UNIVERSITY OF NAIROBI

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SCHOOL OF MATHEMATICS

lAPLACE TRANSFORM IN PROBABILITY DISTRIBUTIONS AND IN PURE BIRTH PROCESSES

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Declaration

Declaration by the Candidate

I the undersigned declare that this dissertation is my original work and to the best of my knowledge has not been presented for the award of a degree in any other university.

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Declaration by the Supervisors

This dissertation has been submitted for examination with my approval as the university supervisor.

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Dedication

This dissertation is dedicated to my loving life partner Ann Mwikali Muli who has taught me that nothing good comes without hard work and perseverance, my father Morris Zakayo, who taught me that the best kind of knowledge to have is that which is learned for its own sake, my mother Agatha Kimari, who taught me that even the largest task can be accomplished if it is done one step at a time, my uncle Duncan Mwashagha Kimari and my aunt Janet Mwadiga.It is also dedicated to my brothers Daniel Mbuva, Duncan Mbuva and sister Mary Mwikali, my late father in law Geoffrey Muli, mum Catherine Mumbua for giving me the best gift ever, siblings Grace Mwongeli, Daniel Mwendwa, Jonathan Musau and the rest of the family. To my late father in law Geoffrey Ndambuki Muli, we miss u a lot!

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Abstract

Transforms such as generating functions, Laplace transform, Mellin transform and Fourier transforms are very useful tools in probability distributions and stochastic processes. The objective of this work is to use Laplace transform in constructing continuous probability distributions and/or obtaining their properties. These distributions and their properties have been expressed explicitly in some cases and in terms of modified Bessel functions of third kind in other cases. Distributions based on sum of independent random variables have been constructed. Mixed probability distributions, in particular Poisson and exponential mixtures have been studied.

Probability distributions emerging from birth processes have also been obtained. The pure birth processes considered are Poisson, simple birth, simple birth with immigration and Polya processes. Laplace transform has been applied in solving the basic difference differential equations for each of the special cases.

Two approaches were considered, first the Laplace transform was applied to the basic difference differential equations directly, this yielded a general expression for $P_n(t)$ after which both the complex inversion formula and the Partial fractions method were used in determining the inverse Laplace transform to obtain the underlying distributions.

Secondly the Probability generating function technique was used and then Laplace transform was applied to the resulting ODE/PDE . In the cases of simple birth, simple birth with immigration and Polya processes two techniques were used to solve the ODE obtained, one was using the Dirac delta function whereas the second technique involved the use of the Gauss hyper geometric function. The different approaches yielded the same results. For the case of Poisson process a Poisson distribution with parameter λt was obtained.

In simple birth process the negative binomial distribution with parameters $r = n_0$ and $p = e^{-\lambda t}$ was obtained when the initial population $X(0)$ is n_0.When the initial population $X(0)$ is 1 a shifted geometric distribution with parameters $p = e^{-\lambda t}$ was obtained.

In the case of simple birth with immigration where v is the immigration rate the negative binomial distribution with parameters $r = n_0 + \frac{v}{\lambda}$ and $p = e^{-\lambda t}$ was obtained when the initial population $X(0)$ is n_0 whereas for the case where the initial population is 1 the negative binomial distribution with parameters $r = 1 + \frac{v}{\lambda}$ and $p = e^{-\lambda t}$ was obtained.

In the case of the Polya process both methods yielded a negative binomial distribution although the parameters differed.When the initial population was considered to be n_o, applying Laplace transform to the finite difference differential equations yielded a negative binomial distribution with parameters $r = n_0 + \frac{1}{a}$ and $p = e^{-\frac{\lambda at}{1+\lambda at}}$ whereas applying Laplace transform to the differential equations based on probability generating function the parameters were obtained as $r = n_0 + \frac{1}{a}$ and $p = \frac{\lambda at}{1+\lambda at}$. When the initial population was assumed to be 1, applying Laplace transform to the finite difference differential equations yielded a negative binomial distribution with parameters $r = 1 + \frac{1}{a}$ and $p = e^{-\frac{\lambda at}{1+\lambda at}}$ whereas applying Laplace transform to the differential equations based on probability generating function the parameters were obtained as $r = 1 + \frac{1}{a}$ and $p = \frac{\lambda at}{1+\lambda at}$.

Abbreviations

ODE: Ordinary Differential Equation

PDE: Partial Differential Equation

I.F: Integrating Factor

PDF: Probability Density Function

PMF: Probability Mass Function

PGF: Probability Generating Function

LT: Laplace Transform

ILT: Inverse Laplace Transform

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Chapter 1

Introduction

1.1 Background Information

Integral transforms, mainly originated in the work of Heaviside (Electro-magnetic theory). They form a pair of operational calculus and are effectively applicable to the boundary value problems. The Laplace transform is one of the integral transforms that is widely used in many fields including Mathematics, Physics and Engineering. It was named after a French mathematician and astronomer Pierre-Simon Laplace who lived between 1749 and 1827.

Similar Integral transforms had been used earlier; in particular Leonhard Euler in around 1744 investigated similar transforms. While working on his work on integrating probability density functions, Joseph Louis Lagrange being an admirer of Euler investigated other forms of integrals which some of the current historians have captured in the current Laplace transform theorem. It is these types of integrals that attracted the attention of Pierre Simon Laplace in 1782 who was working in the spirit of Euler. Laplace used integrals as solutions of equations. Three years later in 1785 He took a critical step where instead of concentrating on solutions in the forms of integrals. He began to examine their applications. He specifically used the z transform.

Laplace realized that Joseph Fourier's way of solving the diffusion equation using of Fourier series worked only to a constrained region of space since the solutions obtained were fluctuating.

In the year 1809, Laplace was motivated to use his transform to find solutions which diffused indefinitely in the space. This coupled with his earlier work on Probability theory that he began in 1770 led to the development of what we now call Laplace transform.

The Laplace transform operates on a function of time and yields another function with complex frequency. It is closely related to the Fourier transform and is used to solve both ordinary and partial differential equations. It does so by transforming the differential equations into simple algebraic equations. These types of differential equations are also found in stochastic processes. Laplace transforms of probability density functions are called expectations of e^{-sx}. They are useful in determining moments of a distribution. It has also been very applicable in the analysis and the design for systems that are linear and time-invariant.

In the early 19th century, transform techniques were used in signal processing at Bell Labs for signal filtering and telephone long-lines communication by H. Bode and other researchers. Transform theory also provided the foundation of the theory of Classical Control Theory as practiced during the World when State Variable techniques began to be used for controls design. The present-day rampant use of the Laplace transform emerged immediately after the Second World War even though Heaviside and Bromwich among others had been used before.

1.2 Problem Statement

The Laplace transform of many distributions cannot be expressed explicitly and thus there is need to use special functions. In pure birth process a lot of concentration has been on the linear differential equations based on probability generating function; it is therefore necessary to consider alternative approaches such as the Laplace transform method. Also, most books have not considered all the special cases of the pure birth process and even for those special cases considered, the books provide only sketchy details.

Another challenge is that many books do not give the inversion of functions but rather rely on the transform pair tables which do not always cater for the type of inversions one may need. For the Laplace transform of distributions, the books only provide results but they do not systematically prove them. It is thus necessary to consolidate all this together.

Some of the undergraduate and graduate degree courses contain a lecture course on stochastic processes, an integral part of which is continuous time discrete state Markov processes which include the pure birth process. Regrettably the short span of time available for these courses doesnt allow extensive coverage but rather permits such processes to be solved only by the usual approach of Lagrange partial-differential equation by the use of auxiliary equations. On the other hand books that have considered alternative approaches provide solutions with scanty explanation in scattered sections. This necessitates the need for detailed alternative approaches.

1.3 Objectives

Main objective

The overarching goal of this dissertation is to apply Laplace transform to various probability distributions and pure birth process.

Specific objectives

1. To derive the Laplace transforms of various probability distributions explicitly and in terms of special functions; in particular in terms of modified Bessel function of the third kind

2. To apply Laplace transform to the finite difference differential equations in all the special cases of pure birth process namely Poisson, simple birth, simple birth with immigration and Polya processes.

3. To apply Laplace transform to the differential equations based on probability generating function specifically for the four cases of pure birth process

1.4 Literature Review

Sarguta (2012) used Laplace transform to construct mixed Poisson distributions. Wakoli (2015) used Laplace transform to construct hazard functions of exponential mixtures. He gave the link between hazard functions of exponential mixtures with mixed Poisson distributions through Laplace transform. He also used the Laplace transform in constructing sums of continuous independent random variables.

Laplace transform has also been used in obtaining the properties of normal mixtures. Barndorff-Nielsen (1982) used the Laplace transform when the mixing distribution is inverse Gaussian and General Inverse Gaussian. . Madan (1990) also used Laplace transform in normal mixtures when the mixing distribution is gamma.

Hougaard (1986) came up with Laplace transforms of sums of independent random variables where the are gamma. Onchere (2013) also used Laplace transform for frailty models. Ogal (2012) used Laplace transform in binomial mixtures. Irungu (2013) also used Laplace transforms in studying negative binomial mixtures.

Morgan (1979) used Laplace transform to study linear birth and death process. Feller (1971) applied Laplace transform to simple birth process

Chapter 2

Definition, Examples and Properties

2.1 Introduction

This chapter covers the definition of Laplace transform, its existence and properties. It will also focus on various methods of finding Laplace transforms with numerous examples some of which will be generalized into a table of transform pairs.

Definition

A function $K(s,t)$ of two variables or parameters s and t such that s (real or complex) is independent of t, and that the integral

$$\int_a^b K(s,t)\, f(t)\, dt$$

is convergent is known as the kernel of the transformation $\overline{f}(s)$ or $F(s)$ or

$$T\{f(t)\} = \int_a^b K(s,t)\, f(t)\, dt$$

We call $\overline{f}(s)$ the integral transform of the function $f(t)$. Choosing different kernels and different values of a and b, we get different integral transforms. They include the transforms of Laplace, Fourier, Hankel and Mellin.

The Laplace-transform is derived by letting $a = 0$ and $b = \infty$ and choosing the kernel of transformation as

$$K(s,t) = \begin{cases} e^{-st} & t \geqslant 0, \\ 0 & t < 0 \end{cases}$$

Thus it is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Where $f(t)$ is a function of t defined for $t \geqslant 0$ and

$$\int_0^\infty e^{-st} f(t)\, dt$$

converges for some values of s

In probability theory, the Laplace transform is derived as an expected value. The assumption made is that X is a random variable with a given pdf or pmf. The Laplace transform of $f(x)$ is then defined by

$$L\{f(x)\} = E\left[e^{-sX}\right]$$

If we replace s by $-t$, then we obtain the moment generating function of X. Replacing s by $-it$ yields the characteristic function.

A function $f(t)$ is said to be sectionally continuous or piecewise continuous in an interval $a \leqslant t \leqslant b$ if the interval can be subdivided into a finite number of subintervals in each of which the function is continuous and has finite right and left hand limits.

If real constants $M > 0$ and exist such that for all $t > N$

$$|e^{-\gamma t} f(t)| < M \text{or} |f(t)| < M e^{\gamma t}$$

Then we say that $f(t)$ is a function of exponential order γ as $t \to \infty$ or briefly is of exponential order. Intuitively, functions of exponential order cannot grow in absolute value more rapidly than $Me^{\gamma t}$ as t increases. In practice however, this is no restriction since M and γ can be as large as desired.

Sufficient Conditions For Existence

Let $f(t)$ defined for $t \to 0$ be a real valued function with the following properties

1. $f(t)$ is piecewise continuous in every finite closed interval $0 \leqslant t \leqslant b$,$b > 0$

2. $f(t)$ is of exponential order there exists an $M > 0$ and $t_0 > 0$ such that

$$|f(t)| \leqslant Me^{\gamma t} t > t_0$$

Then the Laplace transform of the function $f(t)$ defined by

$$\begin{aligned} L\{f(t)\} &= \overline{f}(s) \\ &= \int_0^\infty e^{-st} f(t)\, dt \end{aligned}$$

exists for $s > a$.

If the conditions are not satisfied, however the Laplace transform may or may not exist. Thus these conditions are not necessary for the existence of Laplace transforms.

2.2 Properties

Linearity

If C_1 and C_2 are any constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $\overline{f}_1(s)$ and $\overline{f}_2(s)$ respectively, then

$$\begin{aligned} L\{C_1 f_1(t) + C_2 f_2(t)\} &= C_1 L\{f_1(t)\} + C_2 L\{f_2(t)\} \\ &= C_1 \overline{f}_1(s) + C_2 \overline{f}_2(s) \end{aligned}$$

Proof

By definition

$$\begin{aligned} L\{C_1 f_1(t) + C_2 f_2(t)\} &= \int_0^\infty e^{-st}\left[C_1 f_1(t) + C_2 f_2(t)\right] dt \\ &= C_1 \int_0^\infty e^{-st} f_1(t)\, dt + C_2 \int_0^\infty e^{-st} f_2(t)\, dt \end{aligned}$$

$$= C_1 L\left\{f_1(t)\right\} + C_2 L\left\{f_2(t)\right\}$$

$$= C_1 \overline{f}_1(s) + C_2 \overline{f}_2(s)$$

In general

$$L\left\{\sum_{i=1}^{k} C_i f_i(t)\right\} = \int_0^\infty \left[e^{-st}\sum_{i=1}^{k} C_i f_i(t)\right] dt$$

$$= \sum_{i=1}^{k}\left\{\int_0^\infty e^{-st} C_i f_i(t)\, dt\right\}$$

$$= \sum_{i=1}^{k} C_i \left\{\int_0^\infty e^{-st} f_i(t)\, dt\right\}$$

$$= \sum_{i=1}^{k} C_i L\left\{f_i(t)\right\}$$

$$= \sum_{i=1}^{k} C_i \overline{f}_i(s)$$

ii. First translation or shifting property

$$L\left\{e^{at} f(t)\right\} = \overline{f}(s-a)$$

Proof

By definition

$$L\left\{e^{at} f(t)\right\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)} f(t)\, dt$$

$$= \overline{f}(s-a)$$

Similarly

$$L\left\{e^{-at}f\left(t\right)\right\}=\overline{f}\left(s+a\right)$$

iii. Second translation or shifting property

$$L\left\{f\left(t-a\right)\right\}=\begin{cases}e^{-as}\overline{f}\left(s\right) & t\geqslant a,\\ 0 & t<a\end{cases}$$

Proof

By definition

$$L\left\{f\left(t-a\right)\right\}=\int\limits_{0}^{\infty}e^{-st}f\left(t-a\right)dt$$

Put $u=t-a\Rightarrow t=u+a$ and $du=dt$

Therefore

$$L\left\{f\left(t-a\right)\right\}=\int\limits_{0}^{\infty}e^{-s(u+a)}f\left(u\right)du$$

$$=e^{-as}\int\limits_{0}^{\infty}e^{-su}f\left(u\right)du$$

$$=\begin{cases}e^{-as}\overline{f}\left(s\right) & t\geqslant a,\\ 0 & t<a\end{cases}$$

iv. Change of scale property

$$L\left\{f\left(at\right)\right\}=\frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$$

Proof

$$L\left\{f\left(at\right)\right\}=\int\limits_{0}^{\infty}e^{-st}f\left(at\right)dt$$

Let

$$u=at\Rightarrow t=\frac{u}{a}$$

and

$$dt = \frac{du}{a}$$

We thus have

$$L\left\{f\left(at\right)\right\} = \int\limits_{0}^{\infty} e^{-s\left(\frac{u}{a}\right)} f\left(u\right) \frac{du}{a}$$

$$= \frac{1}{a} \int\limits_{0}^{\infty} e^{-\frac{s}{a}u} f\left(u\right) du$$

$$= \frac{1}{a} \overline{f}\left(\frac{s}{a}\right)$$

v. Laplace transform of derivatives

$$\begin{aligned} L\left\{f^{(n)}\left(t\right)\right\} &= s^n \overline{f}\left(s\right) - s^{n-1} f\left(0\right) - s^{n-2} f'\left(0\right) - ... - s f^{(n-2)}\left(0\right) - f^{(n-1)}\left(0\right) \\ &= s^n \overline{f}\left(s\right) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}\left(0\right) \end{aligned}$$

Proof

a)

$$\begin{aligned} L\left\{f'\left(t\right)\right\} &= \int\limits_{0}^{\infty} e^{-st} \frac{d}{dt} f\left(t\right) dt \\ &= \int\limits_{0}^{\infty} e^{-st} df\left(t\right) \end{aligned}$$

Using Integration by parts, let

$$u = e^{-st} \Rightarrow du = -se^{-st} dt$$

and

$$dv = df(t) \Rightarrow v = f(t)$$

Thus

$$L\left\{f'\left(t\right)\right\} = e^{-st}f(t)\Big|_0^\infty + s\int\limits_0^\infty e^{-st}f\left(t\right)dt$$

$$= -f\left(0\right) + s\overline{f}\left(s\right)$$

$$= s\overline{f}\left(s\right) - f\left(0\right)$$

b)

$$L\left\{f''\left(t\right)\right\} = \int\limits_0^\infty e^{-st}\frac{d^2}{dt^2}f\left(t\right)dt$$

$$= \int\limits_0^\infty e^{-st}\frac{d}{dt}f'\left(t\right)dt$$

$$= \int\limits_0^\infty e^{-st}df'\left(t\right)$$

Using Integration by parts, let

$$u = e^{-st} \Rightarrow du = -se^{-st}$$

and

$$dv = df^{'}(t) \Rightarrow v = f^{'}(t)$$

Therefore

$$L\left\{f''\left(t\right)\right\} = e^{-st}f^{'}(t)\Big|_0^\infty + s\underbrace{\int\limits_0^\infty e^{-st}f'\left(t\right)dt}_{L\{f'(t)\}}$$

$$= -f'\left(0\right) + sL\left\{f'\left(t\right)\right\}$$

$$= -f'\left(0\right) + s\left[s\overline{f}\left(s\right) - f\left(0\right)\right]$$

$$= s^2\overline{f}\left(s\right) - sf\left(0\right) - f'\left(0\right)$$

c)

$$L\left\{f'''(t)\right\} = \int_0^\infty e^{-st} f'''(t)\,dt$$

$$= \int_0^\infty e^{-st} \frac{d}{dt} f''(t)\,dt$$

$$= \int_0^\infty e^{-st} df''(t)\,dt$$

Using Integration by parts, Let

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

and

$$dv = df^{''}(t) \Rightarrow v = f^{''}(t)$$

Therefore

$$L\left\{f'''(t)\right\} = e^{-st} f^{''}(t)\Big|_0^\infty + s \underbrace{\int_0^\infty e^{-st} f''(t)\,dt}_{L\{f''(t)\}}$$

$$= -f''(0) + sL\left\{f''(t)\right\}$$

$$= -f''(0) + s\left[s^2\overline{f}(s) - sf(0) - f'(0)\right]$$

$$= s^3\overline{f}(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$= s^3\overline{f}(s) - \sum_{i=1}^{3} s^{n-i} f^{(i-1)}(0)$$

Using mathematical induction, we assume that

$$L\left\{f^{(n-1)}(t)\right\} = s^{n-1}\overline{f}(s) - \sum_{1=1}^{n-1} s^{n-1-i} f^{(i-1)}(0)$$

