2)}{(1+\alpha)^{k+2}k!}\int_{0}^{\infty}\tfrac{(1+\alpha)^{k+2}\lambda^{k+1}e^{-\lambda(1+\alpha)}}{\Gamma(k+2)}d\lambda$$
$$=\frac{\alpha^{2}(k+1)}{(1+\alpha)^{k+2}}$$

Mean of the Poisson Erlang

$$E(X)=\int_{0}^{\infty}\sum_{k=0}^{\infty}k\tfrac{e^{-\lambda}\lambda^{k}}{k!}\alpha^{2}\lambda e^{-\lambda\alpha}d\lambda$$
$$=\int_{0}^{\infty}\lambda\sum_{k=0}^{\infty}\tfrac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\alpha^{2}\lambda e^{-\lambda\alpha}d\lambda$$
$$=\alpha^{2}\int_{0}^{\infty}\lambda^{2}e^{-\lambda\alpha}d\lambda$$

Using integration by parts we have;

$$E(X)\ =\frac{2}{\alpha}$$

Variance of the Poisson Erlang.

$$Var(K) = E(k(k-1)) + E(k) - [E(k)]^2$$

$$E(K(K-1)) = \int_0^\infty \sum_{k=0}^\infty k(k-1)\tfrac{e^{-\lambda}\lambda^k}{k!}\alpha^2 \lambda e^{-\lambda\alpha} d\lambda$$

$$= \int_0^\infty \lambda^2 \sum_{k=0}^\infty \tfrac{e^{-\lambda}\lambda^{k-2}}{(k-2)!}\alpha^2 \lambda e^{-\lambda\alpha} d\lambda$$

$$= \alpha^2 \int_0^\infty \lambda^3 e^{-\lambda\alpha} d\lambda$$

using integration by parts we have:

$$E(K(K-1)) = \frac{6}{\alpha^2}$$

The posterior distribution of λ given claim history $k_1, \dots, k_t$ will be

$$u(\lambda / k_1, ..., k_t) \propto p(k_1, ..., k_t / \lambda) u(\lambda)$$

$$\propto e^{-\lambda t} \lambda^k \lambda e^{-\lambda\alpha}$$

$$\propto e^{-\lambda(t+\alpha)} \lambda^{k+1}$$

$$\int_0^\infty u(\lambda / k_1, ..., k_t) d\lambda \propto \int_0^\infty e^{-\lambda(t+\alpha)} \lambda^{k+1} d\lambda$$

Hence

$$\int_0^\infty A e^{-\lambda(t+\alpha)} \lambda^{k+1} d\lambda = 1$$

$$A \int_0^\infty \tfrac{(t+\alpha)^{k+2} e^{-\lambda(t+\alpha)} \lambda^{k+1}}{\Gamma(k+2)} d\lambda = 1$$

$$A = \frac{(t+\alpha)^{k+2}}{\Gamma(k+2)}$$

The posterior structure function for λ will therefore be

$$u(\lambda / k_1, ..., k_t) = \frac{(t+\alpha)^{k+2} \lambda^{k+1} e^{-\lambda(t+\alpha)}}{\Gamma(k+2)} \qquad \lambda > 0 \text{ , } \alpha > 0$$

Which is the pdf of Gamma $(k+2,\ t+\alpha)$

The optimal choice of λ_{t+1} for a policyholder with claim history $k_1,...,k_t$ will be the mean of the posterior structure function

$$\lambda_{t+1} = \int_0^\infty \lambda \frac{(t+\alpha)^{k+2}\lambda^{k+1}e^{-\lambda(t+\alpha)}}{\Gamma(k+2)} d\lambda$$

$$= \int_0^\infty \frac{(t+\alpha)^{k+2}\lambda^{k+2}e^{-\lambda(t+\alpha)}}{\Gamma(k+2)} d\lambda$$

$$= \frac{k+2}{t+\alpha}\int_0^\infty \frac{(t+\alpha)^{k+3}\lambda^{k+2}e^{-\lambda(t+\alpha)}}{\Gamma(k+3)} d\lambda$$

$$\hat{\lambda}_{t+1} = \frac{k+2}{t+\alpha} \qquad (3.2.3.1)$$

3.2.4 Poisson Lindley distribution.

Taking the conditional distribution of the number of claims k given the underlying risk λ to be Poisson distributed with pdf

$$p(k/\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$k = 0,1,2,3,... \quad \text{and } \lambda>0$$

Let λ be Lindley distributed with parameter θ

$$u(\lambda) = \frac{\theta^2}{\theta+1}(\lambda+1)e^{-\lambda\theta} \qquad \lambda>0\ ,\ \ \theta>0$$

The unconditional distribution of k will be:

$$p(k) = \int_0^\infty p(k/\lambda).u(\lambda) d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!}\frac{\theta^2(\lambda+1)e^{-\lambda\theta}}{\theta+1} d\lambda$$

$$= \frac{\theta^2}{1+\theta}\int_0^\infty \frac{e^{-\lambda(1+\theta)}\lambda^k(\lambda+1)}{k!} d\lambda$$

$$=\frac{\theta^2}{1+\theta}\left[\int_0^\infty \frac{e^{-\lambda(1+\theta)}\lambda^{k+1}}{k!}d\lambda+\int_0^\infty \frac{e^{-\lambda(1+\theta)}\lambda^{k}}{k!}d\lambda\right]$$

$$=\frac{\theta^2}{1+\theta}\left[\frac{k+1}{(1+\theta)^{k+2}}\int_0^\infty \frac{e^{-\lambda(1+\theta)}\lambda^{k+1}(1+\theta)^{k+2}}{\Gamma(k+2)}d\lambda+\frac{1}{(1+\theta)^{k+1}}\int_0^\infty \frac{e^{-\lambda(1+\theta)}\lambda^{k}(1+\theta)^{k+1}}{\Gamma(k+1)}d\lambda\right]$$

$$=\frac{\theta^2}{1+\theta}\left[\frac{k+1}{(1+\theta)^{k+2}}+\frac{1}{(1+\theta)^{k+1}}\right]$$

$$=\frac{\theta^2(k+2+\theta)}{(1+\theta)^{k+3}}$$

Mean

$$E(K)=\int_0^\infty\left[\sum_{k=0}^\infty k\frac{e^{-\lambda}\lambda^k}{k!}\right]\frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\int_0^\infty\left[\sum_{k=0}^\infty \frac{e^{-\lambda}\lambda^k}{(k-1)!}\right]\frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{1+\theta}\int_0^\infty\left[\lambda\sum_{k=0}^\infty \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\right](1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{1+\theta}\int_0^\infty \lambda(1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{1+\theta}\left[\int_0^\infty \lambda e^{-\theta\lambda}d\lambda+\int_0^\infty \lambda^2 e^{-\theta\lambda}d\lambda\right]$$