But by definition

$$L\left\{f^{(n)}(t)\right\} = \int_0^\infty e^{-st} f^{(n)}(t)\, dt$$

$$= \int_0^\infty e^{-st} \frac{d^n}{dt^n} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \frac{d}{dt} f^{(n-1)}(t)\, dt$$

$$= \int_0^\infty e^{-st} df^{(n-1)}(t)$$

Using Integration by parts, Let

$$u = e^{-st} \Rightarrow du = se^{-st} dt$$

and

$$dv = df^{(n-1)}(t) \Rightarrow v = f^{(n-1)}(t)$$

Thus

$$L\left\{f^{(n)}(t)\right\} = e^{-st} f^{(n-1)}(t)\Big|_0^\infty + s \underbrace{\int_0^\infty e^{-st} f^{(n-1)}(t)\, dt}_{L\left\{f^{(n-1)}(t)\right\}}$$

$$= -f^{(n-1)}(0) + sL\left\{f^{(n-1)}(t)\right\}$$

$$= -f^{(n-1)}(0) + s\left[s^{n-1}\overline{f}(s) - \sum_{1=1}^{n-1} s^{n-1-i} f^{(i-1)}(0)\right]$$

$$= -f^{(n-1)}(0) + s^n\overline{f}(s) - s\sum_{1=1}^{n-1} s^{n-1-i} f^{(i-1)}(0)$$

$$= s^n\overline{f}(s) - \sum_{1=1}^{n-1} s^{n-i} f^{(i-1)}(0) - f^{(n-1)}(0)$$

$$= s^n\overline{f}(s) - \sum_{1=1}^{n} s^{n-i} f^{(i-1)}(0)$$

vi. Laplace transform of Integrals

a)

$$L\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}\overline{f}(s)$$

Proof

Let

$$L\left\{\int_0^t f(x)\,dx\right\} = L\{\phi(t)\}$$

where

$$\phi(t) = \int_0^t f(x)\,dx$$

But by definition

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)\,dt$$

Using integration by parts, We let

$$u = \phi(t) \Rightarrow du = d\phi(t)$$

and

$$dv = e^{-st}dt \Rightarrow v = \frac{e^{-st}}{-s}$$

Therefore

$$L\left\{\phi\left(t\right)\right\} = \left.\frac{\phi\left(t\right)e^{-st}}{-s}\right|_0^\infty + \frac{1}{s}\int\limits_0^\infty e^{-st}d\phi\left(t\right)$$

$$= \left[0 + \phi\left(0\right)\right] + \frac{1}{s}\int\limits_0^\infty e^{-st}d\phi\left(t\right)$$

But

$$\begin{aligned}\phi\left(0\right) &= \int\limits_0^0 f\left(x\right)dx\\ &= 0\end{aligned}$$

Thus

$$L\left\{\phi\left(t\right)\right\} = \frac{1}{s}\int\limits_0^\infty e^{-st}d\phi\left(t\right)$$

But

$$d\phi\left(t\right) = d\int\limits_0^t f\left(x\right)dx$$

By Leibnitz theorem

$$\begin{aligned}d\phi\left(t\right) &= \left[\left.f(x)\right|_0^t\right]dt\\ &= f\left(t\right)dt\end{aligned}$$

Therefore

$$L\left\{\phi\left(t\right)\right\} = \frac{1}{s}\underbrace{\int\limits_0^\infty e^{-st}f\left(t\right)dt}_{L\{f(t)\}}$$

$$= \frac{1}{s}L\left\{f\left(t\right)\right\}$$

$$= \frac{1}{s}\overline{f}\left(s\right)$$

b) For a double integral, We have

$$L\left\{\int_0^t\int_0^u f(x)\,dx du\right\} = \frac{1}{s^2}\overline{f}(s)$$

Proof

$$L\left\{\int_0^t\int_0^u f(x)\,dx du\right\} = L\left\{\int_0^t \phi(u)\,du\right\}$$
$$= L\{\psi(t)\}$$

Where

$$\phi(u) = \int_0^u f(x)\,dx$$

And

$$\psi(t) = \int_0^t \phi(u)\,du$$

But by definition

$$L\{\psi(t)\} = \int_0^\infty e^{-st}\psi(t)\,dt$$

Using integration by parts, We let

$$u = \psi(t) \Rightarrow du = d\psi(t)$$

and

$$dv = e^{-st}dt \Rightarrow v = \frac{e^{-st}}{-s}$$

Therefore

$$L\{\psi(t)\} = \left.\frac{\psi(t)\,e^{-st}}{-s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}d\psi(t)$$

$$= \frac{[0+\psi(0)]}{-s} + \frac{1}{s}\int_0^\infty e^{-st}d\psi(t)$$

But

$$\psi(0) = \int_0^0 \phi(u)\,du$$
$$= 0$$

Thus

$$L\{\psi(t)\} = \frac{1}{s}\int_0^\infty e^{-st}d\psi(t)$$

Also

$$\psi(t) = \int_0^t \phi(u)\,du$$

implies that

$$d\psi(t) = d\int_0^t \phi(u)\,du$$

Which by Leibnitz theorem simplifies to

$$\begin{aligned} d\psi(t) &= d\int_0^t \phi(u)\,du \\ &= [\phi(u)]\,dt \\ &= [\phi(t) - \phi(0)]\,dt \\ &= \phi(t)\,dt \end{aligned}$$

Hence

$$\begin{aligned} L\{\psi(t)\} &= \frac{1}{s}\int_0^\infty e^{-st}d\psi(t) \\ &= \underbrace{\int_0^\infty e^{-st}\phi(t)\,dt}_{L\{\phi(t)\}} \\ &= \frac{1}{s}L\{\phi(t)\} \\ &= \frac{1}{s}\left[\frac{1}{s}\overline{f}(s)\right] \end{aligned}$$

$$= \frac{1}{s^2}\overline{f}(s)$$

vii. Multiplication by powers of t

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{f}(s)$$

Proof

By definition

$$L\{f(t)\} = \overline{f}(s)$$

$$= \int_0^\infty e^{-st} f(t)\, dt$$

Therefore

$$\frac{d}{ds}L\{f(t)\} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt$$

But by Leibnitzs rule for differentiating under the integral sign, we have

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \left[\frac{d}{ds}e^{-st}\right] f(t)\, dt$$

$$= \int_0^\infty -te^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st}\,[-tf(t)]dt$$

$$= L\{-tf(t)\}$$

$$\therefore \frac{d}{ds}\overline{f}(s) = L\{-tf(t)\}$$

Similarly

$$
\begin{aligned}
\frac{d^2}{ds^2}\overline{f}\left(s\right) &= \frac{d}{ds}\left[\frac{d}{ds}\overline{f}\left(s\right)\right] \\
&= \frac{d}{ds}L\left\{-tf\left(t\right)\right\} \\
&= \frac{d}{ds}\int\limits_0^\infty e^{-st}\left[-tf\left(t\right)\right]dt \\
&= \int\limits_0^\infty \left[\frac{d}{ds}e^{-st}\right]\left[-tf\left(t\right)\right]dt \\
&= \int\limits_0^\infty \left[-te^{-st}\right]\left[-tf\left(t\right)\right]dt \\
&= \int\limits_0^\infty e^{-st}\left[\left(-1\right)^2 t^2 f\left(t\right)\right]dt \\
&= L\left\{\left(-1\right)^2 t^2 f\left(t\right)\right\}
\end{aligned}
$$

Extending to the third derivative, we have;

$$
\begin{aligned}
\frac{d^3}{ds^3}\overline{f}\left(s\right) &= \frac{d}{ds}\left[\frac{d^2}{ds^2}\overline{f}\left(s\right)\right] \\
&= \frac{d}{ds}L\left\{\left(-1\right)^2 t^2 f\left(t\right)\right\} \\
&= \frac{d}{ds}\int\limits_0^\infty e^{-st}\left[\left(-1\right)^2 t^2 f\left(t\right)\right]dt \\
&= \int\limits_0^\infty \frac{d}{ds}e^{-st}\left[\left(-1\right)^2 t^2 f\left(t\right)\right]dt \\
&= \int\limits_0^\infty -te^{-st}\left[\left(-1\right)^2 t^2 f\left(t\right)\right]dt \\
&= \int\limits_0^\infty e^{-st}\left[\left(-1\right)^3 t^3 f\left(t\right)\right]dt \\
&= L\left\{\left(-1\right)^3 t^3 f\left(t\right)\right\}
\end{aligned}
$$

By mathematical induction, we assume that

$$\frac{d^{n-1}}{ds^{n-1}}\overline{f}(s) = L\left\{(-1)^{n-1}t^{n-1}f(t)\right\}$$

Then

$$\frac{d^n}{ds^n}\overline{f}(s) = \frac{d}{ds}L\left\{(-1)^{n-1}t^{n-1}f(t)\right\}$$

$$= \frac{d}{ds}\int_0^\infty e^{-st}\left[(-1)^{n-1}t^{n-1}f(t)\right]dt$$

$$= \int_0^\infty \frac{d}{ds}e^{-st}\left[(-1)^{n-1}t^{n-1}f(t)\right]dt$$

$$= \int_0^\infty -te^{-st}\left[(-1)^{n-1}t^{n-1}f(t)\right]dt$$

$$= \int_0^\infty e^{-st}\left[(-1)^{n}t^{n}f(t)\right]dt$$

$$= L\left\{(-1)^{n}t^{n}f(t)\right\}$$

$$\therefore \frac{d^n}{ds^n}\overline{f}(s) = L\left\{(-1)^{n}t^{n}f(t)\right\}$$

viii. LT of Division by t

$$L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \overline{f}(u)\,du$$

Proof

By definition

$$
\begin{aligned}
L\left\{f\left(t\right)\right\} &= \overline{f}\left(s\right) \\
&= \int\limits_{0}^{\infty} e^{-st} f\left(t\right) dt
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int\limits_{s}^{\infty} \overline{f}\left(u\right) du &= \int\limits_{s}^{\infty} \int\limits_{0}^{\infty} e^{-ut} f\left(t\right) dt du \\
&= \int\limits_{0}^{\infty} \left[\int\limits_{s}^{\infty} e^{-ut} du\right] f\left(t\right) dt \\
&= \int\limits_{0}^{\infty} \left[\frac{e^{-ut}}{-t}\right]\Bigg|_{s}^{\infty} f\left(t\right) dt \\
&= \int\limits_{0}^{\infty} \left[0 - \frac{e^{-ut}}{-t}\right] f\left(t\right) dt \\
&= \int\limits_{0}^{\infty} e^{-st} \frac{1}{t} f\left(t\right) dt \\
&= L\left\{\frac{f\left(t\right)}{t}\right\}
\end{aligned}
$$

ix. LT of Periodic Functions

If $f(t)$ has a period T where $T > 0$, then

$$
L\left\{f\left(t\right)\right\} = \frac{1}{1 - e^{-sT}} \int\limits_{0}^{T} e^{-su} f\left(u\right) du
$$

Proof
By definition,

$$L\left\{f\left(t\right)\right\} = \int\limits_{0}^{\infty} e^{-st} f\left(t\right) dt$$

$$= \int\limits_{0}^{T} e^{-st} f\left(t\right) dt + \int\limits_{T}^{2T} e^{-st} f\left(t\right) dt + \int\limits_{T}^{3T} e^{-st} f\left(t\right) dt + ...$$

Now, in the first integral, let $t = u \Rightarrow dt = du$
In the second integral, let $t = u + T \Rightarrow dt = du$
In the third integral, let $t = u + 2T \Rightarrow dt = du$
In the k^{th} integral, let $t = u + (k-1)T \Rightarrow dt = du$
and so on

We therefore have

$$L\left\{f\left(t\right)\right\} = \int\limits_{0}^{T} e^{-su} f\left(u\right) du + \int\limits_{0}^{T} e^{-s(u+T)} f\left(u+T\right) du + \int\limits_{0}^{T} e^{-s(u+2T)} f\left(u+2T\right) du + ...$$

$$= \int\limits_{0}^{T} e^{-su} f\left(u\right) du + e^{-sT} \int\limits_{0}^{T} e^{-su} f\left(u+T\right) du + e^{-2sT} \int\limits_{0}^{T} e^{-su} f\left(u+2T\right) du + ...$$

Since $f(t)$ is of period T, it follows that $f(t+kT) = f(t)$, Thus $f(u+kT) = f(u)$ for all values of k.

Hence we have

$$L\left\{f\left(t\right)\right\} = \int\limits_{0}^{T} e^{-su} f\left(u\right) du + e^{-sT} \int\limits_{0}^{T} e^{-su} f\left(u\right) du + e^{-2sT} \int\limits_{0}^{T} e^{-su} f\left(u\right) du + ...$$

$$= \left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + ...\right) \int_0^T e^{-su} f(u)\, du$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-su} f(u)\, du$$

x. Initial value theorem

The initial value theorem states that the behavior of a time function $f(t)$ in the neighbourhood of $t = 0$ corresponds to the behavior of $sL\{f(t)\}$ in the neighbourhood of $s = \infty$

Proof

We know from property (v) above that

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= s\overline{f}(s) - f(0)$$

Therefore

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\, dt = \lim_{s\to\infty} \left[s\overline{f}(s) - f(0)\right]$$

$$0 = \lim_{s\to\infty} \left[s\overline{f}(s) - f(0)\right]$$

$$\lim_{s\to\infty} s\overline{f}(s) = f(0)$$

$$\therefore \lim_{s\to\infty} s\overline{f}(s) = \lim_{t\to 0} f(0)$$

xi. Final Value theorem

The final value theorem states that the behavior of a time function $f(t)$ in the neighbourhood of $t = \infty$ corresponds to the behavior of $sL\{f(t)\}$ in the neighbourhood of $s = 0$

Proof

From above

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt \\ &= s\overline{f}(s) - f(0) \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt &= \lim_{s\to 0} \left[s\overline{f}(s) - f(0)\right] \\ \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt &= \lim_{s\to 0} \left[s\overline{f}(s) - f(0)\right] \\ \int_0^\infty f'(t)\,dt &= \lim_{s\to 0} s\overline{f}(s) - f(0) \\ \int_0^\infty \frac{d}{dt} f(t)\,dt &= \lim_{s\to 0} s\overline{f}(s) - f(0) \\ \int_0^\infty df(t) &= \lim_{s\to 0} s\overline{f}(s) - f(0) \\ f(t)|_0^\infty &= \lim_{s\to 0} s\overline{f}(s) - f(0) \\ f(\infty) - f(0) &= \lim_{s\to 0} s\overline{f}(s) - f(0) \\ f(\infty) &= \lim_{s\to 0} s\overline{f}(s) \\ \therefore \lim_{t\to\infty} f(t) &= \lim_{s\to 0} s\overline{f}(s) \end{aligned}$$

xii. Convolution

Let $f_1(t)$ and $f_2(t)$ be two functions and $L\{f_1(t)\} = \overline{f}_1(s)$ and $L\{f_2(t)\} = \overline{f}_2(s)$.

Then

$$L\left\{\int_0^t f_1(x)\, f_2(t-x)\, dx\right\} = \overline{f}_1(s) * \overline{f}_2(s)$$

Proof

By definition of Laplace transform

$$L\left\{\int_0^t f_1(x)\, f_2(t-x)\, dx\right\} = \int_0^\infty e^{-st}\left[\int_0^t f_1(x)\, f_2(t-x)\, dx\right] dt$$

Where the RHS is a double integral over the angular region bounded by the lines $x = 0$ and $x = t$ in the first quadrant of the $tx-$plane. Changing the order of integration, we write;

$$L\left\{\int_0^t f_1(x)\, f_2(t-x)\, dx\right\} = \int_0^\infty f_1(x)\left[\int_x^\infty e^{-st} f_2(t-x)\, dx\right] dt$$

Making in the inner integral the substitution $t - x = u$, we obtain

$$\int_x^\infty e^{-st} f_2(t-x)\, dx = \int_0^\infty e^{-(u+x)s} f_2(u)\, du$$

$$= e^{-sx} \underbrace{\int_0^\infty e^{-su} f_2(u)\, du}_{L\{f_2(u)\}}$$

$$= e^{-sx}\overline{f}_2(s)$$

Hence

$$L\left\{\int_0^t f_1(x)\, f_2(t-x)\, dx\right\} = \int_0^\infty f_1(x) e^{-sx}\overline{f}_2(s)\, dx$$

$$= \overline{f}_2(s) \underbrace{\int_0^\infty e^{-sx} f_1(x)\, dx}_{L\{f_1(x)\}}$$

$$= \overline{f}_2(s) * \overline{f}_1(s)$$

Some Special Functions

1. Gamma Function

It is defined as
$\Gamma(n) = \int\limits_0^\infty y^{n-1}e^{-y}dy$ where $n > 0$

Relationship between gamma and factorial moments

By definition of gamma function
$\Gamma(n+1) = \int\limits_0^\infty y^n e^{-y}dy$

Using integration by parts, let

$$u = y^n \Rightarrow du = ny^{n-1}$$

and

$$dv = e^{-y}dy \Rightarrow v = -e^{-y}$$

We now have

$$\begin{aligned}\Gamma(n+1) &= \int\limits_0^\infty y^n e^{-y}dy \\ &= uv - \int vdu \\ &= -y^n e^{-y}\Big|_0^\infty + \int\limits_0^\infty ny^n e^{-y}dy \\ &= 0 + n\underbrace{\int\limits_0^\infty y^n e^{-y}dy}_{\Gamma(n)} \\ &= n\Gamma(n)\end{aligned}$$

Now

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1)\\ &= 1(0!)\\ &= 1\end{aligned}$$

$$\begin{aligned}\Gamma(3) &= 2\Gamma(2)\\ &= 2(1)\\ &= 2!\end{aligned}$$

$$\begin{aligned}\Gamma(4) &= 3\Gamma(3)\\ &= 3(2)(1)\\ &= 3!\end{aligned}$$

$$\begin{aligned}&\Gamma(5)\\ &= 4\Gamma(4)\\ &= 4(3)(2)(1)\\ &= 4!\end{aligned}$$

$$\begin{aligned}\Gamma(6) &= 5\Gamma(5)\\ &= 5(4)(3)(2)(1)\\ &= 5!\end{aligned}$$

And so on, therefore in general $\Gamma(n+1) = n!$

Another property of gamma is that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. The proof is as follows;

$$\Gamma(n) = \int_0^\infty y^{n-1}e^{-y}dy$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \int_0^\infty y^{1/2-1}e^{-y}dy$$

$$= \int_0^\infty y^{-1/2} e^{-y} dy$$

Let $y = u^2 \Rightarrow dy = 2udu$, we thus have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \left(u^2\right)^{-1/2} e^{-u^2} 2udu$$

$$= 2\int_0^\infty e^{-u^2} du$$

But

$$\int_0^\infty e^{-u^2} du = \int_0^\infty e^{-v^2} dv$$

Thus

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left(2\int_0^\infty e^{-u^2} du\right)\left(2\int_0^\infty e^{-v^2} dv\right)$$

$$= 4\int_0^\infty \int_0^\infty e^{-\left(u^2+v^2\right)} dudv$$

To evaluate this integral, we change it from Cartesian to polar coordinates by letting$u = r\cos\theta, \quad v = r\sin\theta$ implying that

$$u^2 + v^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2 \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{1}$$

$$= r^2 dudv$$

$$= rdrd\theta$$

The polar bounds are $0 < \theta < \frac{\pi}{2}$ and $0 < r < \infty$, with this we have,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} rdrd\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^\infty e^{-r^2} rdr\right) d\theta$$

Let

$$x = r^2 \Rightarrow dx = 2rdr$$

and

$$dr = \frac{dx}{2r}$$

This implies that

$$\begin{aligned}
\int_0^\infty e^{-r^2} r dr &= \int_0^\infty e^{-x} r \frac{dx}{2r} \\
&= \frac{1}{2} \int_0^\infty e^{-x} dx \\
&= \frac{1}{2} \left(-e^{-x} \Big|_0^\infty \right) \\
&= \frac{1}{2} (0 + 1) \\
&= \frac{1}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\left[\Gamma \left(\frac{1}{2} \right) \right]^2 &= 4 \int_0^{\pi/2} \frac{1}{2} d\theta \\
&= 2 \int_0^{\pi/2} d\theta \\
&= 2 \left(\theta \Big|_0^{\frac{\pi}{2}} \right) \\
&= \pi
\end{aligned}$$

Taking the square root of both sides yields

$$\Gamma \left(\frac{1}{2} \right) = \sqrt{\pi}$$

2. Bessel Functions

A Bessel function of order n is defined by

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2(4)(2n+2)(2n+4)} - ... \right\}$$

The function satisfies

$$J_{-n}(t) = (-1)^n \, J_n(t) \quad \text{for positive integer} \quad n$$

$$\frac{d}{dt}\left\{t^n J_n(t)\right\} = t^n J_{n-1}(t)$$

$$t^2 \frac{d^2}{dt^2} J_n(t) + t \frac{d}{dt} J_n(t) + \left(t^2 - n^2\right) J_n(t) = 0$$

3. The Error Function

It is denoted by $erf(t)$ and defined as

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

4. The Complementary Error Function

This is the complement of the error function defined as

$$\begin{aligned} erfc(t) &= 1 - erf(t) \\ &= 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du \\ &= \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du \end{aligned}$$

5. Sine Integral
It is defined by

$$Si(t) = \int_0^t \frac{\sin u}{u} du$$

6. Cosine Integral
It is defined by

$$Ci(t) = \int_t^\infty \frac{\cos u}{u} du$$

7. Exponential Integral
It is denoted by $Ei(t)$ and defined by

$$Ei(t) = \int_t^\infty \frac{e^{-u}}{u} du$$

8. Heaviside unit function
It is also known as the unit step function and it is defined as

$$\eta(t-a) = \begin{cases} 0 & t < a, \\ 1 & t > a \end{cases}$$

9. Dirac Delta Function
This function is also called the unit impulse function, it is denoted by $\delta(t)$ and defined by the property

$$\delta(t-c) = \begin{cases} 0 & t \neq c, \\ ''\infty'' & t = c \end{cases}$$

10. Null Function
Let $N(t)$ be a function of t such that

$$\int_0^t N(u)du = 0 \quad \forall t > 0$$

Then $N(t)$ is called a null function.

2.3 Methods of finding Laplace Transforms

There are several methods that can be used to determine the Laplace transforms of given functions. They include;

a) Direct method
This is a primitive approach that requires some basic knowledge of Integral calculus.The

Laplace transform of a function $f(t)$ is obtained directly by evaluating

$$L\left\{f\left(t\right)\right\} = \int\limits_{0}^{\infty} e^{-st} f\left(t\right) dt$$

Examples

1.$f(t) = e^{at}$

$$\begin{aligned} L\left\{e^{at}\right\} &= \int\limits_{0}^{\infty} e^{-st} e^{at} dt \\ &= \int\limits_{0}^{\infty} e^{-t(s-a)} dt \\ &= \left[\frac{1}{-(s-a)} e^{-t(s-a)}\Big|_{0}^{\infty}\right] \\ &= -\frac{1}{s-a}\left[0-1\right] \\ &= \frac{1}{s-a} \qquad s > a \end{aligned}$$

For $a = 1$, We have

$$L\left\{e^{t}\right\} = \frac{1}{s-1} \qquad s > 1$$

2. $f(t) = 1$

$$\begin{aligned} L\left\{1\right\} &= \int\limits_{0}^{\infty} e^{-st}(1) dt \\ &= \int\limits_{0}^{\infty} e^{-st} dt \end{aligned}$$

$$\begin{aligned} &= \left[\frac{1}{-s} e^{-st}\Big|_{0}^{\infty}\right] \\ &= -\frac{1}{s}\left[0-1\right] \\ &= \frac{1}{s} \qquad\qquad s > 0 \end{aligned}$$

3.$f(t) = t$
By definition

$$L\{t\} = \int_0^\infty e^{-st} t dt$$

Using integration by parts, let

$$u = t \Rightarrow du = dt$$

and

$$dv = e^{-st}dt \Rightarrow v = -\frac{1}{s}e^{-st}$$

Thus

$$\begin{aligned}
L\{t\} &= uv - \int vdu \\
&= t\left[-\frac{1}{s}e^{-st}\Big|_0^\infty\right] + \frac{1}{s}\int_0^\infty e^{-st}dt \\
&= \frac{t}{s}\left[-e^{-st}\right]\Big|_0^\infty + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]\Big|_0^\infty \\
&= \frac{t}{s}\left[-e^{-st}\right]\Big|_0^\infty + \frac{1}{s^2}\left[-e^{-st}\right]\Big|_0^\infty \\
&= 0 + \frac{1}{s^2}\left[0 - (-1)\right] \\
&= \frac{1}{s^2} \qquad s > 0
\end{aligned}$$

4.$f(t) = t^n$

$$L\{t^n\} = \int_0^\infty e^{-st} t^n dt$$

Let

$$u = st \Rightarrow t = \frac{u}{s} \text{ and } dt = du/s$$

The limits remain unchanged, thus we now have

$$L\{t^n\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n du$$

But by the definition of a gamma function

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

We therefore have

$$\begin{aligned} L\left\{t^n\right\} &= \frac{1}{s^{n+1}} \int_0^\infty u^{(n+1)-1} e^{-u} du \\ &= \frac{1}{s^{n+1}} \Gamma(n+1) \\ &= \frac{n!}{s^{n+1}} \end{aligned}$$

For $n = \frac{1}{2}$ we have

$$\begin{aligned} L\left\{\sqrt{t}\right\} &= \frac{\Gamma\left(1/2+1\right)}{s^{1/2+1}} \\ &= \frac{\Gamma\left(3/2\right)}{s^{3/2}} \\ &= \frac{1/2\,\Gamma\left(1/2\right)}{s^{3/2}} \\ &= \frac{\sqrt{\pi}}{2s^{3/2}} \end{aligned}$$

5. $f(t) = \sin(at)$

$$L\left\{\sin(at)\right\} = \int_0^\infty e^{-st} \sin(at) dt$$

But it can be shown that

$$\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2+b^2} \left[a \sin(bt) - b \cos(bt)\right]$$

Thus

$$L\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)dt$$

$$= \frac{e^{-st}}{(-s)^2+a^2}\left[-s\sin(at) - a\cos(at)\right]\Big|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}\left[0 - a\right]$$

$$= \frac{a}{s^2+a^2}$$

Similarly

$$L\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)dt$$

But

$$\int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2}\left[a\cos(bt) + b\sin(bt)\right]$$

Thus

$$L\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)dt$$

$$= \frac{e^{-st}}{(-s)^2+a^2}\left[-s\cos(at) + a\sin(at)\right]\Big|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}\left[-s + 0\right]$$

$$= \frac{s}{s^2+a^2}$$

Alternatively from example (1) above we have seen that

$$L\left\{e^{at}\right\} = \frac{1}{s-a} \qquad s > a$$

$$\Rightarrow L\left\{e^{iat}\right\} = \frac{1}{s-ia}$$

$$= \frac{1}{s-ai}\left(\frac{s+ai}{s+ai}\right)$$
$$= \frac{s+ai}{s^2+a^2}$$

But

$$\begin{aligned} e^{ait} &= \cos at + i\sin at \\ L\left\{e^{ait}\right\} &= L\left\{\cos at + i\sin at\right\} \\ &= L\left\{\cos at\right\} + iL\left\{\sin at\right\} \end{aligned}$$

Thus

$$L\left\{\cos at\right\} + iL\left\{\sin at\right\} = \frac{s+ai}{s^2+a^2}$$

Associating real and imaginary parts, it follows that;

$$L\left\{\cos at\right\} = \frac{s}{s^2+a^2}$$

And

$$L\left\{\sin at\right\} = \frac{a}{s^2+a^2}$$

6. $f(t) = \sinh(at)$

$$L\left\{\sinh(at)\right\} = \int_0^\infty e^{-st}\sinh(at)dt$$

But

$$\begin{aligned} \sinh(at) &= \frac{e^{at}-e^{-at}}{2} \\ \Rightarrow L\left\{\sinh(at)\right\} &= \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt \\ &= \frac{1}{2}\int_0^\infty e^{-st}\left(e^{at}-e^{-at}\right)dt \\ &= \frac{1}{2}\left\{\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-(s+a)}\,dt\right\} \end{aligned}$$

$$= \frac{1}{2}\left\{ \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty - \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty \right\}$$
$$= \frac{1}{2}\left\{ \left(0 + \frac{1}{s-a}\right) - \left(0 + \frac{1}{s+a}\right) \right\}$$
$$= \frac{1}{2}\left\{ \frac{1}{s-a} - \frac{1}{s+a} \right\}$$

$$= \frac{1}{2}\left\{ \frac{s+a-(s-a)}{(s-a)(s+a)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{2a}{s^2-a^2} \right\}$$
$$= \frac{a}{s^2-a^2}$$

Similarly

$$L\{\cosh(at)\} = \int_0^\infty e^{-st}\cosh(at)dt$$

But

$$\cosh(at) = \frac{e^{at}+e^{-at}}{2} \Rightarrow L\{\cosh(at)\}$$
$$= \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$$
$$= \frac{1}{2}\int_0^\infty e^{-st}\left(e^{at}+e^{-at}\right) dt$$

$$= \frac{1}{2}\left\{ \int_0^\infty e^{-(s-a)t}dt + \int_0^\infty e^{-(s+a)}\ dt \right\}$$
$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]\Bigg|_0^\infty$$
$$= \frac{1}{2}\left\{ \left(0 + \frac{1}{s-a}\right) + \left(0 + \frac{1}{s+a}\right) \right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+a+(s-a)}{(s-a)(s+a)}\right\}$$
$$= \frac{1}{2}\left\{\frac{2s}{s^2-a^2}\right\}$$
$$= \frac{s}{s^2-a^2}$$

7. $g(t) = \delta(t-c)f(t)$ where $\delta(t-c)$ is the Dirac delta function and c is a constant.
By definition

$$L\{g(t)\} = \int_0^\infty e^{-st}\delta(t-c)f(t)dt$$

But by the definition of Dirac delta function, it can be shown that

$$e^{-st}f(t) = e^{-sc}f(c)$$
$$\Rightarrow L\{\delta(t-c)f(t)\} = \int_0^\infty e^{-sc}f(c)\delta(t-c)dt$$
$$= e^{-sc}f(c)\underbrace{\int_0^\infty \delta(t-c)dt}_{1}$$
$$\therefore L\{\delta(t-c)f(t)\} = e^{-sc}f(c)$$

As a special case when $f(t) = 1$ we have $L\{\delta(t-c)\} = e^{-cs}$
When $c = 0$ we have $L\{\delta(t)\} = 1$

8. $f(t) = \eta(t-c)$ where $\eta(t-c)$ is the Heaviside step function.

By definition

$$L\{\eta(t-c)f(t-c)\} = \int_0^\infty e^{-st}\eta(t-c)f(t-c)dt$$

But since $\eta(t-c)$ a unit step function it takes the value zero for values of $t < c$ and the value 1 for values of $t \geqslant c$ .Thus we can rewrite the above integral as

$$L\{\eta(t-c)f(t-c)\} = \int_c^\infty e^{-st}(1)\,f(t-c)dt$$

$$= \int_c^\infty e^{-st}f(t-c)dt$$

Let $x = t - c \Rightarrow t = x + c$ and $dx = dt$

Changing limits of integration we have
When $t = c, \quad x = 0$
When $t = \infty, \quad x = \infty$
With this we have,

$$L\{\eta(t-c)f(t-c)\} = \int_0^\infty e^{-s(x+c)}f(x)dx$$

$$= e^{-sc}\int_0^\infty e^{-sx}f(x)dx$$

$$= e^{-sc}\overline{f}(s)$$

As a special case when $f(t-c) = f(x) = 1$ we have

$$L\{\eta(t-c)(1)\} = \int_0^\infty e^{-s(x+c)}(1)dx$$

$$= e^{-sc}\int_0^\infty e^{-sx}dx$$

$$= e^{-sc}L\{1\}$$

$$= \frac{e^{-sc}}{s}$$

$$\therefore L\{\eta(t-c)\} = \frac{e^{-sc}}{s}$$

b) Series method

If the given function $f(t)$ can be expressed as a power series expansion say;

$$\begin{aligned} f(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ... \\ &= \sum_{n=0}^{\infty} a_n t^n \end{aligned}$$

Then the Laplace transform of $f(t)$ is obtained by simply evaluating the Laplace transform of each term in the series and then summing up the result. Thus

$$\begin{aligned} L\{f(t)\} &= \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!a_2}{s^3} + \frac{3!a_3}{s^4} + ... \\ &= \sum_{n=0}^{\infty} \frac{n!a_n}{s^{n+1}} \end{aligned}$$

Examples

1. We know the exponential function can be expressed as a series expansion; i.e

$$\begin{aligned} e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ... \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{aligned}$$

Which means that our coefficients are;

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_2 &= \frac{1}{2!} \\ a_3 &= \frac{1}{3!} \\ &\vdots \\ a_n &= \frac{1}{n!} \end{aligned}$$

Thus

$$
\begin{aligned}
L\left\{e^t\right\} &= \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!a_2}{s^3} + \frac{3!a_3}{s^4} + ... \\
&= \frac{1}{s} + \frac{1}{s^2} + \frac{2!}{s^3}\left(\frac{1}{2!}\right) + \frac{3!}{s^4}\left(\frac{1}{3!}\right) + ... \\
&= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + ... \\
&= \frac{1}{s} \qquad \underbrace{\left(1 + \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + ...\right)}_{\textit{geometric series with common ratio } r=1/s}
\end{aligned}
$$

$$
\begin{aligned}
\therefore L\left\{e^t\right\} &= \frac{1}{s}\left[\frac{1}{1 - 1/s}\right] \\
&= \frac{1}{s-1} \qquad\qquad s > 1
\end{aligned}
$$

2. The Bessel function of order n is defined as

$$
J_n(t) = \frac{t^n}{2^n\Gamma(n+1)}\left\{1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2(4)(2n+2)(2n+4)} - ...\right\}
$$

Thus for $n = 0$, we have a Bessel function of order zero defined as

$$
\begin{aligned}
J_0(t) &= \frac{1}{\Gamma(1)}\left\{1 - \frac{t^2}{2(0+2)} + \frac{t^4}{2(4)(0+2)(0+4)} - ...\right\} \\
&= 1 - \frac{t^2}{2(2)} + \frac{t^4}{2(4)(2)(4)} - \frac{t^6}{2(4)(2)(6)(4)(6)} + ... \\
&= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + ...
\end{aligned}
$$

Which means that our coefficients are

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= -\frac{1}{2^2} \\
a_3 &= 0 \\
a_4 &= \frac{1}{2^2 4^2} \\
a_5 &= 0 \\
a_6 &= -\frac{1}{2^2 4^2 6^2} \\
&\vdots
\end{aligned}
$$

and so on

Thus

$$L\left\{J_0(t)\right\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!a_2}{s^3} + \frac{3!a_3}{s^4} + \frac{4!a_4}{s^5} + \frac{5!a_5}{s^6} + \frac{6!a_6}{s^7} + ...$$

$$= \frac{1}{s} + \frac{0}{s^2} - \frac{2!\left(\frac{1}{2^2}\right)}{s^3} + \frac{3!(0)}{s^4} + \frac{4!\left(\frac{1}{2^2*4^2}\right)}{s^5} + \frac{5!(0)}{s^6} - \frac{6!\left(\frac{1}{2^2*4^2*6^2}\right)}{s^7} + ...$$

$$= \frac{1}{s} - \frac{2!\left(\frac{1}{2^2}\right)}{s^3} + \frac{4!\left(\frac{1}{2^2*4^2}\right)}{s^5} - \frac{6!\left(\frac{1}{2^2*4^2*6^2}\right)}{s^7} + ...$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{2}*\frac{3}{4}\left(\frac{1}{s^4}\right) - \frac{1}{2}*\frac{3}{4}*\frac{5}{6}\left(\frac{1}{s^6}\right) ......\right]$$

But from negative binomial expansion

$$
\begin{aligned}
(1+x)^{-r} &= \sum_{j=0}^{\infty} \binom{-r}{j} x^j \\
(1+x)^{-1/2} &= \sum_{j=0}^{\infty} \binom{-1/2}{j} x^j
\end{aligned}
$$

And it can be shown that

$$\begin{aligned}
\binom{-r}{j} &= (-1)^j \binom{r+j-1}{j} \\
\binom{-\frac{1}{2}}{j} &= (-1)^j \binom{\frac{1}{2}+j-1}{j} \\
&= (-1)^j \binom{j-\frac{1}{2}}{j} \\
\Rightarrow (1+x)^{-1/2} &= \sum_{j=0}^{\infty} (-1)^j \binom{j-\frac{1}{2}}{j} x^j
\end{aligned}$$

$$\left(1+\frac{1}{s^2}\right)^{-1/2} = \sum_{j=0}^{\infty} (-1)^j \binom{j-\frac{1}{2}}{j} \left(\frac{1}{s^2}\right)^j$$

$$= 1 - \binom{\frac{1}{2}}{1}\frac{1}{s^2} + \binom{\frac{3}{2}}{2}\frac{1}{s^4} - \binom{\frac{5}{2}}{3}\frac{1}{s^6} + ...$$

Since

$$\binom{n}{j} = \frac{n(n-1)(n-2)...[n-(k-1)]}{1*2*3*...*k}$$

We have

$$\begin{aligned}
\binom{\frac{1}{2}}{1} &= \frac{1/2}{1} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\binom{\frac{3}{2}}{2} &= \frac{3/2\left(3/2-1\right)}{1*2} \\
&= \frac{3}{4}*\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\binom{\frac{5}{2}}{3} &= \frac{5/2\left(5/2-1\right)\left(5/2-2\right)}{1*2*3} \\
&= \frac{5}{6}*\frac{3}{4}*\frac{1}{2}
\end{aligned}$$

$$\Rightarrow \left(1+\frac{1}{s^2}\right)^{-1/2} = 1-\left(\frac{1}{2}\right)\frac{1}{s^2}+\left(\frac{3}{4}*\frac{1}{2}\right)\frac{1}{s^4}-\left(\frac{5}{6}*\frac{3}{4}*\frac{1}{2}\right)\frac{1}{s^6}+..$$

$$\therefore L\left\{J_0(t)\right\} = \frac{1}{s}\left\{\left(1+\frac{1}{s^2}\right)^{-1/2}\right\}$$

$$= \frac{1}{\sqrt{s^2+1}}$$

Similarly $J_n\left(at\right)$ can be obtained directly as follows

$$L\left\{J_n\left(at\right)\right\} = \int_0^\infty J_n\left(at\right)e^{-st}dt(1) \tag{2.1}$$