Using integration by parts we have:

$$=\frac{\theta^2}{1+\theta}\left(\frac{2}{\theta^3}+\frac{1}{\theta^2}\right)$$

$$=\frac{2+\theta}{\theta(1+\theta)}$$

Variance

$$Var(K)=E(K(K-1))+E(K)-\left(E(K)\right)^2$$

$$E(K(K-1))=\int_0^\infty\left[\sum_{k=0}^\infty k(k-1)\frac{e^{-\lambda}\lambda^k}{k!}\right]\left(\frac{\theta^2}{1+\theta}\right)(1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{1+\theta}\int_0^\infty\left[\lambda^2\sum_{k=2}^\infty \tfrac{e^{-\lambda}\lambda^{k-2}}{(k-2)!}\right](1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{1+\theta}\int_0^\infty \lambda^2(1+\lambda)e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{1+\theta}\left[\int_0^\infty \lambda^2 e^{-\theta\lambda}d\lambda+\int_0^\infty \lambda^3 e^{-\theta\lambda}d\lambda\right]$$

$$=\frac{\theta^2}{1+\theta}\left(\frac{2}{\theta^3}+\frac{6}{\theta^4}\right)$$

$$=\frac{2\theta+6}{(1+\theta)\theta}$$

$$Var(K)=\frac{2\theta+6}{(1+\theta)\theta^2}+\frac{2+\theta}{(1+\theta)\theta}-\left(\frac{2+\theta}{(1+\theta)\theta}\right)^2$$

$$=\frac{\theta^3+4\theta^2+8\theta+2}{(1+\theta)^2\theta^2}$$

The posterior structure function for a policyholder with a claim history $k_1,...,k_t$ is given by:

$$u(\lambda / k_1,...,k_t) \propto p(k_1,...,k_t / \lambda).u(\lambda)$$

$$\propto e^{-\lambda t}\lambda^k(\lambda+1)e^{-\lambda\theta}$$

$$\int_0^\infty u(\lambda / k_1,...,k_t)d\lambda \propto \int_0^\infty e^{-\lambda(t+\theta)}\left(\lambda^{k+1}+\lambda^k\right)d\lambda$$

$$\propto \int_0^\infty \lambda^{k+1}e^{-\lambda(t+\theta)}d\lambda+\int_0^\infty \lambda^k e^{-\lambda(t+\theta)}d\lambda$$

$$\int_0^\infty u(\lambda / k_1,...,k_t)d\lambda = A\left[\int_0^\infty \lambda^{k+1}e^{-\lambda(t+\theta)}d\lambda+\int_0^\infty \lambda^k e^{-\lambda(t+\theta)}d\lambda\right]$$

$$A\left[\int_0^\infty \lambda^{k+1}e^{-\lambda(t+\theta)}d\lambda+\int_0^\infty \lambda^k e^{-\lambda(t+\theta)}d\lambda\right]=1$$

$$A=\frac{(t+\theta)^{k+2}}{\Gamma(k+2)+(t+\theta)\Gamma(k+1)}$$

$$u(\lambda / k_1, ..., k_t) = \frac{(t+\theta)^{k+2} e^{-\lambda(t+\theta)} \left(\lambda^{k+1} + \lambda^k\right)}{\Gamma(k+2) + (t+\theta)\Gamma(k+1)}$$

The mean of the posterior structure function will be:

$$\lambda_{t+1}(k_1, ..., k_t) = \int_0^\infty \lambda \frac{(t+\theta)^{k+2} e^{-\lambda(t+\theta)} \left(\lambda^{k+1} + \lambda^k\right)}{\Gamma(k+2) + (t+\theta)\Gamma(k+1)} d\lambda$$

$$= \frac{1}{\Gamma(k+2) + (t+\theta)\Gamma(k+1)} \left[\int_0^\infty (t+\theta)^{k+2} e^{-\lambda(t+\theta)} \lambda^{k+2} d\lambda + \int_0^\infty (t+\theta)^{k+2} e^{-\lambda(t+\theta)} \lambda^{k+1} d\lambda \right]$$

$$= \frac{1}{\Gamma(k+2) + (t+\theta)\Gamma(k+1)} \left[\frac{\Gamma(k+3)}{(t+\theta)} \int_0^\infty \frac{(t+\theta)^{k+3} e^{-\lambda(t+\theta)} \lambda^{k+2}}{\Gamma(k+3)} d\lambda + \Gamma(k+2) \int_0^\infty \frac{(t+\theta)^{k+2} e^{-\lambda(t+\theta)} \lambda^{k+1}}{\Gamma(k+2)} d\lambda \right]$$

$$= \frac{1}{\Gamma(k+2) + (t+\theta)\Gamma(k+1)} \left[\frac{\Gamma(k+3)}{(t+\theta)} + \Gamma(k+2) \right]$$

$$\hat{\lambda}_{t+1} = \frac{(k+1)\left[(k+2) + (t+\theta)\right]}{(t+\theta)\left[(k+1) + (t+\theta)\right]} \qquad (3.2.4.1)$$

3.2.5 Estimation of parameters

We estimate the frequency distribution parameters using the method of moment and maximum likelihood method. We use the Newton's approximations for the non-linear equations.

3.2.5.1 Estimation of Negative Binomial Distribution parameters

Using methods of moments

$E(K) = \frac{\alpha}{\tau}$

Therefore:

$\bar{k} = \frac{\alpha}{\tau}$

$\alpha = \bar{k}\tau$

$$Var(K) = s^2 = \tfrac{\alpha}{\tau}\left(1+\tfrac{1}{\tau}\right)$$

$$s^2 = \overline{\mathrm{k}}\left(1+\tfrac{1}{\tau}\right)$$

$$\tfrac{1}{\tau} = \tfrac{s^2-\overline{\mathrm{k}}}{\overline{\mathrm{k}}}$$

$$\hat{\tau} = \tfrac{\overline{\mathrm{k}}}{s^2-\overline{\mathrm{k}}}$$

$$\hat{\alpha} = \tfrac{\overline{\mathrm{k}}^2}{s^2-\overline{\mathrm{k}}}$$

Using the maximum likelihood method

According to Lemaire (1995) we estimate the parameter as follows:

$$L(\alpha,\tau) = \prod_{i=1}^{n}\left[\frac{(k+\alpha-1)!}{k!\,(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^{\alpha}\left(\frac{1}{1+\tau}\right)^{k}\right]$$