But $J_n\left(x\right)$ can be written as an integral as follows

$$J_n\left(x\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\theta - x\sin\theta)}d\theta$$

$$\Rightarrow J_n\left(at\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\theta - at\sin\theta)}d\theta$$

Putting this in equation (2.1) yields

$$L\left\{J_n\left(at\right)\right\} = \frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi} e^{i(n\theta - at\sin\theta)}e^{-st}d\theta dt$$

Changing the order of integration yields

$$L\left\{J_n\left(at\right)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\int_0^\infty e^{-ati\sin\theta}e^{-st}dtd\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\int_0^\infty e^{-(s+ia\sin\theta)t}dtd\theta \tag{2.2}$$

But

$$\int_0^\infty e^{-(s+ia\sin\theta)t}dt = \left[\frac{e^{-(s+ia\sin\theta)t}}{-(s+ia\sin\theta)} \Big|_0^\infty \right]$$
$$= \frac{1}{-(s+ia\sin\theta)} \left[e^{-(s+ia\sin\theta)t} \Big|_0^\infty \right]$$
$$= \frac{1}{-(s+ia\sin\theta)} [0-1]$$
$$= \frac{1}{s+ia\sin\theta}$$

Therefore equation (2.2) now becomes

$$L\{J_n(at)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \left(\frac{1}{s+ia\sin\theta} \right) d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{in\theta}}{s+ia\sin\theta} d\theta \tag{2.3}$$

Let

$$z = e^{i\theta}$$
$$dz = ie^{i\theta} d\theta$$
$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

Also

$$\therefore \left(z - \frac{1}{z} \right) = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$
$$= 2i\sin\theta$$
$$\frac{1}{2}\left(z - \frac{1}{z} \right) = i\sin\theta$$

With this $L\{J_n(at)\}$ now becomes

$$
\begin{aligned}
L\{J_n(at)\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(e^{i\theta}\right)^n}{s+a(i\sin\theta)}d\theta \\
&= \frac{1}{2\pi}\int_{|z|=1}\frac{z^n}{s+a\left(\frac{1}{2}\left[z-\frac{1}{z}\right]\right)}\frac{dz}{iz} \\
&= \frac{1}{2\pi i}\int_{|z|=1}\frac{z^n}{s+\frac{a}{2}\left(z-\frac{1}{z}\right)}\frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_{|z|=1}\frac{z^n}{\frac{2s+a}{2}\left(\frac{z^2-1}{z}\right)}\frac{dz}{z} \\
&= \frac{2}{2\pi i}\int_{|z|=1}\frac{z^n}{2zs+a\left(z^2-1\right)}dz \\
&= \frac{2}{2\pi i}\int_{|z|=1}\frac{z^n}{2zs+az^2-a}dz \\
&= \frac{2}{2\pi i}\int_{|z|=1}\frac{z^n}{a\left(z^2+2\left(\frac{s}{a}\right)z-1\right)}dz \\
&= \frac{1}{\pi i}\int_{|z|=1}\frac{z^n}{a\left(z-z^+\right)\left(z-z^-\right)}dz \qquad (2.4)
\end{aligned}
$$

Where z^+ and z^- are the roots of

$$
\begin{aligned}
&z^2+2\left(\frac{s}{a}\right)z-1=0 \\
&\Rightarrow az^2+2sz-a=0
\end{aligned}
$$

Hence

$$
\begin{aligned}
z &= \frac{-2s\pm\sqrt{(2s)^2-4a(-a)}}{2a} \\
&= \frac{-2s\pm\sqrt{4s^2+4a^2}}{2a} \\
&= \frac{-2s\pm\sqrt{4\left(s^2+a^2\right)}}{2a}
\end{aligned}
$$

$$= \frac{-2s \pm 2\sqrt{s^2 + a^2}}{2a}$$
$$= \frac{2\left(-s \pm \sqrt{s^2 + a^2}\right)}{2a}$$
$$= \frac{-s \pm \sqrt{s^2 + a^2}}{a}$$

$$\therefore z^+ = \frac{-s + \sqrt{s^2 + a^2}}{a}, z^- = \frac{-s - \sqrt{s^2 + a^2}}{a}$$

Here, in the contour integration of expression (2.4) only the root of z^+ lies outside the unit circle. Hence applying Cauchys Integral theorem we have

$$\int\limits_{|z|=1} \frac{z^n}{a\left(z - z^+\right)\left(z - z^-\right)} dz = 2\pi i * [\text{Residue of Integrand}]$$
$$= 2\pi i \lim_{z \to z^+} \frac{z^n}{a\left(z - z^+\right)\left(z - z^-\right)}\left(z - z^+\right)$$
$$= 2\pi i \lim_{z \to z^+} \frac{z^n}{a\left(z - z^-\right)}$$

Therefore

$$L\left\{J_n\left(at\right)\right\} = \frac{1}{\pi i} \int\limits_{|z|=1} \frac{z^n}{a\left(z - z^+\right)\left(z - z^-\right)} dz$$
$$= \frac{1}{\pi i}\left[2\pi i \lim_{z \to z^+} \frac{z^n}{a\left(z - z^-\right)}\right]$$
$$= 2 \lim_{z \to z^+} \frac{z^n}{a\left(z - z^-\right)}$$
$$= \frac{2\left(z^+\right)^n}{a\left(z^+ - z^-\right)}$$

$$= \frac{2\left(\frac{-s+\sqrt{s^2+a^2}}{a}\right)^n}{a\left(\frac{-s+\sqrt{s^2+a^2}}{a} - \frac{-s-\sqrt{s^2+a^2}}{a}\right)}$$

$$= \frac{2\left(\frac{-s+\sqrt{s^2+a^2}}{a}\right)^n}{a\left(-\frac{s}{a}+\frac{\sqrt{s^2+a^2}}{a}+\frac{s}{a}+\frac{\sqrt{s^2+a^2}}{a}\right)}$$

$$= \frac{2\left(\frac{-s+\sqrt{s^2+a^2}}{a}\right)^n}{2a\left(\frac{\sqrt{s^2+a^2}}{a}\right)}$$

$$= \frac{\frac{1}{a^n}\left(\sqrt{s^2+a^2}-s\right)^n}{\left(\sqrt{s^2+a^2}\right)}$$

$$= \frac{\left(\sqrt{s^2+a^2}-s\right)^n}{a^n\left(\sqrt{s^2+a^2}\right)}$$

$3.f(t) = erf\sqrt{t}$

Where

$$erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx$$

By series expansion

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...$$

We thus have

$$erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...\right)dx$$

$$= \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{(5)2!} - \frac{x^7}{(7)3!} + ...\Big|_0^{\sqrt{t}}\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(\sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3} + \frac{\left(\sqrt{t}\right)^5}{(5)2!} - \frac{\left(\sqrt{t}\right)^7}{(7)3!} + ...\right)$$

Therefore

$$L\left\{erf\sqrt{t}\right\} = L\left\{\frac{2}{\sqrt{\pi}}\left(\sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3} + \frac{\left(\sqrt{t}\right)^5}{(5)2!} - \frac{\left(\sqrt{t}\right)^7}{(7)3!} + ...\right)\right\}$$

$$= \frac{2}{\sqrt{\pi}} L\left\{\sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3} + \frac{\left(\sqrt{t}\right)^5}{(5)2!} - \frac{\left(\sqrt{t}\right)^7}{(7)3!} + ...\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left(L\left\{t^{1/2}\right\} - \frac{1}{3}L\left\{t^{3/2}\right\} + \frac{1}{(5)2!}L\left\{t^{5/2}\right\} - \frac{1}{(7)3!}L\left\{t^{7/2}\right\} + ...\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{\Gamma\left(3/2\right)}{s^{3/2}} - \frac{\Gamma\left(5/2\right)}{3s^{5/2}} + \frac{\Gamma\left(7/2\right)}{(5)2!s^{7/2}} - \frac{\Gamma\left(9/2\right)}{(7)3!s^{9/2}} + ...\right)$$

$$= \frac{2}{\sqrt{\pi}}\left\{\begin{matrix}\frac{1/2\,\Gamma\left(1/2\right)}{s^{3/2}} - \frac{3/2\left(1/2\right)\Gamma\left(1/2\right)}{3s^{5/2}} + \frac{5/2\left(3/2\right)\left(1/2\right)\Gamma\left(1/2\right)}{(5)2!s^{7/2}} \\ -\frac{7/2\left(5/2\right)\left(3/2\right)\left(1/2\right)\Gamma\left(1/2\right)}{(7)3!s^{9/2}} + ...\end{matrix}\right\}$$

$$= \frac{2\Gamma\left(1/2\right)}{s^{3/2}\sqrt{\pi}}\left(\frac{1}{2} - \frac{1}{4s} + \frac{3}{16s^2} - \frac{15}{96s^3} + ...\right)$$

$$= \frac{1}{s^{3/2}}\underbrace{\left[1 - \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{1}{s^2}\right) - \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{1}{s^3}\right) + ...\right]}_{\left(1+\frac{1}{s}\right)^{-1/2}}$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2}$$

$$\therefore L\left\{erf\sqrt{t}\right\} = \frac{1}{s\sqrt{s+1}}$$

c) Method of differential equations

If a given function satisfies a differential equation. Then to obtain the desired Laplace transform $\overline{f}\,(s)$. The Laplace transform is applied to the differential equation and its properties are used to obtain the required solution. This is achieved by making $\overline{f}\,(s)$ the subject of the formula.

Examples

1. The Bessel function of order n satisfies the following differential equation.

$$t^2 J_n''(t) + tJ_n'(t) + \left(t^2 - n^2\right) J_n(t) = 0$$

This implies that $J_0(t)$ will satisfy the differential equation

$$tJ_0''(t) + J_0'(t) + tJ_0(t) = 0$$

We now take the Laplace transform of both sides of the above equation

$$L\left\{tJ_0''(t)\right\} + L\left\{J_0'(t)\right\} + L\left\{tJ_0(t)\right\} = 0$$

Let

$$y = L\left\{J_0(t)\right\}$$

Then by the property of derivatives,

$$L\left\{J_0'(t)\right\} = sy - J_0(0)$$

And

$$L\left\{J_0''(t)\right\} = s^2y - sJ_0(0) - J_0'(0)$$

$$\Rightarrow L\left\{tJ_0''(t)\right\} = -\frac{d}{ds}\left(s^2y - sJ_0(0) - J_0'(0)\right)$$

$$-\frac{d}{ds}\left(s^2y - sJ_0(0) - J_0'(0)\right) + \left(sy - J_0(0)\right) - \frac{dy}{ds} = 0$$

But

$$J_0(0) = 1 \text{ and } J_0'(0) = 0$$

Substituting this back to the equation yields

$$-\frac{d}{ds}\left(s^2 y - s\right) + (sy - 1) - \frac{dy}{ds} = 0$$

$$-2sy - s^2\frac{dy}{ds} - \frac{d}{ds}(-s) + (sy - 1) - \frac{dy}{ds} = 0$$

$$-2sy - \frac{dy}{ds}\left(s^2 + 1\right) + 1 + (sy - 1) = 0$$

$$-\frac{dy}{ds}\left(s^2 + 1\right) - sy = 0$$

$$-\frac{dy}{ds}\left(s^2 + 1\right) = sy$$

$$\frac{dy}{ds} = -\frac{sy}{(s^2 + 1)}$$

$$\frac{dy}{y} = -\frac{s}{(s^2 + 1)}ds$$

Integrating both sides yields

$$\int \frac{dy}{y} = \int -\frac{s}{(s^2 + 1)}ds$$

$$\ln y = -\int \frac{s}{(s^2 + 1)}ds$$

Let $u = s^2 + 1 \Rightarrow du = 2sds$

Then

$$\ln y = -\int \frac{s}{u}\frac{du}{2s}$$

$$= -\frac{1}{2}\int \frac{du}{u}$$

$$
\begin{aligned}
&= -\frac{1}{2}\ln u \\
&= -\frac{1}{2}\ln\left(1+s^2\right) \\
&= \ln\left(1+s^2\right)^{-1/2} \\
\Rightarrow y &= e^{\ln\left(1+s^2\right)^{-1/2}} \\
&= \frac{1}{\sqrt{1+s^2}}
\end{aligned}
$$

$$
\therefore L\left\{J_0(t)\right\} = \frac{1}{\sqrt{1+s^2}}
$$

2. The trigonometric function $\sin\left(\sqrt{t}\right)$ satisfies the differential equation

$$
4\frac{d^2}{dt^2}t\sin\left(\sqrt{t}\right) + 2\frac{d}{dt}\sin\left(\sqrt{t}\right) + \sin\left(\sqrt{t}\right) = 0
$$

Applying Laplace transform to both sides of the differential equation yields,

$$
L\left\{4\frac{d^2}{dt^2}t\sin\left(\sqrt{t}\right) + 2\frac{d}{dt}\sin\left(\sqrt{t}\right) + \sin\left(\sqrt{t}\right)\right\} = L\left\{0\right\}
$$

$$
4L\left\{\frac{d^2}{dt^2}t\sin\left(\sqrt{t}\right)\right\} + 2L\left\{\frac{d}{dt}\sin\left(\sqrt{t}\right)\right\} + L\left\{\sin\left(\sqrt{t}\right)\right\} = 0
$$

Let $Y(t) = \sin\left(\sqrt{t}\right)$ and $y = L\left\{\sin\left(\sqrt{t}\right)\right\}$

With this substitution the above equation becomes

$$
4L\left\{tY''(t)\right\} + 2L\left\{Y'(t)\right\} + L\left\{Y(t)\right\} = 0
$$

By the property of derivatives

$$
\begin{aligned}
L\left\{Y'(t)\right\} &= sL\left\{Y(t)\right\} - Y(0) \\
&= sy - \sin\left(\sqrt{0}\right) \\
&= sy
\end{aligned}
$$

$$\Rightarrow L\{Y''(t)\} = s^2 L\{Y(t)\} - sY(0) - Y'(0)$$
$$= s^2 y - sY(0) - Y'(0)$$

And by the multiplication by t property

$$L\{tY(t)\} = -\frac{d}{ds} L\{Y(t)\}$$
$$\Rightarrow L\{tY''(t)\} = -\frac{d}{ds}\left[s^2 y - sY(0) - Y'(0)\right]$$

By implicit differentiation, this becomes

$$L\{tY''(t)\} = -2sy - s^2\frac{dy}{ds} + Y(0)$$

With this now, the equation

$$4L\{tY''(t)\} + 2L\{Y'(t)\} + L\{Y(t)\} = 0$$

becomes,

$$4\left[-2sy - s^2\frac{dy}{ds} + Y(0)\right] + 2sy + y = 0$$

$$-8sy - 4s^2\frac{dy}{ds} + 0 + 2sy + y = 0$$

$$-4s^2\frac{dy}{ds} - 6sy + y = 0$$

$$-4s^2\frac{dy}{ds} - y(6s - 1) = 0$$

$$\frac{dy}{ds} + \frac{(6s-1)}{4s^2} y = 0$$

Which is a linear differential equation . Using integrating factor method we have

$$I.F = e^{\int \frac{(6s-1)}{4s^2} ds}$$

But

$$\int \frac{(6s-1)}{4s^2} ds = \frac{1}{4}\int \frac{(6s-1)}{s^2} ds$$

$$= \frac{1}{4}\int\left(\frac{6}{s} - \frac{1}{s^2}\right)ds$$
$$= \frac{1}{4}\left[6\ln s + \frac{1}{s}\right]$$
$$= \frac{2}{3}\ln s + \frac{1}{4s}$$
$$= \ln s^{\frac{2}{3}} + \frac{1}{4s}$$

Thus

$$I.F = e^{\int \frac{(6s-1)}{4s^2}ds}$$
$$= e^{\ln s^{\frac{2}{3}} + \frac{1}{4s}}$$
$$= e^{\ln s^{\frac{2}{3}}} e^{\frac{1}{4s}}$$
$$= s^{\frac{2}{3}} e^{\frac{1}{4s}}$$

Multiplying both sides of the differential equation by the Integrating factor yields

$$\frac{dy}{ds} s^{\frac{2}{3}} e^{\frac{1}{4s}} + \frac{(6s-1)}{4s^2} y s^{\frac{2}{3}} e^{\frac{1}{4s}} = 0$$

$$\Rightarrow \frac{d}{ds}\left(y s^{\frac{2}{3}} e^{\frac{1}{4s}}\right) = 0$$

Integrating both sides we get

$$\int d\left(y s^{\frac{2}{3}} e^{\frac{1}{4s}}\right) = \int 0 ds$$

$$y s^{2/3} e^{1/4s} = c$$

$$\Rightarrow y = \frac{c}{s^{2/3}} e^{-\frac{1}{4s}}$$

Where c is a constant of integration.
But it can be shown that $\sin\sqrt{t} \sim \sqrt{t}$ for small values of t and

$$y = \frac{c}{s^{2/3}}$$

for large values of s. From previous examples

$$L\left\{\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Comparing the results, it follows that

$$c = \frac{\sqrt{\pi}}{2}$$

Thus

$$\begin{aligned} L\left\{\sin\sqrt{t}\right\} &= \frac{c}{s^{2/3}} e^{-1/4s} \\ &= \frac{\sqrt{\pi}}{2s^{2/3}} e^{-1/4s} \end{aligned}$$

d) Differentiation with respect to a parameter

This method employs Leibnitzs rule for differentiating under an Integral sign along with the various rules and theorems pertaining to Laplace transforms to arrive at the desired transform.

Examples

1. $f(t) = \cos(at)$

From a previous example we had seen that

$$\begin{aligned} L\{\cos at\} &= \int_0^\infty e^{-st}\cos at dt \\ &= \frac{s}{s^2+a^2} \end{aligned}$$

Using Leibnitzs rule

$$\begin{aligned} \frac{d}{da}\int_0^\infty e^{-st}\cos at dt &= \int_0^\infty e^{-st}\{-t\sin at\}\,dt \\ &= -L\{t\sin at\} \end{aligned}$$

$$\frac{d}{da}\left\{\frac{s}{s^2+a^2}\right\} = -L\left\{t\sin at\right\}$$

Thus

$$\begin{aligned} L\left\{t\sin at\right\} &= -\frac{d}{da}s\left(s^2+a^2\right)^{-1} \\ &= 2as\left(s^2+a^2\right)^{-2} \\ &= \frac{2as}{\left(s^2+a^2\right)^2} \end{aligned}$$

2. $f(t) = \ln t$

By definition

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$$

Differentiating both sides with respect to α yields,

$$\Gamma'(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\ln y \; dy$$

Setting $\alpha = 1$ implies

$$\Gamma'(1) = \int_0^\infty e^{-y}\ln y \; dy$$

Making the substitution $y = st \Rightarrow dy = sdt$ we get

$$\begin{aligned} \Gamma'(1) &= \int_0^\infty e^{-st}\ln\left(st\right)sdt \\ \frac{\Gamma'(1)}{s} &= \int_0^\infty e^{-st}\ln st dt \\ &= \int_0^\infty e^{-st}\left(\ln s + \ln t\right)dt \end{aligned}$$

$$\Rightarrow \frac{\Gamma'(1)}{s} - \int_0^\infty e^{-st} \ln s dt = \int_0^\infty e^{-st} \ln t dt$$

$$\Rightarrow \underbrace{\int_0^\infty e^{-st} \ln t dt}_{L\{\ln t\}} = \frac{\Gamma'(1)}{s} - \ln s \underbrace{\int_0^\infty e^{-st} dt}_{L\{1\}={}^1/_s}$$

$$\begin{aligned} \therefore L\{\ln t\} &= \frac{\Gamma'(1) - \ln s}{s} \\ &= \frac{-\gamma - \ln s}{s} \end{aligned}$$

Where γ is the Euler constant.

e) Miscellaneous Methods

This methods involves the use of the properties of Laplace transform.

Examples

1. By the division by t property

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(u) du$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$\begin{aligned} \therefore L\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{u^2+1} du \\ &= \tan^{-1}\left({}^1/_s\right) \end{aligned}$$

2. By the property of derivatives

$$L\left\{f^{(n)}(t)\right\} = s\overline{f}(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

In particular

$$L\{f'(t)\} = s\overline{f}(s) - f(0) \tag{2.5}$$

Letting $f(t) = 1$ implies $f(0) = 1$ and $f'(t) = 0$. With this, equation (2.5) becomes

$$\begin{aligned} L\{0\} &= sL\{1\} - 1 \\ \Rightarrow 0 &= sL\{1\} - 1 \\ \therefore L\{1\} &= \frac{1}{s} \qquad s > 0 \end{aligned}$$

Similarly Letting $f(t) = t$ implies $f(0) = 0$ and $f'(t) = 1$. With this, equation (2.5) becomes

$$\begin{aligned} L\{1\} &= sL\{t\} - 0 \\ \Rightarrow \frac{1}{s} &= sL\{t\} \\ \therefore L\{t\} &= \frac{1}{s^2} \qquad s > 0 \end{aligned}$$

For $f(t) = t^2$ we have $f(0) = 0$ and $f'(t) = 2t$. With this, equation (2.5) becomes

$$\begin{aligned} L\{2t\} &= sL\left\{t^2\right\} - 0 \\ 2L\{t\} &= sL\left\{t^2\right\} \\ \Rightarrow \frac{2}{s} &= sL\left\{t^2\right\} \\ \therefore L\left\{t^2\right\} &= \frac{2}{s^2} \\ &= \frac{\Gamma(3)}{s^2} \qquad s > 0 \end{aligned}$$

For $f(t) = t^3$ we have $f(0) = 0$ and $f'(t) = 3t^2$. With this, equation (2.5) becomes

$$\begin{aligned} L\left\{3t^2\right\} &= sL\left\{t^3\right\} - 0 \\ 3L\left\{t^2\right\} &= sL\left\{t^3\right\} \\ \Rightarrow \frac{3}{s^2} &= sL\left\{t^3\right\} \\ \therefore L\left\{t^3\right\} &= \frac{3}{s^3} \\ &= \frac{\Gamma(4)}{s^3} \qquad s > 0 \end{aligned}$$

By mathematical induction, we assume that for $f(t) = t^{n-1}$

$$L\{f(t)\} = \frac{\Gamma(n)}{s^n}$$

Then for $f(t) = t^n$ we have $f(0) = 0$ and $f'(t) = nt^{n-1}$. With this, equation (2.5) becomes

$$\begin{aligned} L\left\{nt^{n-1}\right\} &= sL\{t^n\} - 0 \\ nL\left\{t^{n-1}\right\} &= sL\{t^n\} \\ \Rightarrow \frac{n\Gamma(n)}{s^n} &= sL\{t^n\} \\ \therefore L\{t^n\} &= \frac{n\Gamma(n)}{s^{n+1}} \\ &= \frac{\Gamma(n+1)}{s^n} \qquad s > 0 \end{aligned}$$

3. From a previous example, we had

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

But by the change of scale property if

$$L\{f(t)\} = \overline{f}(s)$$

then

$$L\{f(at)\} = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$$

Thus

$$\begin{aligned} L\{J_0(at)\} &= \frac{1}{a}\frac{1}{\sqrt{\left({}^s\!/_a\right)^2+1}} \\ &= \frac{1}{\sqrt{s^2+a^2}} \end{aligned}$$

4. $f(t) = \int\limits_0^t \frac{\sin u}{u} du$

We have $f(0) = 0$ and

$$f'(t) = \frac{\sin t}{t} \Rightarrow tf'(t) = \sin t$$

Thus

$$L\{tf'(t)\} = L\{\sin t\}$$

But from a previous example we had

$$L\{\sin t\} = \frac{1}{s^2+1}$$

By multiplication by t and the property of derivatives we have

$$L\{tf'(t)\} = -\frac{d}{ds}\left[s\overline{f}(s) - f(0)\right]$$

$$-\frac{d}{ds}\left[s\overline{f}(s) - f(0)\right] = \frac{1}{s^2+1}$$

$$\frac{d}{ds}\left[s\overline{f}(s)\right] = -\frac{1}{s^2+1}$$

Integrating both sides yields

$$sf(s) = -\tan^{-1} s + c$$

But by the initial value theorem

$$\begin{aligned}
\lim_{s\to\infty} s\overline{f}(s) &= \lim_{t\to 0} f(t) \\
&= f(0) \\
&= 0 \\
\Rightarrow c &= \frac{\pi}{2}
\end{aligned}$$

Thus

$$\begin{aligned}
s\overline{f}(s) &= -\tan^{-1} s + \frac{\pi}{2} \\
&= \tan^{-1}\frac{1}{s} \\
\Rightarrow \overline{f}(s) &= \frac{1}{s}\tan^{-1}\frac{1}{s}
\end{aligned}$$

$$\therefore L\{Si(t)\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

5. $f(t) = \int\limits_t^\infty \frac{\cos u}{u} du$

we have $f(0) = 0$ and

$$f'(t) = -\frac{\cos t}{t} \Rightarrow tf'(t) = -\cos t$$

Thus

$$L\{tf'(t)\} = L\{\sin t\}$$

But from a previous example we had

$$L\{\cos t\} = \frac{s}{s^2+1}$$

Using multiplication by t and the property of derivatives we have

$$L\{tf'(t)\} = -\frac{d}{ds}\left[s\overline{f}(s) - f(0)\right]$$

$$-\frac{d}{ds}\left[s\overline{f}(s) - f(0)\right] = \frac{-s}{s^2+1}$$

$$\frac{d}{ds}\left[s\overline{f}(s)\right] = \frac{s}{s^2+1}$$

Integrating both sides yields

$$s\overline{f}(s) = \frac{1}{2}\ln\left(s^2+1\right) + c$$

But by the final value theorem

$$\begin{aligned} \lim_{s\to 0} s\overline{f}(s) &= \lim_{t\to\infty} f(t) \\ &= f(\infty) \\ &= 0 \\ \Rightarrow c &= 0 \end{aligned}$$

Thus

$$s\overline{f}(s) = \frac{1}{2}\ln\left(s^2+1\right)$$
$$\Rightarrow \overline{f}(s) = \frac{\ln\left(s^2+1\right)}{2s}$$

$$\therefore L\left\{Ci(t)\right\} = \frac{\ln\left(s^2+1\right)}{2s}$$

6.

$$f(t) = \int_t^\infty \frac{e^{-u}}{u}du$$

we have $f(0) = 0$ and

$$f'(t) = -\frac{e^{-t}}{t} \Rightarrow tf'(t) = -e^{-t}$$

Thus

$$L\left\{tf'(t)\right\} = -L\left\{e^{-t}\right\}$$

But from a previous example we had

$$L\left\{e^{-t}\right\} = \frac{1}{s+1}$$

Using multiplication by t and the property of derivatives we have

$$L\left\{tf'(t)\right\} = -\frac{d}{ds}\left[s\overline{f}(s) - f(0)\right]$$

$$-\frac{d}{ds}\left[s\overline{f}(s) - f(0)\right] = \frac{-1}{s+1}$$

$$\frac{d}{ds}\left[s\overline{f}(s)\right] = \frac{1}{s+1}$$

Integrating both sides yields

$$s\overline{f}(s) = \ln\left(s+1\right) + c$$

But by the final value theorem

$$\begin{aligned}
\lim_{s\to 0} s\overline{f}(s) &= \lim_{t\to\infty} \\
f(t) &= f(\infty) \\
&= 0 \\
\Rightarrow c &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
s\overline{f}(s) &= \ln(s+1) \\
\Rightarrow \overline{f}(s) &= \frac{\ln(s+1)}{s}
\end{aligned}$$

$$\therefore L\{Ei(t)\} = \frac{\ln(s+1)}{s}$$

f) Use of tables

The results of the above examples can be generalized into a table of transform pairs which can be used to determine the Laplace transform of a given function.The table of transform pairs is as shown below

f(t)	$\overline{f}(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

Table 2.1: Table of transform pairs

$\sinh(at)$	$\frac{a}{s^2 - a^2}$
$\cosh(at)$	$\frac{s}{s^2 - a^2}$
$\delta(t-c)f(t)$	$e^{-sc}f(c)$
$\delta(t-c)$	e^{-sc}
$\delta(t)$	1
$u(t-c)f(t-c)$	$e^{-sc}\overline{f}(s)$
$u(t-c)$	$\frac{e^{-sc}}{s}$
$J_0(t)$	$\frac{1}{\sqrt{s^2+1}}$
$J_n(at)$	$\frac{\left(\sqrt{s^2+a^2}-s\right)^n}{a^n\left(\sqrt{s^2+a^2}\right)}$
$erf\sqrt{t}$	$\frac{1}{s\sqrt{s+1}}$

Chapter 3

Inverse Laplace Transform

3.1 Introduction

This chapter covers the definition of the Inverse Laplace transform, its properties. It will also focus on various methods of finding Inverse Laplace transform with various examples.

Definition

If $\overline{f}(s)$ is the Laplace transform of a given function $f(t)$ then $f(t)$ is called the inverse Laplace transform of $\overline{f}(s)$ .That is

$$L\left\{f(t)\right\} = \overline{f}(s) \Rightarrow L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Uniqueness of inverse Laplace transforms

If $N(t)$ is a null function, then

$$L\left\{N(t)\right\} = 0$$

Let

$$\begin{aligned} L\left\{f(t)\right\} &= \overline{f}(s) \\ L\left\{f(t) + N(t)\right\} &= L\left\{f(t)\right\} + L\left\{N(t)\right\} \text{ (by linearity)} \\ &= \overline{f}(s) + 0 = \overline{f}(s) \end{aligned}$$

From this we deduce that we can have two different functions with the same Laplace transform if null functions are permitted. In such cases the inverse Laplace transform is not unique. On the other hand, if null functions are not allowed then inverse Laplace transform will always be unique.

Lerchs theorem

For sectionally continuous functions of t say $f(t)$ in all finite intervals $0 \leqslant t \leqslant N$ such that $t < N$, these functions are of exponential order. The inverse Laplace transform of $\overline{f}(s)$ will always be unique.

3.2 Properties of Inverse Laplace transform

The properties of inverse Laplace transform are analogous to those of Laplace transforms.

i. Linearity Property

Suppose c_1 and c_2 are an two constants and $f_1(t)$ and $f_2(t)$ are functions of t with Laplace transform $\overline{f}_1(s)$ and $\overline{f}_2(s)$ respectively, then

$$\begin{aligned} L^{-1}\left\{c_1\overline{f}_1(s)+c_2\overline{f}_2(s)\right\} &= c_1L^{-1}\left\{\overline{f}_1(s)\right\}+c_2L^{-1}\left\{\overline{f}_2(s)\right\} \\ &= c_1f_1(t)+c_2f_2(t) \end{aligned}$$

ii. First translation or shifting property

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Then

$$L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at}f(t)$$

iii. Second translation or shifting property

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Then

$$L^{-1}\left\{e^{as}\overline{f}(s)\right\} = \begin{cases} f(t-a) & t < a \\ 0 & t > a \end{cases}$$

iv. Change of scale property

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Then

$$L^{-1}\left\{\overline{f}(ks)\right\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

v. Inverse Laplace transform of derivatives

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Then

$$L^{-1}\left\{\overline{f}^{n}(s)\right\} = L^{-1}\left\{\frac{\partial^n}{\partial s^n}\overline{f}(s)\right\} = (-1)^n t^n f(t)$$

vi. Inverse Laplace transform of integrals

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Then

$$L^{-1}\left\{\int_{s}^{\infty} \overline{f}(s)du\right\} = \frac{f(t)}{t}$$

vii. Multiplication by s^n

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

and $\overline{f}(0) = 0$.Then

$$L^{-1}\left\{s\overline{f}(s)\right\} = f'(t)$$

That is by multiplication by s has the effect of differentiating $f(t)$. If$\overline{f}(0) \neq 0$. Then

$$L^{-1}\left\{s\overline{f}(s) - \overline{f}(0)\right\} = f'(t) \Rightarrow L^{-1}\left\{s\overline{f}(s)\right\} = f'(t) + \overline{f}(0)\delta(t)$$

Where $\delta(t)$ is the Dirac delta function or unit impulse function.

viii. Division by s

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

Then

$$L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)du$$

Thus division by s has the effect of integrating $f(t)$ from 0 to t .

ix. Convolution property

If

$$L^{-1}\left\{\overline{f}(s)\right\} = f(t)$$

and if

$$L^{-1}\left\{\overline{g}(s)\right\} = g(t)$$

Then

$$L^{-1}\left\{\overline{f}(s)\overline{g}(s)\right\} = \int_0^t f(u)g(t-u)du = f * g$$

We call $f * g$ the convolution or faltung of f and g. we note that $f * g = g * f$

3.3 Methods Of Finding Inverse Laplace Transforms

a) Partial Fractions Method

Partial fraction expansion is also known as partial fraction decomposition. It is performed in situations where a complicated fraction is involved and there is need to express the fraction as a sum or difference of simpler fractions.

Let

$$\frac{P(s)}{Q(s)} = \frac{a_0 s^M + a_1 s^{M-1} + a_2 s^{M-2} + a_3 s^{M-3} + ... + a_M}{b_0 s^N + b_1 s^{N-1} + b_2 s^{N-2} + b_3 s^{N-3} + ... + b_N}$$

be any rational algebraic fraction . Again suppose that the degree of the denominator is less than that of the numerator, then $\frac{P(s)}{Q(s)}$ is called a proper fraction. In order to resolve this proper fraction into partial fractions we first resolve the denominator into simple factors. Some factors maybe linear, some quadratic and some of them may be repeated.. The Inverse Laplace transform of $\frac{P(s)}{Q(s)}$ is then obtained as the sum or difference of the Inverse Laplace transform of its corresponding partial fractions. There are various ways of decomposing a fraction depending on the nature of its denominator. In cases where $\frac{P(s)}{Q(s)}$ is improper that is the degree of $P(s)$ is more than that of $Q(s)$, one has to change it to a proper fraction before decomposing it into partial fractions. This can be achieved by long division as well as other methods. $\frac{P(s)}{Q(s)}$ can be decomposed into partial fractions according as

Case 1

To every linear factor $s - a$ in the denominator there corresponds a partial fraction of the form $\frac{A}{s-a}$

Case 2

To every linear factor $(s - b)^r$ repeated r times in the denominator we have partial fractions of the form

$$\frac{B_1}{s-b} + \frac{B_2}{(s-b)^2} + \frac{B_3}{(s-b)^3} + ... + \frac{B_r}{(s-b)^r}$$

Case 3

To every quadratic factor of the form $s^2 + ps + q$ in the denominator there corresponds a partial fraction of the form

$$\frac{Cs + D}{s^2 + ps + q}$$

Case 4

To every quadratic factor of the form $(s^2 + ks + \rho)^m$ repeated m times in the denominator there corresponds partial fractions of the form

$$\frac{E_1 s + F_1}{s^2 + ks + \rho} + \frac{E_2 s + F_2}{(s^2 + ks + \rho)^2} + \frac{E_3 s + F_3}{(s^2 + ks + \rho)^3} + ... + \frac{E_m s + F_m}{(s^2 + ks + \rho)^m}$$

After resolving in the way as given in the cases above we multiply both sides by the denominator. Then to evaluate the constants we equate the coefficients of different powers of and solve the resulting equations. With these constants we can now express $\frac{P(s)}{Q(s)}$ as a sum or difference of its corresponding partial fractions. A method related to this uses the Heaviside expansion formula.

Examples

1. Consider a function $f(t)$ whose Laplace transform is given as

$$\overline{f}(s) = \frac{s + 3}{s^3 + 7s^2 + 10s}$$

We need to determine $f(t)$. Here $P(s) = s + 3$ and $Q(s) = s^3 + 7s^2 + 10s$.

$Q(s)$ can be factorized as $s^3 + 7s^2 + 10s = s(s + 2)(s + 5)$. Thus $Q(s)$ has three distinct roots namely $s = 0, s = -2$ and $s = -5$. With this $\overline{f}(s)$ can be decomposed as follows.

$$\begin{aligned} \overline{f}(s) &= \frac{s + 3}{s(s + 2)(s + 3)} \\ &= \frac{A_1}{s} + \frac{A_2}{s + 2} + \frac{A_3}{s + 5} \end{aligned}$$

The next step is to determine the coefficients A_1, A_2 and A_3 . For A_1 we multiply both sides of the above equation by s which yields

$$\frac{s+3}{(s+2)(s+5)} = A_1 + \frac{sA_2}{s+2} + \frac{sA_3}{s+5}$$

Setting $s = 0$ implies

$$\frac{3}{(2)(5)} = A_1$$
$$\therefore A_1 = \frac{3}{10}$$

For A_2 we multiply both sides of the above equation by $s + 2$ which yields

$$\frac{s+3}{s(s+5)} = \frac{A_1(s+2)}{s} + A_2 + \frac{A_3(s+2)}{s+5}$$

Setting $s = -2$ implies

$$\frac{(-2+3)}{-2(-2+5)} = A_2$$
$$\therefore A_2 = -\frac{1}{6}$$

Similarly to obtain A_3 we multiply both sides of the above equation by $s+5$ which yields

$$\frac{s+3}{s(s+2)} = \frac{A_1(s+5)}{s} + \frac{A_2(s+5)}{(s+2)} + A_3$$

Setting $s = -5$ implies

$$\frac{(-5+3)}{-5(-5+2)} = A_3$$
$$\therefore A_3 = -\frac{2}{15}$$

With these coefficients $\overline{f}(s)$ can now be expressed as

$$\overline{f}(s) = \frac{3}{10}\left(\frac{1}{s}\right) - \frac{1}{6}\left(\frac{1}{s+2}\right) - \frac{2}{15}\left(\frac{1}{s+5}\right)$$

Applying inverse Laplace transform to both sides yields

$$f(t) = \frac{3}{10}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+2}\right\} - \frac{2}{15}L^{-1}\left\{\frac{1}{s+5}\right\}$$

Which from the table of transform pairs implies that

$$f(t) = \frac{3}{10} - \frac{1}{6}e^{-2t} - \frac{2}{15}e^{-5t}$$

2.

$$\overline{f}(s) = \frac{s^2 - 10s + 13}{(s-1)(s-2)(s-3)}$$

Here $P(s)$ is $s^2 - 10s + 13$ and $Q(s)$ is $(s-1)(s-2)(s-3)$. We can decompose $\overline{f}(s)$ into partial fractions as shown below.

$$\frac{s^2 - 10s + 13}{(s-1)(s-2)(s-3)} = \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{A_3}{s-3}$$

Multiplying both sides by $Q(s)$ yields

$$\begin{aligned} s^2 - 10s + 13 &= A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2) \\ \Rightarrow s^2 - 10s + 13 &= A_1\left(s^2 - 5s + 6\right) + A_2\left(s^2 - 4s + 3\right) + A_3\left(s^2 - 3s + 2\right) \\ &= (A_1 + A_2 + A_3)s^2 + (-5A_1 - 4A_2 - 3A_3)s + (6A_1 + 3A_2 + 2A_3) \end{aligned}$$

Equating the coefficients of powers of s on both sides, we have the following equations.