$$\hat{\tau} = \frac{\alpha}{\overline{k}} \qquad (1)$$

$$\frac{\partial \ln L(\alpha,\tau)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\sum_{i=1}^{n}\ln(k+\alpha-1)! - \frac{\partial}{\partial\alpha} n\ln(\alpha-1)! + n\ln\tau - n\ln(1+\tau) = 0$$

$$\ln L(\alpha,\tau) = \sum_{i=1}^{n}\ln\left[\frac{(k+\alpha-1)!}{k!\,(\alpha-1)!}\left(\frac{\tau}{1+\tau}\right)^{\alpha}\left(\frac{1}{1+\tau}\right)^{k}\right]$$

$$= \sum_{i=1}^{n}\ln(k+\alpha-1)! - \sum_{i=1}^{n}\ln k! - \sum_{i=1}^{n}\ln(\alpha-1)! + n\alpha\ln\tau - n\alpha\ln(1+\tau) - \sum_{i=1}^{n}k_i\ln(1+\tau)$$

$$\frac{\partial \ln L(\alpha,\tau)}{\partial\tau} = \frac{n\alpha}{\tau} - \frac{n\alpha}{1+\tau} - \frac{\sum_{i=1}^{n}k_i}{1+\tau} = 0$$

$$\frac{n\alpha}{\tau} - \frac{n\alpha}{1+\tau} - \frac{n\overline{k}}{1+\tau} = 0$$

Using stirling approximation

That is:

$$\ln x! = \ln\left[(2\pi x)^{0.5}\left(\tfrac{x}{e}\right)^{x}\right]$$

$$\sum_{i=1}^{n}\ln(k+\alpha-1)! = \sum_{i=1}^{n}\ln\left[\left(2\pi(k+\alpha-1)\right)^{0.5}\left(\tfrac{k+\alpha-1}{e}\right)^{k+\alpha-1}\right]$$

$$= \sum_{i=1}^{n}\left[0.5\ln 2\pi + 0.5\ln(k+\alpha-1) + (k+\alpha-1)\ln(k+\alpha-1) - (k+\alpha-1)\right]$$

$$\frac{\partial}{\partial\alpha}\sum_{i=1}^{n}\ln(k+\alpha-1)! = \sum_{i=1}^{n}\left[\tfrac{1}{2(k+\alpha-1)} + \ln(k+\alpha-1)\right]$$

$$= \sum_{i=1}^{n}\tfrac{1}{2(k+\alpha-1)} + \sum_{i=1}^{n}\ln(k+\alpha-1)$$

$$\frac{\partial}{\partial\alpha}\ln(\alpha-1)! = \frac{\partial}{\partial\alpha}\ln\left[\left(2\pi(\alpha-1)\right)^{0.5}\left(\tfrac{(\alpha-1)!}{e}\right)^{\alpha-1}\right]$$

$$= \frac{\partial}{\partial\alpha}\left[0.5\ln 2\pi + 0.5\ln(\alpha-1) + (\alpha-1)\ln(\alpha-1) - (\alpha-1)\right]$$

$$= \tfrac{1}{2(\alpha-1)} + \ln(\alpha-1)$$

Therefore

$$\frac{\partial \ln L(\alpha,\tau)}{\partial\alpha} = \sum_{i=1}^{n}\tfrac{1}{2(k_i+\alpha-1)} + \sum_{i=1}^{n}\ln(k_i+\alpha-1) + \tfrac{1}{2(\alpha-1)} + \ln(\alpha-1) + n\ln\tau - n\ln(1+\tau) = 0 \qquad (2)$$

replacing (1) in (2)

$$\frac{\partial \ln L(\alpha)}{\partial\alpha} = \sum_{i=1}^{n}\tfrac{1}{2(k_i+\alpha-1)} + \sum_{i=1}^{n}\ln(k_i+\alpha-1) + \tfrac{1}{2(\alpha-1)} + \ln(\alpha-1) + n\ln\alpha - n\ln(\bar{k}+\alpha) = 0 \qquad (3)$$

Equation (3) is nonlinear in unknown α and the solution needs to be found by numerical methods. We consider one important algorithm for finding such a solution, Newton's method.

$$\alpha = \alpha_0 - \frac{g(\alpha_0)}{g'(\alpha_0)}$$

where:

α_0 is the initial estimate using the method of moments

$$g(\alpha) = \frac{\partial \ln L(\alpha)}{\partial\alpha}$$

3.2.5.2 Estimation of the geometric distribution parameters

Using the maximum likelihood estimation

$$L(\theta)=\prod_{i=1}^{n}\left(\tfrac{\theta}{1+\theta}\right)\left(\tfrac{1}{1+\theta}\right)^{k_i}$$

$$\ln L(\theta)=\sum_{i=1}^{n}\ln\left[\left(\tfrac{\theta}{1+\theta}\right)\left(\tfrac{1}{1+\theta}\right)^{k_i}\right]$$

$$=\text{n}\ln\theta\text{-n}\ln(1+\theta)\text{-}\sum_{i=1}^{n}k_i\ln(1+\theta)$$

$$=\text{n}\ln\theta\text{-n}\ln(1+\theta)\text{-n}\bar{\text{k}}\ln(1+\theta)$$

$$\frac{\partial\ln L(\theta)}{\partial\theta}=\frac{n}{\theta}-\frac{n}{1+\theta}-\frac{n\bar{k}}{1+\theta}=0$$

$$\hat{\theta}=\frac{1}{\bar{k}}$$

3.2.5.3 Estimation of Poisson Erlang parameter

Method of moments

$$\bar{K}=\frac{2}{\alpha}$$

$$\hat{\alpha}=\frac{2}{\bar{K}}$$

Maximum likelihood estimate

$$L(\alpha)=\prod_{i=1}^{n}\frac{\alpha^2(k_i+1)}{(1+\alpha)^{k+2}}$$

$$\ln L(\alpha)=\sum_{i=1}^{n}\ln\left(\frac{\alpha^2(k_i+1)}{(1+\alpha)^{k_i+2}}\right)$$

$$=\sum_{i=1}^{n}\left(2\ln\alpha+\ln(k_i+1)-(k_i+2)\ln(1+\alpha)\right)$$

$$=2n\ln\alpha+\sum_{i=1}^{n}\ln(k_i+1)-\sum_{i=1}^{n}(k_i+2)\ln(1+\alpha)$$

$$=2n\ln\alpha+\sum_{i=1}^{n}\ln(k_i+1)-n(\bar{k}+2)\ln(1+\alpha)$$

$$\frac{\partial \ln L(\alpha)}{\partial \alpha} = \frac{2n}{\alpha} - \frac{n(\bar{k}+2)}{(1+\alpha)} = 0$$

$$2(1+\alpha)\text{-}(\bar{k}+2)\alpha = 0$$

$$2+2\alpha = \bar{k}\alpha + 2\alpha$$

$$\hat{\alpha} = \frac{2}{\bar{k}}$$

3.2.5.4 Estimation of Poisson Lindley parameter

Using method of moments

$$\frac{2+\theta}{\theta(1+\theta)} = \bar{k}$$

$$\bar{k}(\theta+\theta^2) = 2+\theta$$

$$\bar{k}(\theta+\theta^2) - 2 - \theta = 0$$

$$\bar{k}\theta^2 + (\bar{k}-1)\theta - 2 = 0$$

$$\hat{\theta} = \frac{-(\bar{k}-1) + \sqrt{(\bar{k}-1)^2 + 8\bar{K}}}{2\bar{K}}$$

For more on the moments of Poisson Lindley See Shanker and Fesshaye (2015)

Maximum likelihood estimate

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^2 (k_i+2+\theta)}{(1+\theta)^{k+3}}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln\left[\frac{\theta^2 (k_i+2+\theta)}{(1+\theta)^{k_i+3}}\right]$$

$$= 2n \ln\theta + \sum_{i=1}^{n} \ln(k_i + 2 + \theta) - \sum_{i=1}^{n} (k+3)\ln(1+\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{2n}{\theta} + \sum_{i=1}^{n} \frac{1}{(k_i+2+\theta)} - \frac{n(\bar{k}+3)}{(1+\theta)} = 0$$

$$\frac{2n}{\theta} + \sum_{i=1}^{n} \frac{1}{(k_i+2+\theta)} - \frac{n(\bar{k}+3)}{(1+\theta)} = 0 \qquad (5)$$

Equation five is nonlinear and can be solved by numerical method such as Newton's method assuming the method of moment estimate as the initial estimate.

3.3 Severity component

In an insurance portfolio, in addition to many small claim severities, high claim severities can also be observed. Therefore, long tail distributions such as Lognormal, Weibull, Pareto, Burr, etc. are widely used to model claim severity data.

In this study we use the Exponential Inverse Gamma mixture (Pareto distribution) to model the frequency component.

Let X be the size of claim each insured and Y be the mean claim size of each insured.

We assume that the conditional distribution of the claim size X given the mean claim size Y is exponential distribution with parameter $\frac{1}{y}$

Therefore:

$$f(x/y) = \frac{1}{y} e^{-\frac{x}{y}} \qquad x>0\ ,y>0$$

The mean of the exponential is E(X/Y)=y and the variance is Var(X/Y)=y^2

The mean claim size is different for different policyholders and takes different values therefore it's reasonable to express y as a distribution. Let the prior distribution for y be Inverse Gamma with parameters s and m and probability density function

$$g(y) = \frac{\frac{1}{m} e^{-\frac{m}{y}}}{\left(\frac{y}{m}\right)^{s+1} \Gamma s}$$

The expected value of y will be $E(X) = \frac{m}{s-1}$

The unconditional distribution of the claim size x can be obtained as follows

$$p(X=x)=\int_0^\infty f(x/y).g(y)dy$$

$$=\int_0^\infty \frac{\frac{1}{y}e^{-\frac{x}{y}}\frac{1}{m}e^{-\frac{m}{y}}}{\left(\frac{y}{m}\right)^{s+1}\Gamma s}dy$$

$$=\int_0^\infty \frac{e^{-\frac{1}{y}(x+m)}}{y^{s+2}\left(\frac{1}{m}\right)^{s}\Gamma s}$$

$$=\int_0^\infty \frac{e^{-\frac{1}{y}(x+m)}\frac{1}{x+m}}{y^{s+2}\left(\frac{1}{x+m}\right)^{s+2}\Gamma(s+1)}m^s s(x+m)^{-s-1}dy$$

$$=m^s s(x+m)^{-s-1}\int_0^\infty \frac{e^{-\frac{1}{y}(x+m)}\frac{1}{x+m}}{\left(\frac{y}{x+m}\right)^{s+2}\Gamma(s+1)}dy$$

$$=m^s s(x+m)^{-s-1}$$

which is the pdf of Pareto distribution with parameters s and m.

Mean of the Pareto.

$$E(X)=\int_0^\infty xsm^s\left(x+m\right)^{-s-1}dx$$

$$=\text{sm}^s\int_0 x(x+m)^{-s-1}dx$$

using integration by parts

$$\text{E(X)}=\text{sm}^s\left\{\left[\frac{-x}{s}(x+m)^{-s}\right]_0^\infty+\frac{1}{s}\int_0^\infty(x+m)^{-s}dx\right\}$$

$$=\text{sm}^s\left[\frac{1}{s}\frac{(x+m)^{-s+1}}{-(s-1)}\right]_0^\infty$$

$$=\frac{m}{s-1}$$

Variance of Pareto:

$$\text{var}(X) = E(x^2) - \left(E(x)\right)^2$$

$$E(X^2) = \int_0^\infty x^2 sm^s (x+m)^{-s-1} dx$$

$$= sm^s \int_0^\infty x^2 (x+m)^{-s-1} dx$$

using integration by parts

$$E(X^2) = sm^s \left\{ \left[\frac{x^2 (x+m)^{-s}}{-s} \right]_0^\infty - 2\int_0^\infty \frac{x(x+m)^{-s}}{-s} dx \right\}$$

$$= 2m^s \int_0^\infty \frac{x(x+m)^{-s}}{-s} dx$$

$$= 2m^s \left\{ \left[\frac{-x(x+m)^{-s+1}}{s-1} \right]_0^\infty + \frac{1}{s-1} \int_0^\infty (x+m)^{-s+1} dx \right\}$$

$$= \frac{2m^s}{(s-1)} \left[\frac{(x+m)^{-s+1}}{s-1} \right]_0^\infty$$

$$= \frac{2m^s m^{-s+2}}{(s-1)(s-2)}$$

$$= \frac{2m^2}{(s-1)(s-2)}$$

$$\text{var}(X) = E(x^2) - \left[E(x)\right]^2$$

$$= \frac{2m^2}{(s-1)(s-2)} - \left[\frac{m}{s-1}\right]^2$$

$$= \frac{2m^2}{(s-1)(s-2)} - \frac{m^2}{(s-1)^2}$$

$$= \frac{m^2 s}{(s-2)(s-1)^2}$$

The relatively tame exponential distribution gets transformed in the heavily-tailed Pareto distribution which is a better candidate to model claim severity.

In order to obtain an optimal BMS that will take into account the size of loss in each claim , we have to find the posterior distribution of the mean claim size y given the information we have about the claim size for each policyholder for the time period he is in the portfolio.

Consider a policyholder who has been in the portfolio for t years.