For s^2

$$1 = A_1 + A_2 + A_3 \qquad (i)$$

For s we have

$$-10 = -5A_1 - 4A_2 - 3A_3 \qquad (ii)$$

Equating constants yields

$$13 = 6A_1 + 3A_2 + 2A_3 \qquad (iii)$$

Putting $A_1 = 1 - A_2 - A_3$ from (i) in (ii) and (iii) we get

$$-10 = -5(1 - A_2 - A_3) - 4A_2 - 3A_3$$

And

$$\begin{aligned} 13 &= 6(1 - A_2 - A_3) + 3A_2 + 2A_3 \\ \Rightarrow -10 &= -5 + 5A_2 + 5A_3 - 4A_2 - 3A_3 \\ 13 &= 6 - 6A_2 - 6A_3 + 3A_2 + 2A_3 \end{aligned}$$

$$\Longrightarrow \left.\begin{aligned} -5 &= A_2 + 2A_3 \\ 7 &= -3A_2 - 4A_3 \end{aligned}\right\} \qquad (iv)$$

$$\begin{array}{r} 2A_2 + 4A_3 = -10 \\ -3A_2 - 4A_3 = \quad 7 \\ \hline A_2 = \ -3 \end{array}$$

$$\Rightarrow \boxed{A_2 = 3}$$

From (iv) we had

$$\begin{aligned} -5 &= A_2 + 2A_3 \\ -5 &= 3 + 2A_3 \\ -8 &= 2A_3 \\ \Rightarrow &\boxed{A_3 = -4} \end{aligned}$$

Therefore from (i)

$$\begin{aligned} 1 &= A + B + C \\ 1 &= A + 3 - 4 \\ \Rightarrow &\boxed{A = 2} \end{aligned}$$

$$\therefore \frac{s^2 - 10s + 13}{(s-1)(s-2)(s-3)} = \frac{2}{s-1} + \frac{3}{s-2} - \frac{4}{s-3}$$

Applying inverse Laplace transform to both sides yields

$$L^{-1}\left\{\frac{s^2-10s+13}{(s-1)(s-2)(s-3)}\right\} = L^{-1}\left\{\frac{2}{s-1}+\frac{3}{s-2}-\frac{4}{s-3}\right\}$$
$$= 2L^{-1}\left\{\frac{1}{s-1}\right\}+3L^{-1}\left\{\frac{1}{s-2}\right\}-4L^{-1}\left\{\frac{1}{s-3}\right\}$$

Which from the table of transform pairs implies that

$$L^{-1}\left\{\frac{s^2-10s+13}{(s-1)(s-2)(s-3)}\right\} = 2e^t+3e^{2t}-4e^{3t}$$

3.

$$\overline{f}(s) = \frac{4s+5}{(s-1)^2(s-2)}$$

In this case $P(s) = 4s+5$ and $Q(s) = (s-1)^2(s-2)$. $Q(s)$ has one distinct root at and one repeated root at $s = 1$.

We therefore split $\overline{f}(s)$ into its corresponding partial fractions as follows

$$\frac{4s+5}{(s-1)^2(s-2)} = \frac{A_1}{(s-1)^2}+\frac{A_2}{s-1}+\frac{A_3}{s+2}$$

Multiplying both sides by $(s-1)^2(s-2)$ yields

$$\Rightarrow 4s+5 = A_1(s+2)+A_2(s-1)(s+2)+A_3(s-1)^2$$
$$\Rightarrow 4s+5 = A_1(s+2)+A_2\left(s^2+s-2\right)+A_3\left(s^2-2s+1\right)$$
$$4s+5 = (A_2+A_3)s^2+(A_1+A_2-2A_3)s+(2A_1-2A_2+A_3)$$

We now equate the coefficients of s corresponding powers of on both sides of the above equation.
For s^2 we have

$$0 = A_2+A_3 \qquad (i)$$

Equating coefficients of s we have

$$4 = A_1+A_2-2A_3 \qquad (ii)$$

And equating constants yields

$$5 = 2A_1 - 2A_2 + A_3 \qquad (iii)$$

Using equation (i) in equations (ii) and (iii) we get

$$4 = A_1 - A_3 - 2A_3$$

And

$$5 = 2A_1 - 2(-A_3) + A_3$$

$$\implies \left.\begin{aligned} 4 &= A_1 - 3A_3 \\ 5 &= 2A_1 + 3A_3 \end{aligned}\right\} \qquad (iv)$$

$$\begin{aligned} A_1 - 3A_3 &= 4 \\ 2A_1 + 3A_3 &= 5 \\ \hline 3A_1 &= 9 \end{aligned}$$

$$\Rightarrow \boxed{A_1 = 3}$$

From (iv) we had

$$4 = A_1 - 3A_3$$

$$3A_3 = A_1 - 4$$

$$3A_3 = 3 - 4$$

$$\Rightarrow \boxed{A_3 = -\frac{1}{3}}$$

Therefore from (i)

$$A_2 = -A_3$$

$$\Rightarrow \boxed{A_2 = \frac{1}{3}}$$

With this we now have

$$\frac{4s+5}{(s-1)^2(s-2)} = \frac{3}{(s-1)^2} + \frac{1}{3}\left(\frac{1}{s-1}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

Thus

$$\begin{aligned} L^{-1}\left\{\frac{4s+5}{(s-1)^2(s-2)}\right\} &= L^{-1}\left\{\frac{3}{(s-1)^2} + \frac{1}{3}\left(\frac{1}{s-1}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)\right\} \\ &= 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\} \end{aligned}$$

Using the table of transform pairs it follows that

$$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s-2)}\right\} = 3te^t + \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

4.

$$\overline{f}(s) = \frac{5s+3}{(s-1)\left(s^2+2s+5\right)}$$

In this case $P(s) = 5s+3$ and $Q(s) = (s-1)(s^2+2s-5)$. $Q(s)$ is a product of a linear term and a quadratic term. We therefore split $\overline{f}(s)$ into its corresponding partial fractions as follows

$$\frac{5s+3}{(s-1)\left(s^2+2s+5\right)} = \frac{A_1s+A_2}{s^2+2s+5} + \frac{A_3}{s-1}$$

Multiplying both sides of the above equation by $(s-1)\left(s^2+2s+5\right)$ yields

$$\begin{aligned} 5s+3 &= (As+B)(s-1) + C\left(s^2+2s+5\right) \\ 5s+3 &= As^2 + Bs - As - B + Cs^2 + 2Cs + 5C \\ 5s+3 &= (A+C)s^2 + (B-A+2C)s + (5C-B) \end{aligned}$$

To obtain A_1, A_2 and A_3 we equate the coefficients of the corresponding powers of s

Equating coefficients of s^2 we have

$$0 = A + C \qquad\qquad (i)$$

Equating coefficients of s we have

$$5 = B - A + 2C \qquad (ii)$$

And equating constants yields

$$3 = 5C - B \qquad (iii)$$

Putting $A = -C$ from (i) in (ii) and (iii) we get

$$5 = B - (-C) + 2C$$

And

$$3 = 5C - B$$

$$\Longrightarrow \left.\begin{aligned} 5 &= B + 3C \\ 3 &= -B + 5C \end{aligned}\right\} \qquad (iv)$$

$$\begin{array}{r} B + 3C = 5 \\ -B + 5C = 3 \\ \hline 8C = 8 \end{array}$$

$$\Rightarrow \boxed{C = 1}$$

Therefore from (i)

$$A = -C$$
$$\Rightarrow \boxed{A = -1}$$

From (iii) we had

$$3 = 5C - B$$
$$3 = 5(1) - B$$
$$\Rightarrow \boxed{B = 2}$$

With these coefficients we have

$$\frac{5s+3}{(s-1)\left(s^2+2s+5\right)}=\frac{2-s}{s^2+2s+5}+\frac{1}{s-1}$$

$$\therefore \quad \frac{5s+3}{(s-1)\left(s^2+2s+5\right)}=\frac{2-s}{s^2+2s+5}+\frac{1}{s-1}$$

Thus

$$L^{-1}\left\{\frac{5s+3}{(s-1)\left(s^2+2s+5\right)}\right\}=L^{-1}\left\{\frac{2-s}{s^2+2s+5}+\frac{1}{s-1}\right\}$$

$$=L^{-1}\left\{\frac{3-(s+1)}{s^2+2s+5}+\frac{1}{s-1}\right\}$$

$$=L^{-1}\left\{\frac{3-(s+1)}{(s+1)^2+4}+\frac{1}{s-1}\right\}$$

$$=L^{-1}\left\{\frac{3}{(s+1)^2+4}-\frac{s+1}{(s+1)^2+4}+\frac{1}{s-1}\right\}$$

$$=3L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}-L^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}+L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$=\frac{3}{2}L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}-L^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}+L^{-1}\left\{\frac{1}{s-1}\right\}$$

But from the table of transform pairs we have $L\left(\cos\ at\right)=\frac{s}{s^2+a^2}$ and $L\left(\sin\ at\right)=\frac{a}{s^2+a^2}$. Relating this to the above equation, it follows that

$$L^{-1}\left\{\frac{5s+3}{(s-1)\left(s^2+2s+5\right)}\right\}=\frac{3}{2}e^{-t}\sin 2t-e^{-t}\cos 2t+e^{t}$$

5.

$$\overline{f}(s)=\frac{s}{s^4+4a^4}$$

By factorizing s^4+4a^4 , $\overline{f}(s)$ can be expressed as

$$\frac{s}{s^4+4a^4}=\frac{s}{(s^2-2as+2a^2)\,(s^2+2as+2a^2)}$$

In this case $Q(s)$ is a product of two quadratic terms, we thus decompose

$$\frac{s}{s^4+4a^4}$$

into its corresponding partial fractions as follows

$$\frac{s}{s^4+4a^4}=\frac{A_1s+A_2}{s^2-2as+2a^2}+\frac{A_3s+A_4}{s^2+2as+2a^2}$$

Multiplying both sides by s^4+4a^4 yields

$$s=(A_1s+A_2)\left(s^2+2as+2a^2\right)+(A_3s+A_4)\left(s^2-2as+2a^2\right)$$

$$s=A_1s^3+2A_1as^2+2A_1sa^2+A_2s^2+2A_2as+2A_2a^2+A_3s^3-2A_3as^2$$
$$+\,2A_3sa^2+A_4s^2-2A_4as+2A_4a^2$$

$$s=(A_1+A_3)\,s^3+(2A_1a+A_2-2A_3a+A_4)\,s^2+\left(2A_1a^2+2A_3a+2A_2a^2-2A_4a\right)s$$
$$+\left(2A_2a^2+2A_4a^2\right)$$

To obtain A_1, A_2, A_3 and A_4 we equate the coefficients of the corresponding powers of s
Equating coefficients of s^3

$$0=A_1+A_3 \qquad (i)$$

Equating coefficients of s^2

$$0=2A_1a+A_2-2A_3a+A_4 \qquad (ii)$$

Equating coefficients of s we have

$$1=2A_1a^2+2A_2a+2A_3a^2-2A_4a \qquad (iii)$$

And equating constants yields

$$0 = 2A_2a^2 + 2A_4a^2 \qquad (iv)$$

Putting $A = -C$ from (i) in (iii) we have

$$\begin{aligned}1 &= 2(-A_3)a^2 + 2A_2a + 2A_3a^2 - 2A_4a \\ 1 &= 2A_2a - 2A_4a \qquad (v)\end{aligned}$$

But from (iv)

$$A_2 = -A_4 \qquad (vi)$$

Substituting $A_2 = -A_4$ in (iv) yields

$$\begin{aligned}1 &= 2(-A_4)a - 2A_4a \\ 1 &= -4aA_4 \\ &\Rightarrow \boxed{A_4 = -\frac{1}{4a}}\end{aligned}$$

From (vi) we had

$$\begin{aligned}&A_2 = -A_4 \\ &\Rightarrow \boxed{A_2 = \frac{1}{4a}}\end{aligned}$$

From equation (ii) we obtained $0 = 2A_1a + A_2 - 2A_3a + A_4$ but if $A_2 = -A_4$ then

$$\begin{aligned}&0 = 2A_1a + (-A_4) - 2A_3a + A_4 \\ &2A_3a = 2A_1a \\ &\Rightarrow A_1 = A_3\end{aligned}$$

But from (i) $A_1 = -A_3$ hence

$$\Rightarrow \boxed{A_1 = A_3 = 0}$$

$$\therefore \quad \frac{s}{s^4+4a^4} = \frac{(0)s+\left(\frac{1}{4a}\right)}{s^2-2as+2a^2} + \frac{(0)s-\left(\frac{1}{4a}\right)}{s^2+2as+2a^2}$$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s-a)^2+a^2}\right]$$

$$= \frac{1}{4a^2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s-a)^2+a^2}\right]$$

Thus

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = L^{-1}\left\{\frac{1}{4a^2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s-a)^2+a^2}\right]\right\}$$

$$= \frac{1}{4a^2}L^{-1}\left\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s-a)^2+a^2}\right\}$$

$$= \frac{1}{4a^2}\left[L^{-1}\left\{\frac{a}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{a}{(s-a)^2+a^2}\right\}\right]$$

But from the table of transform pairs we have

$$L\left(\sin\ \mathrm{a}t\right) = \frac{a}{s^2+a^2}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{4a^2}\left[e^{-at}\sin at - e^{at}\sin at\right]$$

b) Series method

If $\overline{f}(s)$ has a series of expansion in inverse powers of s given by

$$\overline{f}(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \frac{a_4}{s^5} + ...$$

Then under suitable conditions we can invert term by term to obtain

$$f(t) = a_0 + a_1 t + a_2\left(\frac{t^2}{2}\right) + a_3\left(\frac{t^3}{3}\right) + a_4\left(\frac{t^4}{4}\right) + ...$$

Example

$$\overline{f}(s) = \frac{e^{-1/s}}{s}$$

$\overline{f}(s)$ can be expressed as an infinite series as follows

$$\frac{e^{-1/s}}{s} = \frac{1}{s}\left[1 - \frac{1}{s} + \frac{1}{2!s^2} - \frac{1}{3!s^3} + \frac{1}{4!s^4} + ...\right]$$

$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{2!s^3} - \frac{1}{3!s^4} + \frac{1}{4!s^5} - ....$$

$$L^{-1}\left\{\frac{e^{-1/s}}{s}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s^2} + \frac{1}{2!s^3} - \frac{1}{3!s^4} + \frac{1}{4!s^5} - ....\right\}$$

Here $a_0 = 1, a_1 = -1, a_2 = \frac{1}{2!}, a_3 = -\frac{1}{3!}, a_4 = \frac{1}{4!}, ...$ thus we have

$$L^{-1}\left\{\frac{e^{-\frac{1}{s}}}{s}\right\} = 1 - t + \frac{1}{2!}\left(\frac{t^2}{2}\right) - \frac{1}{3!}\left(\frac{t^3}{3}\right) + \frac{1}{4!}\left(\frac{t^4}{4}\right)....$$

$$= 1 - t + \frac{t^2}{4} - \frac{t^3}{18} + \frac{t^4}{96}....$$

2.

$$\overline{f}(s) = e^{-\sqrt{s}}$$

By infinite series expansion, we have

$$e^{-\sqrt{s}} = 1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \frac{s^{5/2}}{5!} + ...$$

This implies

$$L^{-1}\left\{e^{-\sqrt{s}}\right\} = L^{-1}\left\{1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \frac{s^{5/2}}{5!} + ...\right\}$$

$$L^{-1}\left\{e^{-\sqrt{s}}\right\} = \left\{\begin{array}{c} L^{-1}\left\{1\right\} - L^{-1}\left\{s^{1/2}\right\} + L^{-1}\left\{\frac{s}{2!}\right\} - L^{-1}\left\{\frac{s^{3/2}}{3!}\right\} + \\ L^{-1}\left\{\frac{s^2}{4!}\right\} - L^{-1}\left\{\frac{s^{5/2}}{5!}\right\} + ... \end{array}\right\} \tag{3.1}$$

From the table of Laplace transform pairs we have

$$L\left\{t^n\right\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

Letting $n = -p - \frac{3}{2}$ implies that

$$\begin{aligned} L\left\{t^{-p-3/2}\right\} &= \frac{\Gamma\left(-p - 3/2 + 1\right)}{s^{-p-3/2+1}} \\ &= \frac{\Gamma\left(-p - 1/2\right)}{s^{-p-1/2}} \end{aligned}$$

$$\Rightarrow \frac{L\left\{t^{-p-3/2}\right\}}{\Gamma\left(-p - 1/2\right)} = \frac{1}{s^{-p-1/2}}$$

$$\therefore L^{-1}\left\{s^{p+1/2}\right\} = \frac{t^{-p-3/2}}{\Gamma\left(-p - 1/2\right)}$$

But using the property that for $P \geqslant 0$,

$$\Gamma\left(-p - \frac{1}{2}\right) = (-1)^{p+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\left(\frac{2}{7}\right)...\left(\frac{2}{2p+1}\right)\sqrt{\pi}$$

We have

$$L^{-1}\left\{s^{p+1/2}\right\} = \frac{t^{-p-3/2}}{(-1)^{p+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\left(\frac{2}{7}\right)...\left(\frac{2}{2p+1}\right)\sqrt{\pi}}$$

$$= \frac{(-1)^{p+1}}{\sqrt{\pi}}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)...\left(\frac{2p+1}{2}\right)t^{-p-3/2} \tag{3.2}$$

But $L^{-1}(s^p) = 0$, Then using (2.1) on (2.2) we have

$$L^{-1}\left(e^{-\sqrt{s}}\right) = \frac{t^{-3/2}}{2\sqrt{\pi}} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{t^{-5/2}}{3!\sqrt{\pi}} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{t^{-7/2}}{5!\sqrt{\pi}} - ...$$

$$= \frac{1}{2\sqrt{\pi}t^{3/2}}\underbrace{\left\{1 - \left(\frac{1}{2^2t}\right) + \frac{1}{2!}\left(\frac{1}{2^2t}\right)^2 - \frac{1}{3!}\left(\frac{1}{2^2t}\right)^3 + ...\right\}}_{e^{-1/4t}}$$

$$= \frac{1}{2\sqrt{\pi}t^{3/2}}e^{-\frac{1}{4t}}$$

c) Method of differential equations

This method is applied whenever a function $\overline{f}(s)$ satisfies a differential equation.