Let x_k denote the claim amount for the k^{th} claim, where $k = 1, 2, 3, ..., K$

$\sum_{k=1}^{K} x_k$ is the total claim amount for a policyholder who has been in the portfolio for t years.

We obtain the posterior distribution of the claim size Y given the claim size history of the policy holder $x_1, ..., x_k$ using the Bayes theorem

$$g(y / x_1,...x_k) \propto f(x_1,...x_k / y) g(y)$$

$$f(x_1,...x_k / y) = \prod_{k=1}^{K} \tfrac{1}{y} e^{-\frac{x}{y}}$$

$$= \left(\tfrac{1}{y}\right)^K e^{-\frac{1}{y}\sum_{k=1}^{K} x_k}$$

$$g(y / x_1,...x_k) \propto \left(\tfrac{1}{y}\right)^K e^{-\frac{1}{y}\sum_{k=1}^{K} x_k} \frac{e^{-\frac{m}{y}}}{(y)^{s+1}}$$

$$\propto \left(\tfrac{1}{y}\right)^{K+s+1} e^{-\frac{1}{y}\left(m+\sum_{k=1}^{K} x_k\right)}$$

$$\int_0^{\infty} g(y / x_1,...x_k) dy \propto \int_0^{\infty} \left(\tfrac{1}{y}\right)^{K+s+1} e^{-\frac{1}{y}\left(m+\sum_{k=1}^{K} x_k\right)} dy$$

$$1 = \int_0^{\infty} A \left(\tfrac{1}{y}\right)^{K+s+1} e^{-\frac{1}{y}\left(m+\sum_{k=1}^{K} x_k\right)} dy$$

where $A = \dfrac{\left(m + \sum_{k=1}^{K} x_k\right)^{k+s}}{\Gamma(K+s)}$

Therefore:

$$g(y/x_1,...x_k)dy = \frac{\frac{1}{\left(m+\sum_{k=1}^{K} x_k\right)} e^{-\frac{1}{y}\left(m+\sum_{k=1}^{K} x_k\right)}}{\left(\frac{y}{m+\sum_{k=1}^{K} x_k}\right)^{k+s+1} \Gamma(K+s)}$$

Which is the pdf of an inverse Gamma $\left(s+K, m+\sum_{k=1}^{K} x_k\right)$

The optimal choice of y_{t+1} for a policyholder reporting claim amounts $x_k,\ k=1,2,3,...,K$ over t years is estimated as:

$$\hat{y}_{t+1}(x_1,...,x_k) = \int_0^{\infty} y \frac{\frac{1}{\left(m+\sum_{k=1}^{K} x_k\right)} e^{-\frac{1}{y}\left(m+\sum_{k=1}^{K} x_k\right)}}{\left(\frac{y}{m+\sum_{k=1}^{K} x_k}\right)^{k+s+1} \Gamma(K+s)} dy$$

$$= \frac{m+\sum_{k=1}^{K} x_k}{s+K-1} \int_0^{\infty} \frac{\frac{1}{\left(m+\sum_{k=1}^{K} x_k\right)} e^{-\frac{1}{y}\left(m+\sum_{k=1}^{K} x_k\right)}}{\left(\frac{y}{m+\sum_{k=1}^{K} x_k}\right)^{k+s} \Gamma(K+s)} dy$$

$$\hat{y}_{t+1}(x_1,...,x_k) = \frac{m+\sum_{k=1}^{K} x_k}{s+K-1}$$

3.3.1 Estimation of Pareto distribution parameters

We consider Hogg and Klugman (1984) to estimate the Pareto parameters.

Using method of moments

$$E(X) = \frac{m}{s-1}$$

$$\bar{x} = \frac{m}{s-1}$$

$$m = \bar{x}(s-1)$$

$$Var(X) = S^2 = \frac{m^2 s}{(s-2)(s-1)^2}$$

$$S^2 = \frac{\bar{x}^2 (s-1)^2 s}{(s-2)(s-1)^2}$$

$$S^2 = \frac{\bar{x}^2 s}{(s-2)}$$

$$\hat{s} = \frac{2S^2}{S^2 - \bar{x}^2}$$

$$\hat{m} = \bar{x}\left(\frac{2S^2}{S^2 - \bar{x}^2} - 1\right)$$

$$\hat{m} = \bar{x}\left(\frac{S^2 + \bar{x}^2}{S^2 - \bar{x}^2}\right)$$

Using the maximum likelihood method

$$L(s,m) = \prod_{i=1}^{n} sm^s (x+m)^{-s-1}$$

$$\ln L(s,m) = \sum_{i=1}^{n} \ln\left[sm^s (x+m)^{-s-1} \right]$$

$$= \sum_{i=1}^{n} \left[\ln s + s \ln m - (s+1)\ln(x_i + m) \right]$$

$$= \text{nlns+nslnm-(s+1)} \sum_{i=1}^{n} \ln(x_i + m)$$

$$\frac{\partial \ln L(s,m)}{\partial s} = \frac{n}{s} + n \ln m - \sum_{i=1}^{n} \ln(x_i + m) = 0 \qquad (4)$$

$$\frac{\partial \ln L(s,m)}{\partial m} = \frac{ns}{m} - (s+1)\sum_{i=1}^{n} \frac{1}{(x_i + m)} = 0 \qquad (5)$$

Clearly equations (4) and (5) are nonlinear functions in the unknowns sand m. we use the Newton's method to find the solution.

Say the preliminary guess is (s_0, m_0) , the linear equations equated to zero are

$$g_1(s_0,m_0)+g_{11}(s_0,m_0)(s-s_0)+g_{12}(s_0,m_0)(m-m_0)=0$$
$$g_2(s_0,m_0)+g_{21}(s_0,m_0)(s-s_0)+g_{22}(s_0,m_0)(m-m_0)=0$$

where:

$$g_1=\frac{\partial \ln L(s,m)}{\partial s}$$

$$g_{11}=\frac{\partial g_1}{\partial s}$$

$$g_{12}=\frac{\partial g_1}{\partial m}$$

$$g_2=\frac{\partial \ln L(s,m)}{\partial m}$$

$$g_{21}=\frac{\partial g_2}{\partial s}$$

$$g_{22}=\frac{\partial g_2}{\partial m}$$

We take s_0 and m_0 to be the initial estimates using the method of moments.

3.4 Calculation of premiums

If the risk premium is determined not only by taking the number of claims into account but also the total amount of the claims, then the risk premium to be paid at time t+1 for a policyholder whose claim number history is $k_1,...,k_t$ and whose claim amount history is $x_1,...,x_k$ can be calculated according to the net premium principle as the product of the of the mean of posterior structure function for the frequency component and the mean of the posterior structure function for the severity component. The estimated premium assuming each of the frequency distributions discussed above and assuming the severity component is Pareto would be:

Assuming the frequency component is Negative Binomial distribution.

$$premium = \frac{\alpha + k}{t + \tau} \cdot \frac{m + \sum_{k=1}^{K} x_k}{s + K - 1} \qquad (3.4.1)$$

Assuming the frequency component is Geometric distribution.

$$premium = \frac{k + 1}{t + \theta} \cdot \frac{m + \sum_{k=1}^{K} x_k}{s + K - 1} \qquad (3.4.2)$$

Assuming the frequency component is Poisson Erlang distribution.

$$premium = \frac{(k+1)\left[(k+2) + (t+\theta)\right]}{(t+\theta)\left[(k+1) + (t+\theta)\right]} \cdot \frac{m + \sum_{k=1}^{K} x_k}{s + K - 1} \qquad (3.4.3)$$

Assuming the frequency component is Poisson Lindley distribution.

$$premium = \frac{k + 2}{t + \alpha} \cdot \frac{m + \sum_{k=1}^{K} x_k}{s + K - 1} \qquad (3.4.4)$$

Therefore risk premium that must be paid depends on the parameter of the posterior structure function for the frequency component, the parameters of the posterior structure function for severity component, the number of year's t that the policyholder is under observation, and his/her total number of claims K and the total amount of claims $\sum_{k=1}^{K} x_k$

CHAPTER 4: DATA ANALYSIS.

We consider the data presented by Walhin and Paris (2000).

Table 1: Observed Claim frequency distribution

Number of accidents	Number of policyholders
0	103704
1	14075
2	1766
3	255
4	45
5	6
6	2

The mean and variance of the data is obtained as

$$Mean = E(K) = 0.15514$$
$$Var(K) = S^2 = 0.179314$$

We need to estimate the frequency distributions parameters using the data. This is summarized in the following table

Table 2: Frequency distribution parameters

Distribution	**Parameter**	**Estimated value**
Negative binomial	α	0.9956
	τ	6.4176
Geometric	θ	6.4458
Poisson Erlang	α	12.8916
Poisson Lindley	θ	7.2291

4.1 Estimation of premium based on the frequency component

First we estimate the premium charged to a policyholder based on the frequency component only as in Lemaire (1995)

We consider the various frequency distribution independently:

4.1.1 Negative binomial distribution

We apply the Negative Binomial parameter estimates into equation 3.2.1.1 and obtain the optimal BMS as presented in table 3. This optimal BMS can be considered generous with good drivers and strict with bad drivers. For example, for a policyholder with a no claim in the first year is awarded a bonus of 13.49% on the basic premium charged while as for a driver with one claim in the first year has to pay a malus of 73.04% of the basic premium.

Table 3: Optimal BMS based on Negative binomial

	Number of claims					
Year	0	1	2	3	4	5
0	100					
1	86.51855	173.0371	259.5557	346.0742	432.5928	519.1113
2	76.24026	152.4805	228.7208	304.961	381.2013	457.4416
3	68.14475	136.2895	204.4343	272.579	340.7238	408.8685
4	61.60344	123.2069	184.8103	246.4138	308.0172	369.6206
5	56.20796	112.4159	168.6239	224.8318	281.0398	337.2478
6	51.68148	103.363	155.0445	206.7259	258.4074	310.0889
7	47.82972	95.65943	143.4891	191.3189	239.1486	286.9783

4.1.2 Geometric distribution

We use equation 3.2.2.1 to estimate the optimal BMS as presented in table 4. We observe that a policyholder with a first claim free year enjoys a bonus of 13.43% of the basic premium while for a policyholder with one claim in the first year is penalized 73.14% of the basic premium.

Table 4: Optimal BMS based on Geometric distribution.

	Number of claims					
Year t	0	1	2	3	4	5
0	100					
1	86.56961	173.1392	259.7088	346.2784	432.848	519.4177
2	76.31959	152.6392	228.9588	305.2784	381.598	457.9175
3	68.23985	136.4797	204.7196	272.9594	341.1993	409.4391
4	61.7071	123.4142	185.1213	246.8284	308.5355	370.2426
5	56.31585	112.6317	168.9476	225.2634	281.5793	337.8951
6	51.79097	103.5819	155.3729	207.1639	258.9548	310.7458
7	47.93913	95.87827	143.8174	191.7565	239.6957	287.6348

4.1.3 Poisson Erlang distribution

We estimate the optimal BMS using equation 3.2.3.1 as presented in table 5. Here we find that the system is not generous to good drivers as compared to the negative Binomial and the Geometric distributions since the bonus awarded for the first free claim year is 7.20% of the basic premium. However it's lenient with bad drivers by imposing a malus of 39.20% of the basic premium.

Table 5: Optimal BMS based on Poisson Erlang distribution.

Year	Number of claims					
T	0	1	2	3	4	5
0	100					
1	92.80141	139.2021	185.6028	232.0035	278.4042	324.8049
2	86.56961	129.8544	173.1392	216.424	259.7088	302.9936
3	81.1221	121.6832	162.2442	202.8053	243.3663	283.9274
4	76.31959	114.4794	152.6392	190.799	228.9588	267.1186
5	72.05392	108.0809	144.1078	180.1348	216.1618	252.1887
6	68.23985	102.3598	136.4797	170.5996	204.7196	238.8395
7	64.80927	97.2139	129.6185	162.0232	194.4278	226.8324

4.1.4 Poisson Lindley distribution

We estimate the optimal BMS using equation 3.2.4.1 as presented in table 6.we observe that this system is generous with good policyholders but also lenient with bad policyholders. For example a policyholder with first free claim year is given a bonus of 13.18% of the basic bonus and for a policyholder with one claim in the first year a malus of 8.96% of basic premium.

Table 6: Optimal BMS based on Poisson Lindley distribution.

Year	Number of claims					
T	0	1	2	3	4	5
0	100					
1	86.81664	108.9594	141.1096	180.8121	226.3543	276.5049
2	76.66999	94.72119	121.2425	154.3132	192.5523	234.9418
3	68.6261	83.62569	105.8843	133.8713	166.4585	202.7953
4	62.09664	74.75919	93.71028	117.7111	145.8283	177.3449
5	56.69305	67.52592	83.85824	104.6734	129.1909	156.8014
6	52.14873	61.52217	75.74505	93.97296	115.5468	139.9445
7	48.27474	56.46534	68.96371	85.06091	104.1955	125.917

From the above analysis we see that use Geometric distribution, as the claim frequency distribution is the strictest to bad drivers with a malus of 73.14% and the most lenient is the Poisson Lindley with a malus of 8.96% for a policy holder with one claim in the first year.

The most generous frequency distribution is the Negative binomial with a bonus of 13.48% for a policy holder with first free claim year.