Example

$$\overline{f}(s) = e^{-\sqrt{s}}$$

Differentiating $\overline{f}(s)$ twice we have

$$\begin{aligned}
\frac{d}{ds}\overline{f}(s) &= \frac{d}{ds}e^{-\sqrt{s}} \\
&= -\frac{e^{-\sqrt{s}}}{2\sqrt{s}} \\
\frac{d^2}{ds^2}\overline{f}(s) &= \frac{d^2}{ds^2}e^{-\sqrt{s}} \\
&= \frac{d}{ds}\left(-\frac{e^{-\sqrt{s}}}{2\sqrt{s}}\right) \\
&= \frac{e^{-\sqrt{s}}\big/2\sqrt{s}\,(2\sqrt{s}) + 1\big/\sqrt{s}e^{-\sqrt{s}}}{\left(2\sqrt{s}\right)^2} \\
&= \frac{e^{-\sqrt{s}} + 1\big/\sqrt{s}e^{-\sqrt{s}}}{\left(2\sqrt{s}\right)^2} \\
&= \frac{e^{-\sqrt{s}}\left(1 + 1\big/\sqrt{s}\right)}{\left(2\sqrt{s}\right)^2} \\
&= \frac{e^{-\sqrt{s}}}{4s} + \frac{e^{-\sqrt{s}}}{4\left(\sqrt{s}\right)^3}
\end{aligned}$$

It can be seen that

$$4s\frac{d^2}{ds^2}\overline{f}(s) + \frac{d}{ds}\overline{f}(s) - \overline{f}(s) = 0$$

But by the property of derivatives

$$L^{-1}\left\{\frac{d^n}{ds^n}\overline{f}(s)\right\} = (-1)^n\, t^n f(t)$$

Thus when $n = 2$ we have

$$\begin{aligned} L^{-1}\left\{\frac{d^2}{ds^2}\overline{f}(s)\right\} &= (-1)^2\, t^2 f(t) \\ &= t^2 f(t) \end{aligned}$$

Similarly when $n = 1$ we have

$$\begin{aligned} L^{-1}\left\{\frac{d}{ds}\overline{f}(s)\right\} &= (-1)\, t f(t) \\ &= -t f(t) \end{aligned}$$

Again by the multiplication by s^n property we have

$$L^{-1}\left\{s\overline{f}(s)\right\} = f'(t)$$

Thus

$$\begin{aligned} &\frac{d^2}{ds^2}\overline{f}(s) = L\left\{t^2 f(t)\right\} \\ &\Rightarrow s\frac{d^2}{ds^2}\overline{f}(s) = L\left\{\frac{d}{dt}\left[t^2 f(t)\right]\right\} \end{aligned}$$

Using implicit differentiation, we have

$$s\frac{d^2}{ds^2}\overline{f}(s) = L\left\{\frac{d}{dt}2tf(t) + t^2 f(t)\right\}$$

With this, the differential equation becomes

$$4L\left\{\frac{d}{dt}2tf(t) + t^2 f(t)\right\} - 2L\left\{tf(t)\right\} - L\left\{f(t)\right\} = 0$$

Letting $Y = f(t)$, we have

$$\begin{aligned} 4L\left\{2tY + t^2Y'\right\} - 2L\left\{tY\right\} - L\left\{Y\right\} = 0 \\ \Rightarrow 4t^2Y' + 6tY - Y = 0 \\ \Rightarrow \frac{dY}{dt} + \left(\frac{6t-1}{4t^2}\right)Y = 0 \\ \Rightarrow \frac{dY}{Y} + \left(\frac{6t-1}{4t^2}\right)dt = 0 \\ \Rightarrow \frac{dY}{Y} + \frac{3}{2t}dt - \frac{1}{4t^2}dt = 0 \end{aligned}$$

Integrating both sides yields

$$\begin{aligned} \ln Y + \frac{3}{2}\ln t + \frac{1}{4t} = c \\ \Rightarrow Y = \frac{c}{t^{3/2}}e^{-1/4t} \end{aligned}$$

Now $tY = \frac{c}{t^{1/2}}e^{-1/4t}$
Thus

$$\begin{aligned} L\left\{tY\right\} &= -\frac{d}{ds}L\left\{Y\right\} \\ &= -\frac{d}{ds}\left(e^{-\sqrt{s}}\right) \\ &= \frac{e^{-\sqrt{s}}}{2\sqrt{s}} \end{aligned}$$

For Large t , $tY \sim \frac{c}{t^{\frac{1}{2}}}$ and $L\left\{tY\right\} \sim \frac{c\sqrt{\pi}}{s^{\frac{1}{2}}}$. For small s , $\frac{e^{-\sqrt{s}}}{2\sqrt{s}} \sim \frac{1}{2s^{\frac{1}{2}}}$. Hence by the final value theorem,$c\sqrt{\pi} = \frac{1}{2}$ or $c = \frac{1}{2\sqrt{\pi}}$. It follows that

$$L^{-1}\left\{e^{-\sqrt{s}}\right\} = \frac{1}{2\sqrt{\pi}t^{3/2}}e^{-1/4t}$$

d) Differentiation with respect to a parameter

In this method, the inverse laplace transform of a given function is obtained by differentiating a known Laplace transform of a function with respect to a parameter of interest and relating it to the given function.

Examples

1.

$$\overline{f}(s) = \frac{s}{\left(s^2+a^2\right)^2}$$

Differentiating $\frac{s}{s^2+a^2}$ with respect to a, we have

$$\frac{d}{da}\left(\frac{s}{s^2+a^2}\right) = \frac{-2as}{\left(s^2+a^2\right)^2}$$

Taking the inverse laplace transform of both sides yields

$$L^{-1}\left\{\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)\right\} = L^{-1}\left\{\frac{-2as}{\left(s^2+a^2\right)^2}\right\}$$

$$\Rightarrow \frac{d}{da}L^{-1}\left\{\left(\frac{s}{s^2+a^2}\right)\right\} = -2aL^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} = -\frac{1}{2a}\frac{d}{da}L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$$

From the table of transform pairs we have

$$L\left\{\cos at\right\} = \frac{s}{s^2+a^2}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

Thus

$$\begin{aligned} L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} &= -\frac{1}{2a}\frac{d}{da}L^{-1}\left\{\frac{s}{s^2+a^2}\right\} \\ &= -\frac{1}{2a}\frac{d}{da}\cos at \\ &= -\frac{1}{2a}\left(-t\sin at\right) \\ \therefore L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} &= \frac{t\sin at}{2a} \end{aligned}$$

2.

$$\overline{f}(s) = \frac{1}{(s^2+a^2)^{\frac{3}{2}}}$$

From the table of transform pairs we have

$$L\{J_0(at)\} = \frac{1}{\sqrt{s^2+a^2}}$$

Differentiating both sides with respect to a yields

$$\begin{aligned}
\frac{d}{da}L\{J_0(at)\} &= \frac{d}{da}\frac{1}{\sqrt{s^2+a^2}}\\
L\left\{\frac{d}{da}J_0(at)\right\} &= \frac{-a}{(s^2+a^2)^{\frac{3}{2}}}\\
L\{tJ_0'(at)\} &= \frac{-a}{(s^2+a^2)^{\frac{3}{2}}}\\
\frac{a}{(s^2+a^2)^{\frac{3}{2}}} &= L\{-tJ_0'(at)\}\\
\Rightarrow L^{-1}\left\{\frac{a}{(s^2+a^2)^{\frac{3}{2}}}\right\} &= -tJ_0'(at)\\
\Rightarrow L^{-1}\left\{\frac{1}{(s^2+a^2)^{\frac{3}{2}}}\right\} &= -\frac{t}{a}J_0'(at)
\end{aligned}$$

But by the derivatives property of the Bessel function we had

$$J_0'(a) = -J_1(a)$$

Thus

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^{\frac{3}{2}}}\right\} = \frac{tJ_1(at)}{a}$$

3.

$$\overline{f}(s) = \ln\left(\frac{s^2+1}{s^2+4}\right)$$

Taking inverse Laplace transform of both sides yields

$$L^{-1}\{\overline{f}(s)\} = \frac{1}{t}L^{-1}\left\{\overline{f}'(s)\right\}$$

$$
\begin{aligned}
\overline{f}(s) &= \ln\left(\frac{s^2+1}{s^2+4}\right)\\
\Rightarrow \overline{f}'(s) &= \left(\frac{s^2+4}{s^2+1}\right)\left[\frac{2s\left(s^2+4\right)-2s\left(s^2+1\right)}{\left(s^2+4\right)^2}\right]\\
&= \frac{6s}{\left(s^2+1\right)\left(s^2+4\right)}
\end{aligned}
$$

$$
L^{-1}\left\{\overline{f}'(s)\right\} = L^{-1}\left\{3\left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^2+4}\right)\right\}
$$

Let

$$
\begin{aligned}
\overline{f}_1(s) &= \frac{s}{s^2+1}\\
\overline{f}_2(s) &= \frac{2}{s^2+4}\\
\therefore L^{-1}\left\{\overline{f}'(s)\right\} &= 3L^{-1}\left\{\overline{f}_1(s)\overline{f}_2(s)\right\}
\end{aligned}
$$

From table of transform pairs

$$
\begin{aligned}
L^{-1}\left\{\frac{s}{s^2+a^2}\right\} &= \cos at\\
L^{-1}\left\{\frac{a}{s^2+a^2}\right\} &= \sin at
\end{aligned}
$$

Consequently

$$
\begin{aligned}
f_1(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} &= \cos t\\
f_2(t) = L^{-1}\left\{\frac{2}{s^2+4}\right\} &= \sin 2t
\end{aligned}
$$

Thus

$$
\begin{aligned}
L^{-1}\left\{\overline{f}_1(s)\overline{f}_2(s)\right\} &= f_1(t) * f_2(t)\\
&= \cos t * \sin 2t
\end{aligned}
$$

Therefore

$$L^{-1}\left\{\overline{f}(s)\right\} = \frac{1}{t}L^{-1}\left\{\overline{f}'(s)\right\}$$
$$= \frac{3}{t}\left(\cos t * \sin 2t\right)$$

By convolution property of inverse Laplace transforms we have

$$\frac{3}{t}\left(\cos t * \sin 2t\right) = \frac{3}{t}\int_0^t \cos(t-\tau)\sin 2t d\tau$$
$$= \frac{6}{t}\int_0^t \left(\cos t\cos\tau + \sin t\sin\tau\right)\cos\tau\sin\tau d\tau$$
$$= \frac{6}{t}\left[\int_0^t \left(\cos^2\tau\sin\tau\right)\cos t d\tau + \int_0^t \left(\sin^2\tau\cos\tau\right)\sin t d\tau\right]$$
$$= \frac{6}{t}\left[\int_0^t \left(\cos^2\tau\sin\tau\right)\cos t d\tau + \int_0^t \left(\sin^2\tau\cos\tau\right)\sin t d\tau\right]$$
$$= \frac{6}{t}\left[\cos t\left(\frac{\cos^3\tau}{3}\right) + \sin t\left(\frac{\sin^3\tau}{3}\right)\right]$$
$$= \frac{6}{t}\left[\cos^4 t - \sin^4 t - 1\right]$$

e)Miscellaneous method

This method utilizes the properties of inverse Laplace transforms as well as the table of transform pairs .

1.

$$\overline{f}(s) = \frac{3s}{(3s)^2 + 9}$$

From the table of transform pairs we have;

$$L^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos at$$
$$\Rightarrow f(t) = L^{-1}\left\{\frac{s}{s^2 + 9}\right\} = \cos 3t$$

But by the change of scale property we have

$$
\begin{aligned}
L^{-1}\left\{\overline{f}(as)\right\} &= \frac{1}{a}f\left(\frac{t}{a}\right) \\
f(t) = \cos 3t \ \Rightarrow f\left(\frac{t}{a}\right) &= \cos t \\
\Rightarrow \frac{1}{a}f\left(\frac{t}{a}\right) &= \frac{1}{3}\cos t \\
\therefore L^{-1}\left\{\frac{3s}{(3s)^2+9}\right\} &= \frac{1}{3}\cos t
\end{aligned}
$$

2.

$$
\overline{f}(s) = \frac{3!}{(s-2)^4}
$$

Letting

$$
\frac{3!}{(s-2)^4} = \overline{f}(s-a) \Rightarrow f(s) = \frac{3!}{s^4}
$$

By first shifting property of Laplace transforms we have

$$
L^{-1}\left\{f(s-a)\right\} = e^{at}f(t) \quad \text{in our case} \alpha = 2
$$

From tables of transform pairs we have

$$
\begin{aligned}
& L\left\{t^n\right\} = \frac{n!}{s^{n+1}} \\
& \Rightarrow f(t) = L\left\{t^3\right\} = \frac{3!}{s^4} \\
& \therefore L^{-1}\left\{\frac{3!}{(s-2)^4}\right\} = e^{2t}f(t) = e^{2t}t^3
\end{aligned}
$$

3.

$$\overline{f}(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+2}$$

By completing the square method the function yields

$$\begin{aligned}\frac{2(s-1)}{s^2-2s+2} &= \frac{2(s-1)}{(s^2-2s+1)+2-1} \\ &= \frac{2(s-1)}{(s-1)^2+1}\end{aligned}$$

From tables of transform pairs we have

$$f(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

By first shifting property of Laplace transforms we have $L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at}f(t)$ in our case $a = 1$

Consequently

$$L^{-1}\left\{\frac{(s-1)}{(s-1)^2+1}\right\} = e^t \cos t$$

From tables of transform pairs we have

$$L\left\{\mu(t-c)f(t-c)\right\} = e^{-cs}\overline{f}(s)$$

If we let

$$f(t) = e^t \cos t \Rightarrow \overline{f}(s) = \frac{(s-1)}{(s-1)^2+1}$$

Our expression which was $\frac{2(s-1)e^{-2s}}{s^2-2s+2}$ will now be $2e^{-2s}\overline{f}(s)$

Hence $c = 2$

$$2L^{-1}\left\{e^{-2s}\overline{f}(s)\right\} = 2\mu(t-c)f(t-c)$$

Now

$$\begin{aligned}f(t) &= e^t \cos t \\ \Rightarrow f(t-2) &= e^{t-2}\cos(t-2)\end{aligned}$$

$$L^{-1}\left\{\frac{2(s-1)e^{-2s}}{s^2-2s+2}\right\} = 2\mu(t-2)e^{t-2}\cos(t-2)$$

4.

$$\overline{f}(s) = \frac{3}{s}\left(e^{-2s} - e^{-3s}\right)$$

This can be expressed otherwise as

$$\begin{aligned}\overline{f}(s) &= \frac{3}{s}\left(e^{-2s} - e^{-3s}\right)\\ &= 3\left(\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\right)\end{aligned}$$

Consequently

$$L^{-1}\left\{\overline{f}(s)\right\} = 3\left(L^{-1}\left\{\frac{e^{-2s}}{s}\right\} - L^{-1}\left\{\frac{e^{-3s}}{s}\right\}\right)$$

But from tables of transform pairs we have

$$\begin{aligned}L^{-1}\left\{\frac{e^{-as}}{s}\right\} &= \mu(t-a)\\ \Rightarrow L^{-1}\left\{\frac{e^{-2s}}{s}\right\} &= \mu(t-2)\\ L^{-1}\left\{\frac{e^{-3s}}{s}\right\} &= \mu(t-3)\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{3}{s}\left(e^{-2s} - e^{-3s}\right)\right\} = 3\left[\mu\left(t-3\right) - \mu\left(t-3\right)\right]$$

5.

$$\overline{f}(s) = \frac{s}{\left(s^2+a^2\right)^2}$$

From the table of transform pairs we have

$$\begin{aligned}L^{-1}\left\{\frac{s}{s^2+a^2}\right\} &= \cos at\\ L^{-1}\left\{\frac{a}{s^2+a^2}\right\} &= \sin at \Rightarrow L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}\end{aligned}$$

We can write

$$\frac{s}{\left(s^2+a^2\right)^2} = \frac{s}{\left(s^2+a^2\right)} \times \frac{1}{\left(s^2+a^2\right)}$$

By the convolution theorem it follows that

$$\begin{aligned}
L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= \int_0^t \cos au.\frac{\sin a(t-u)}{a}du \\
&= \frac{1}{a}\int_0^t (\cos au)(\sin at\cos au - \cos at\sin au)\,du \\
&= \frac{1}{a}\sin at\int_0^t \cos^2 audu - \frac{1}{a}\cos at\int_0^t \sin au\cos audu \\
&= \frac{1}{a}\sin at\int_0^t \left(\frac{1+\cos 2au}{2}\right)du - \frac{1}{a}\cos at\int_0^t \frac{\sin 2au}{2}du \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right) \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right) - \frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right) \\
&= \frac{t\sin at}{2a}
\end{aligned}$$

f) The complex inversion formula

This formula is also known as the bromwichs integral formula. By utilizing concepts of the complex variable theory, the formula provides a very powerful tool for determing inverse Laplace transform of given functions. If $\overline{f}(s)$ is the Laplace transform of a given function $f(t)$. Then by this formula we have

$$f(t) = \begin{cases} \frac{1}{2\pi}\int\limits_{c-i\infty}^{c+i\infty} e^{st}\overline{f}(s)ds & t>0 \\ 0 & t<0 \end{cases}$$

The integral above is to be evaluated along a line $s=c$ in the complex plane where s is a complex number. One chooses the real number c such that the line $s=c$ will lie to the right side of all singularities of $\overline{f}(s)$. Evaluation of the integral uses contour integration where the contour integral of interest is the bromwhich contour illustrated in the figure below.

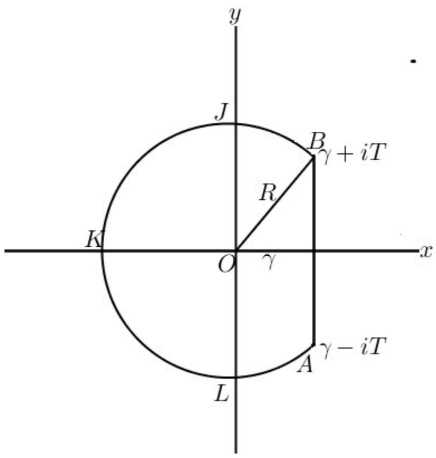

Figure 3.1

We thus have

$$f(t) = \begin{cases} \frac{1}{2\pi} \oint_c \overline{f}(s) ds & t > 0 \\ 0 & t < 0 \end{cases}$$

Where c denotes the Bromwich contour. It is composed of the line AB and an arc $BJKLA$ of a circle centered at the origin with radius R. Let Λ denote the arc $BJKLA$, then since by pythagorous theorem $T = \sqrt{R^2 - c^2}$ we have

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} e^{st}\overline{f}(s)ds$$
$$= \lim_{R\to\infty} \left\{ \frac{1}{2\pi i} \oint_c e^{st}\overline{f}(s)ds - \frac{1}{2\pi i} \int_{\Lambda} e^{st}\overline{f}(s)ds \right\}$$

Cauchy theorem

Let C be a simple closed curve. If $\overline{f}(s)$ is analytic within the region bounded by C as well as on C, then we have Cauchys theorem that

$$\oint_c \overline{f}(s)ds = 0$$

Residue theorem

This theorem forms a backbone of the complex inversion formula. A function $\overline{f}(s)$ is said to have a singular point at a certain value of s for which the function fails to be analytic despite being analytic at every other point. The two major types of singularities are

1. Isolated singularity

If $\overline{f}(s)$ is not analytic at an interior point say at $s = a$, then the point $s = a$ is called an isolated singularity of $\overline{f}(s)$. As an example consider a function $\overline{f}(s)$ defined as

$$\overline{f}(s) = \frac{1}{(s-2)^2}$$

Then the point $s = 2$ is an isolated singularity of $\overline{f}(s)$.