4.2 Estimation of premium based on both claim frequency and Claim severity.

We consider the data presented by Walhin and Paris (2000) but we add some values on the right so that the Pareto distribution fits well.

Table 7: Observed Claim severity distribution (“000”)

6	6	10	11	17	18	20	26	27	34
42	44	47	54	59	60	61	61	61	61
64	64	65	66	67	68	71	71	73	75
76	81	85	87	93	94	101	103	105	109
110	110	113	116	116	129	134	134	141	141
151	154	156	159	167	171	172	173	174	179
181	183	185	187	195	195	203	226	235	240
251	255	273	340	361	429	465	531	646	923
1,043	1,226	1,398	1,423	1,569	1,702	1,929	2,081	2,265	2,545

Mean claim amount= 321422.22

Variance on claim amount= 2.85637E+11

We estimate the parameters s and m for the Pareto distribution using the maximum likelihood method to be:

$\hat{s} = 3.13327749$

$\hat{m} = 685,682.79$

Here we will illustrate only two cases that the aggregate claim amount of a policyholder is equal to Ksh 250,000, and Ksh1, 000,000. However we can use the net premium formula with any value that the aggregate claim amount can take. We use these values of the aggregate claim amount for illustration on how the model works. The premiums are not divided with the premium when t = 0, as it will be interesting to see the variation of the premiums paid for various number of claims and claim sizes in comparison not with the premium paid when t = 0 but with the specific claim sizes. This is the basic advantage of this BMS in comparison with the one that takes under consideration only the frequency component, the differentiation according the severity of the claim.

For us to compute the premiums we need to have information on:

a. Number of years the policy has been in existence, t
b. Total number of claims K that the policyholder has made in the t years
c. Aggregate claim amount $\sum_{k=1}^{K} x_k$

We will consider calculation of premium assuming each of the frequency distributions discussed above independently.

4.2.1 Negative binomial distribution.

We use equation 3.3.1 to estimate the optimal BMS. In table 8 we use an aggregate amount of 250,000 and in table 9 we use aggregate amount of 1,000,000.

We illustrate using an example on how the BMS work. Consider a policyholder with a claim amount of 250,000 in the first year of observation he will pay a premium of 81,406.08 (see table 8). If in the second year of observation he makes a claim of 750,000, then his aggregate amount for the two years he has been under observation will be 1,000,000 and thus the premium charged will be 126,179.14 (see table 9).

Table 8: Negative Binomial Optimal BMS based on frequency and severity component (Aggregate claim amount 250,000)

	Number of claims					
Year	0	1	2	3	4	5
0	48,176.26					
1	41,681.41	81,406.08	108,002.73	129,062.97	146,152.54	160,297.64
2	36,729.71	71,735.14	95,172.15	113,730.46	128,789.81	141,254.49
3	32,829.60	64,118.01	85,066.37	101,654.08	115,114.37	126,255.50
4	29,678.24	57,963.23	76,900.73	91,896.17	104,064.38	114,136.06
5	27,078.90	52,886.57	70,165.45	83,847.53	94,950.00	104,139.55
6	24,898.21	48,627.57	64,514.97	77,095.21	87,303.59	95,753.11
7	23,042.57	45,003.41	59,706.73	71,349.39	80,796.95	88,616.73

Table 9: Negative Binomial Optimal BMS based on frequency and severity component (Aggregate claim amount 1,000,000)

	Number of claims					
Year	0	1	2	3	4	5
0	48,176.26					
1	41,681.41	107,928.10	143,189.91	171,111.56	193,768.89	212,522.45
2	36,729.71	95,106.38	126,179.14	150,783.73	170,749.40	187,275.06
3	32,829.60	85,007.59	112,780.91	134,772.88	152,618.52	167,389.41
4	29,678.24	76,847.59	101,954.91	121,835.84	137,968.45	151,321.47
5	27,078.90	70,116.96	93,025.29	111,164.96	125,884.61	138,068.11
6	24,898.21	64,470.38	85,533.88	102,212.75	115,747.01	126,949.37
7	23,042.57	59,665.47	79,159.13	94,594.94	107,120.51	117,487.96

4.2.2 Geometric distribution.

We apply equation 3.4.2 in estimating the optimal BMS. Table 10 gives the estimated premiums for an aggregate claim amount of 250,000 while table 11 give the estimated premium for an aggregate amount of 1,000,000. For example for a policyholder who have been under observation for the last three years and has made two claims whose aggregate claim amount of 250,000 will have to pay a premium of 84,936.99. If in the fourth year he makes two claims whose aggregate amount is 750,000 then the estimated premium would be 103,874.85 (see table 11).

Table 10: Geometric distribution Optimal BMS based on frequency and severity component (Aggregate claim amount 250,000)

	Number of claims					
year	0	1	2	3	4	5
0	48,177.48					
1	41,707.05	81,276.57	107,751.72	128,715.75	145,727.25	159,807.73
2	36,768.85	71,653.26	94,993.70	113,475.54	128,472.84	140,886.16
3	32,876.24	64,067.53	84,936.99	101,462.21	114,871.79	125,970.95
4	29,728.92	57,934.20	76,805.78	91,749.00	103,874.85	113,911.46
5	27,131.56	52,872.59	70,095.39	83,733.05	94,799.49	103,959.21
6	24,951.58	48,624.36	64,463.33	77,005.23	87,182.50	95,606.26
7	23,095.86	45,008.04	59,669.02	71,278.15	80,698.50	88,495.77

Table 11: Geometric distribution Optimal BMS based on the frequency and severity component (Aggregate claim amount 1,000,000)

	Number of claims					
year	0	1	2	3	4	5
0	48,177.48					
1	41,707.05	107,756.40	142,857.12	128,715.75	145,727.25	159,807.73
2	36,768.85	94,997.82	125,942.55	113,475.54	128,472.84	140,886.16
3	32,876.24	84,940.67	112,609.37	101,462.21	114,871.79	125,970.95
4	29,728.92	76,809.11	101,829.02	91,749.00	103,874.85	113,911.46
5	27,131.56	70,098.43	92,932.39	83,733.05	94,799.49	103,959.21
6	24,951.58	64,466.13	85,465.42	77,005.23	87,182.50	95,606.26
7	23,095.86	59,671.61	79,109.13	71,278.15	80,698.50	88,495.77

4.2.3 Poisson Erlang distribution

Here we apply equation 3.4.3 to estimate the premium for a policyholder with aggregate claim size of 250,000 (table 12) and aggregate claim of 1,000,000 (table 13)

Table 12: Poisson Erlang Optimal BMS based on the frequency and severity component (Aggregate claim size 250,000)

	Number of claims					
Year	0	1	2	3	4	5
0	48,177.48					
1	44,709.38	65,345.51	77,005.55	86,238.42	93,730.54	99,931.79
2	41,707.05	60,957.43	71,834.48	80,447.34	87,436.35	93,221.17
3	39,082.58	57,121.60	67,314.20	75,385.09	81,934.30	87,355.11
4	36,768.85	53,739.94	63,329.13	70,922.21	77,083.71	82,183.60
5	34,713.76	50,736.30	59,789.53	66,958.22	72,775.33	77,590.18
6	32,876.24	48,050.65	56,624.66	63,413.88	68,923.07	73,483.05
7	31,223.47	45,635.02	53,777.99	60,225.91	65,458.14	69,788.88

Table 13: Poisson Erlang Optimal BMS based on frequency and severity component. (Aggregate claim amount of 1,000,000)

	Number of claims					
Year	0	1	2	3	4	5
0	48,177.48					
1	44,709.38	86,635.01	102,093.89	114,334.81	124,267.85	132,489.46
2	41,707.05	80,817.30	95,238.08	106,657.01	115,923.02	123,592.54
3	39,082.58	75,731.76	89,245.10	99,945.47	108,628.41	115,815.31
4	36,768.85	71,248.36	83,961.70	94,028.60	102,197.50	108,958.93
5	34,713.76	67,266.14	79,268.90	88,773.14	96,485.46	102,868.98
6	32,876.24	63,705.50	75,072.91	84,074.06	91,378.14	97,423.76
7	31,223.47	60,502.87	71,298.81	79,847.45	86,784.34	92,526.02

4.2.4 Poisson Lindley distribution

We apply equation 3.4.4 to estimate premiums for a policyholder with aggregate claim of 250,000 and 1,000,000. This is presented in table 14 and table 15 respectively.

Table 14: Poisson Lindley Optimal BMS based on frequency and severity component (Aggregate claim amount of 250,000)

	Number of claims					
year	0	1	2	3	4	5
0	48,262.18					
1	41,899.60	80,856.02	106,344.06	126,184.78	142,045.46	155,000.84
2	37,002.61	71,523.27	94,187.33	111,869.39	126,028.87	137,610.14
3	33,120.45	64,099.64	84,493.97	100,434.49	113,217.97	123,685.93
4	29,969.19	58,057.64	76,589.26	91,095.88	102,743.84	112,291.60
5	27,361.30	53,046.84	70,023.16	83,329.19	94,024.43	102,798.98
6	25,168.11	48,825.54	64,484.28	76,770.76	86,655.47	94,771.31
7	23,298.44	45,221.81	59,750.46	71,160.63	80,347.57	87,895.61

Table 15: Poisson Lindley Optimal BMS based on frequency and severity component (Aggregate claim size of 1,000,000)

	Number of claims					
Year	0	1	2	3	4	5
0	48,262.18					
1	41,899.60	107,157.59	140,936.60	167,231.29	188,251.28	205,420.89
2	37,002.61	94,789.00	124,825.42	148,259.27	167,024.66	182,373.19
3	33,120.45	84,950.54	111,978.93	133,104.72	150,046.52	163,919.60
4	29,969.19	76,943.15	101,502.90	120,728.37	136,165.27	148,818.81
5	27,361.30	70,302.39	92,800.92	110,435.25	124,609.54	136,238.35
6	25,168.11	64,707.95	85,460.31	101,743.44	114,843.54	125,599.37
7	23,298.44	59,931.96	79,186.62	94,308.40	106,483.74	116,487.07

If we consider both the claim frequency and claim severity in computing premium, it's evident from the above analysis that a policy holder with a larger claim amount will pay higher premium compared to a policy holder with a smaller claim size but with the same number of claims.

The negative binomial is the strictest with a bad policyholder paying the highest premium. It's also the most generous frequency distribution with good policyholders paying the least premium.

CHAPTER 5: SUMMARY, CONCLUSION AND RECOMMENDATIONS.

5.1 Introduction

In this paper we have developed the design of an optimal estimate of premiums paid by an automobile insured by considering the claim frequency and the claim severity. Compare this with BMS based on the frequency component only and make comparison when we use different Poisson mixtures in modeling the frequency component. In this chapter we make a discussion of the findings, summary of the main findings, conclusion giving recommendations and areas of further studies.

5.2 Discussion of findings

The study findings matched what has been studied in the past specifically, Lemaire (1995) pointed out that optimal BMS based on the frequency and severity components was fair to policy holders as compared to BMS based on frequency component only, this was same findings by Mehmet Mert and Yasemin Saykan (2005), Frangos, N. E., and Vrontos, S. D. (2001), and Ade Ibiwoye, I. A. Adeleke & S. A. Aduloju (2011). All this studies suggested optimal BMS using Poisson mixture as the frequency distribution and exponential mixture as the severity distribution. This is because of the thick tails of the mixtures as compared to the conditional distribution. Also the Poisson mixtures were found to have a variance greater than the mean a quality desirable by the insurer as compared to the Poisson whose variance is equal to the mean.

5.3 Summary of the findings.

First we considered the design of an optimal BMS based on the frequency component and fit this using Poisson mixtures. In this case we considered negative binomial (Poisson Gamma), Geometric (Poisson exponential), Poisson Erlang and Poisson Lindely distributions as the claim frequency distribution. We observe that the Geometric is the strictest with bad drivers and Negative Binomial is the most generous with good drivers.

Second we consider design of optimal BMS based on claim frequency and claim severity. We fit claim severity using Pareto (exponential Inverse Gamma). In an application, the risk premium is calculated using the net premium principle as the product of the mean of the posterior structure functions of the frequency and severity components. The results obtained using the claim frequency and by using both the claim frequency and claim severity are compared.

5.4 Conclusion

From the findings of the study, it is concluded that it is fairer to charge policyholders premiums which not only take into account the number of claims, but also the aggregate amount of the claims the years he/she have been under observation.

The study also concludes that different frequency and severity distributions gives different level of strictness by the insurer.

5.5 Recommendations

5.5.1 Policy

The study recommends the following:

Premium charged to policyholders should be based both on the frequency and severity components as this creates fairness to all policyholders.

Insurers should choose the frequency and the severity components distributions that yields an optimal BMS as defined by Frangos, N. E., and Vrontos, S. D. (2001). The choice of distributions should also ensure that the insurer remains competitive in the market.

5.5.2 Research.

The study only investigated the effect of different frequency distributions on the level of severity in design of optimal BMS. The study recommends similar studies on the severity distribution.

The study further recommends an investigation of a link between the Poisson and exponential mixtures. This will enable a simplified and extensive analysis on the effect of different claim frequency and claim severity distributions on the design of optimal BMS.

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