In particular if $\overline{f}(s)$ has an isolated singularity at the point $s = a$ but there exists an integer n for which$(s-a)^n \overline{f}(s)$ becomes analytic at $s = a$, then we say that $\overline{f}(s)$ has a pole of order n. In the above example the point $s = 2$ is a pole of order 2 since $(s-2)^2 \overline{f}(s) = 1$ and thus analytic.

Such a singularity is termed as a removable singularity. On the other hand if $\overline{f}(s)$ has an isolated singularity at $s = a$ which is not removable (not a pole), we call this singular point an essential singular point. A good example is the case where $\overline{f}(s) = \sin\left(\frac{1}{s}\right)$. In this case the point $s = 0$ is an essential singular point.

2. Branch point

If $\overline{f}(s)$ is a multivalued function, any point which cannot be an interior point of the region of definition of a single-valued branch of $f(z)$ is a singular branch point for example $\overline{f}(s) = \frac{e^{-a\sqrt{s}}}{s}$ has a branch point at $s = a$.

Residues

The residues of a function $\overline{f}(s)$ at the pole $s = a$ denoted by a_{-i} is given by

$$a_{-i} = \lim_{s \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} (s-a)^n \overline{f}(s)$$

If $s = a$ is a pole of order n. For simple poles, the residue will be given by

$$a_{-i} = \lim_{s \to a} (s-a) \overline{f}(s)$$

Use of Residue theorem in the complex inversion Formula

The residue theorem helps us to simplify the complex inversion formula. We consider two cases. In the first case we suppose that the only singular points of $\overline{f}(s)$ are poles all of which lie to the left of the line $s = c$ for some real constant c . Further assume that the integral around Λ in figure (2.1) approaches zero as $R \to \infty$. Then by Residue theorem we have $f(t)$ as the sum of residues of $e^{st}\overline{f}(s)$ at poles of $\overline{f}(s)$.

The second case is where $\overline{f}(s)$ has branch points, in such a case modifications are made to the Bromwich contour. As an example if $\overline{f}(s)$ has one branch point at the point $s = 0$, then the following modified contour may be used.

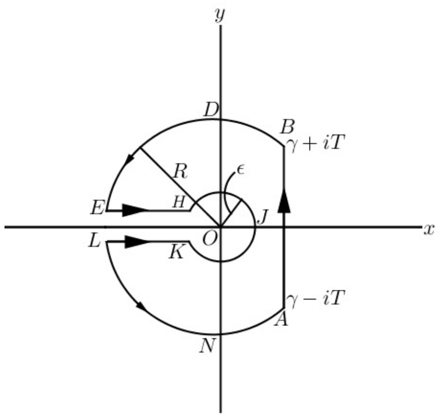

Figure 3.2

Examples

1.

$$\overline{f}(s) = \frac{s^2 - 10s + 13}{(s-1)(s-2)(s-3)}$$

It can be seen that $\overline{f}(s)$ is analytical at every other point except when $s = 1, 2, 3$
Thus $\overline{f}(s)$ has 3 simple poles as listed below;

$$s = 1$$
$$s = 2$$
$$s = 3$$

We now obtain the Residues of $e^{st}\overline{f}(s)$ at poles of $\overline{f}(s)$ as follows

At $s = 1$

$$\operatorname{Re}sidue\left[e^{st}\overline{f}(s)\right] = \lim_{s\to 1}(s-1)\left[\frac{s^2-10s+13}{(s-1)(s-2)(s-3)}\right]e^{st}$$

$$= \lim_{s\to 1}\left[\frac{s^2-10s+13}{(s-2)(s-3)}\right]e^{st}$$

$$= \left[\frac{1^2-10(1)+13}{(1-2)(1-3)}\right]e^{t}$$

$$= \frac{4}{2}e^{t}$$

$$= 2e^{t}$$

At $s = 2$

$$\operatorname{Re}sidue\left[e^{st}\overline{f}(s)\right] = \lim_{s\to 2}(s-2)\left[\frac{s^2-10s+13}{(s-1)(s-2)(s-3)}\right]e^{st}$$

$$= \lim_{s\to 2}\left[\frac{s^2-10s+13}{(s-1)(s-3)}\right]e^{st}$$

$$= \left[\frac{2^2-10(2)+13}{(2-1)(2-3)}\right]e^{2t}$$

$$= \frac{-3}{-1}e^{2t}$$

$$= 3e^{2t}$$

At $s = 3$

$$\operatorname{Re} sidue \left[e^{st}\overline{f}(s)\right] = \lim_{s \to 3} (s-3) \left[\frac{s^2 - 10s + 13}{(s-1)(s-2)(s-3)}\right] e^{st}$$

$$= \lim_{s \to 3} \left[\frac{s^2 - 10s + 13}{(s-1)(s-2)}\right] e^{st}$$

$$= \left[\frac{3^2 - 10(3) + 13}{(3-1)(3-2)}\right] e^{3t}$$

$$= \frac{-8}{2} e^{3t}$$

$$= -4e^{3t}$$

Therefore summing the residues at each pole yields,

$$L^{-1}\left\{\frac{s^2 - 10s + 13}{(s-1)(s-2)(s-3)}\right\} = \sum \operatorname{Re} sidue \left(e^{st}\overline{f}(s)\right)$$
$$= 2e^t + 3e^{2t} - 4e^{3t}$$

2.

$$\overline{f}(s) = \frac{3s + 1}{(s-1)\left(s^2 + 1\right)}$$

It can be seen that $\overline{f}(s)$ is analytical at every other point except when $s = 1, \ i, \ -i$
Thus $\overline{f}(s)$ has 3 simple poles as listed below;

$$s = 1$$
$$s = i$$
$$s = -i$$

Thus the Residues of $e^{st}\overline{f}(s)$ at poles of $\overline{f}(s)$ are;

At $s = 1$

$$\operatorname{Re} sidue\left[e^{st}\overline{f}(s)\right] = \lim_{s\to 1}(s-1)\left[\frac{3s+1}{(s-1)\left(s^2+1\right)}\right]e^{st}$$

$$= \lim_{s\to 1}\left[\frac{3s+1}{s^2+1}\right]e^{st}$$

$$= \left[\frac{3(1)+1}{(1)^2+1}\right]e^{t}$$

$$= \frac{4}{2}e^{t}$$

$$= 2e^{t}$$

At $s = i$

$$\operatorname{Re} sidue\left[e^{st}\overline{f}(s)\right] = \lim_{s\to i}(s-i)\left[\frac{3s+1}{(s-1)\,(s+i)(s-i)}\right]e^{st}$$

$$= \lim_{s\to i}\left[\frac{3s+1}{(s-1)\,(s+i)}\right]e^{st}$$

$$= \left[\frac{3(i)+1}{(i-1)\,(i+i)}\right]e^{it}$$

$$= \frac{3i+1}{2i(i-1)}e^{it}$$

$$= \frac{3i+1}{-2-2i}e^{it}$$

At $s = -i$

$$\operatorname{Re} sidue\left[e^{st}\overline{f}(s)\right] = \lim_{s \to -i}(s+i)\left[\frac{3s+1}{(s-1)(s+i)(s-i)}\right]e^{st}$$

$$= \lim_{s \to -i}\left[\frac{3s+1}{(s-1)(s-i)}\right]e^{st}$$

$$= \left[\frac{3(-i)+1}{(-i-1)(-i-i)}\right]e^{-it}$$

$$= \frac{-3i+1}{-2i(-i-1)}e^{-it}$$

$$= \frac{-3i+1}{-2+2i}e^{-it}$$

Therefore summing the residues at each pole yields,

$$L^{-1}\left\{\frac{s^2-10s+13}{(s-1)(s-2)(s-3)}\right\} = \sum \operatorname{Re} sidue\left(e^{st}\overline{f}(s)\right)$$
$$= 2e^t + \frac{-3i+1}{-2+2i}e^{-it} + \frac{3i+1}{-2-2i}e^{it}$$

But

$$\frac{3i+1}{-2-2i}\left(\frac{-2+2i}{-2+2i}\right) = \frac{6i^2-6i+2i-2}{4-(4i^2)}$$

$$= \frac{-8-4i}{8}$$
$$= -1 - \frac{1}{2}i$$

And

$$\frac{-3i+1}{-2+2i}\left(\frac{-2-2i}{-2-2i}\right) = \frac{6i^2+6i-2i-2}{4-(4i^2)}$$

$$= \frac{-8+4i}{8}$$
$$= -1+\frac{1}{2}i$$

$$\begin{aligned}2e^t + \frac{-3i+1}{-2+2i}e^{-it} + \frac{3i+1}{-2-2i}e^{it} &= 2e^t + \left(-1-\frac{1}{2}i\right)e^{-it} + \left(-1+\frac{1}{2}i\right)e^{it} \\ &= 2e^t + \left(-1-\frac{1}{2}i\right)(\cos t + i\sin t) + \left(-1+\frac{1}{2}i\right)(\cos t - i\sin t) \\ &= 2e^t - \cos t - i\sin t - \frac{1}{2}i\cos t - \frac{1}{2}i^2\sin t - \cos t + i\sin t \\ &\quad + \frac{1}{2}i\cos t - \frac{1}{2}i^2\sin t \\ &= 2e^t - 2\cos t + \sin t\end{aligned}$$

$$\therefore L^{-1}\left\{\frac{s^2-10s+13}{(s-1)(s-2)(s-3)}\right\} = 2e^t - 2\cos t + \sin t$$

3.

$$\overline{f}(s) = \frac{e^{-a\sqrt{s}}}{s}$$

It can be seen that $\overline{f}(s)$ has a branch point at $s=0$ we thus use the modified Bromwich contour.

$$\begin{aligned}f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\overline{f}(s)ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st-a\sqrt{s}}}{s}ds\end{aligned}$$

We thus have

$$\frac{1}{2\pi i}\oint\limits_{c}\frac{e^{st-a\sqrt{s}}}{s}ds=\left\{\begin{array}{l}\frac{1}{2\pi i}\int\limits_{AB}\frac{e^{st-a\sqrt{s}}}{s}ds+\frac{1}{2\pi i}\int\limits_{BDE}\frac{e^{st-a\sqrt{s}}}{s}ds+\frac{1}{2\pi i}\int\limits_{EH}\frac{e^{st-a\sqrt{s}}}{s}ds\\ \\ +\frac{1}{2\pi i}\int\limits_{HJK}\frac{e^{st-a\sqrt{s}}}{s}ds+\frac{1}{2\pi i}\int\limits_{KL}\frac{e^{st-a\sqrt{s}}}{s}ds+\frac{1}{2\pi i}\int\limits_{LNA}\frac{e^{st-a\sqrt{s}}}{s}ds\end{array}\right\}$$

Since the only singularity (branch point) of $\overline{f}(s)$ is not inside C, by Cauchys theorem the integrand on the left side is zero. It follows that

$$\begin{aligned}f(t)&=\lim_{\substack{R\to 0\\ \epsilon\to 0}}\frac{1}{2\pi i}\int\limits_{AB}\frac{e^{st-a\sqrt{s}}}{s}ds\\ &=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{e^{st-a\sqrt{s}}}{s}ds\\ &=\lim_{\substack{R\to 0\\ \epsilon\to 0}}\frac{1}{2\pi i}\left\{\int\limits_{EH}\frac{e^{st-a\sqrt{s}}}{s}ds+\int\limits_{HJK}\frac{e^{st-a\sqrt{s}}}{s}ds+\int\limits_{KL}\frac{e^{st-a\sqrt{s}}}{s}ds\right\}\end{aligned}\tag{3.3}$$

Along EH, $s=xe^{\pi i}$, $\sqrt{s}=\sqrt{x}e^{\pi i/2}=i\sqrt{x}$ and as s goes from $-R$ to $-\epsilon$, x goes from R to ϵ . Hence we have

$$\begin{aligned}\int\limits_{EH}\frac{e^{st-a\sqrt{s}}}{s}ds&=\int\limits_{-R}^{\epsilon}\frac{e^{st-a\sqrt{s}}}{s}ds\\ &=\int\limits_{R}^{\epsilon}\frac{e^{xt+ai\sqrt{x}}}{x}ds\end{aligned}$$

Similarly, along KL ,$s=xe^{-\pi i}$, $\sqrt{s}=\sqrt{x}e^{-\pi i/2}=-i\sqrt{x}$ and as s goes from $-\epsilon$ to $-R$, x goes from ϵ to R . Then

$$\begin{aligned}\int\limits_{KL}\frac{e^{st-a\sqrt{s}}}{s}ds&=\int\limits_{-\epsilon}^{-R}\frac{e^{st-a\sqrt{s}}}{s}ds\\ &=\int\limits_{\epsilon}^{R}\frac{e^{xt+ai\sqrt{x}}}{x}ds\end{aligned}$$

Along HJK, $s = \epsilon e^{i\theta}$ and we have

$$\int\limits_{HJK} \frac{e^{st-a\sqrt{s}}}{s} ds = \int\limits_{\pi}^{-\pi} \frac{e^{\varepsilon e^{i\theta}t-a\sqrt{\varepsilon e^{i\theta/2}}}}{\varepsilon e^{i\theta}} d\theta$$

$$= i\int\limits_{\pi}^{-\pi} e^{\varepsilon e^{i\theta}t-a\sqrt{\varepsilon e^{i\theta/2}}} d\theta$$

Thus equation (2.3) now becomes

$$f(t) = -\lim_{\substack{R\to 0\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{\int\limits_{R}^{\epsilon} \frac{e^{-xt-ai\sqrt{x}}}{x} dx + \int\limits_{\epsilon}^{R} \frac{e^{-xt+ai\sqrt{x}}}{x} dx + i\int\limits_{\pi}^{-\pi} e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}e^{i}}{}^{\theta}/2\, d\theta\right\}$$

$$= -\lim_{\substack{R\to 0\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{\int\limits_{\epsilon}^{R} \frac{e^{-xt}\left(e^{ai\sqrt{x}} - e^{-ai\sqrt{x}}\right)}{x} dx + i\int\limits_{\pi}^{-\pi} e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}e^{i}}{}^{\theta}/2\, d\theta\right\}$$

$$= -\lim_{\substack{R\to 0\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{2i\int\limits_{\epsilon}^{R} \frac{e^{-xt}\sin a\sqrt{x}}{x} dx + i\int\limits_{\pi}^{-\pi} e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}e^{i}} d\theta\right\}$$

Since the limit can be taken underneath the integral sign, we have

$$\lim_{\epsilon\to 0}\int\limits_{\pi}^{-\pi} e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}e}{}^{i\theta}/2\, d\theta = \int\limits_{\pi}^{-\pi} 1 d\theta$$

$$= -2\pi$$

And so we find

$$f(t) = 1 - \frac{1}{\pi}\int\limits_{0}^{\infty} \frac{e^{-xt}\sin a\sqrt{x}}{x}$$

$$= 1 - erf\left(\frac{\alpha}{2\sqrt{t}}\right)$$

$$= erfc\left(\frac{\alpha}{2\sqrt{t}}\right)$$

g) The Heaviside expansion formula

Let $P(S)$ and $Q(S)$ be polynomials where $P(S)$ has degree less than that of $Q(S)$. Suppose that $Q(S)$ has n distinct roots $\alpha_i, i = 1, 2, 3, ..., n$ .Then we have

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)}$$

Proof

Since $Q(S)$ is a polynomial with n distinct roots , we can write according to the method of partial fractions

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \frac{A_3}{s-\alpha_3} + ... + \frac{A_i}{s-\alpha_i} + ... + \frac{A_n}{s-\alpha_n}$$

Multiplying both sides by $s - \alpha_i$ and letting $s \to \alpha_i$ we find using L hospitals rule

$$\begin{aligned} A_i &= \lim_{s\to\alpha_i} \frac{P(s)}{Q(s)}(s-\alpha_i) \\ &= \lim_{s\to\alpha_i} P(s)\left\{\frac{s-\alpha_i}{Q(s)}\right\} \\ &= \lim_{s\to\alpha_i} P(s) \lim_{s\to\alpha_i}\left\{\frac{s-\alpha_i}{Q(s)}\right\} \\ &= P(\alpha_i) \lim_{s\to\alpha_i}\left\{\frac{1}{Q'(s)}\right\} \\ &= \frac{P(\alpha_i)}{Q'(\alpha_i)} \end{aligned}$$

Thus we have

$$\frac{P(s)}{Q(s)} = \left\{\begin{array}{c} \frac{P(\alpha_1)}{Q'(\alpha_1)}\left[\frac{1}{s-\alpha_1}\right] + \frac{P(\alpha_2)}{Q'(\alpha_2)}\left[\frac{1}{s-\alpha_2}\right] + \frac{P(\alpha_3)}{Q'(\alpha_3)}\left[\frac{1}{s-\alpha_3}\right] + ... \\ \\ ... + \frac{P(\alpha_i)}{Q'(\alpha_i)}\left[\frac{1}{s-\alpha_i}\right] + ... + \frac{P(\alpha_n)}{Q'(\alpha_n)}\left[\frac{1}{s-\alpha_n}\right] \end{array}\right\}$$

Taking the inverse Laplace transforms, we have as required

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)}e^{\alpha_3 t} + ... + \frac{P(\alpha_i)}{Q'(\alpha_i)}e^{\alpha_i t} + ... + \frac{P(\alpha_n)}{Q'(\alpha_n)}e^{\alpha_n t}$$
$$= \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)}e^{\alpha_i t}$$

Examples

1.

$$\overline{f}(s) = \frac{2s^2 - 4}{(s+1)(s-2)(s-3)}$$

In this case $P(s) = 2s^2 - 4$ and $Q(s) = (s+1)(s-2)(s-3)$. Therefore the roots of $Q(s)$ are $\alpha_1 = -1, \alpha_2 = 2$ and $\alpha_3 = 3$. But

$$\begin{aligned} Q(s) &= (s+1)(s-2)(s-3) \\ &= (s+1)\left(s^2 - 5s + 6\right) \\ &= s^3 - 4s^2 + s + 6 \\ \Rightarrow Q'(s) &= 3s^2 - 8s + 1 \end{aligned}$$

Thus

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{3} \frac{P(s)}{Q'(s)}e^{\alpha_i t}$$

$$= \frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= \left[\frac{2(-1)^2 - 4}{3(-1)^2 - 8(-1) + 1}\right]e^{-t} + \left[\frac{2(2)^2 - 4}{3(2)^2 - 8(2) + 1}\right]e^{2t} + \left[\frac{2(3)^2 - 4}{3(3)^2 - 8(3) + 1}\right]e^{3t}$$

$$= \frac{-2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

2.

$$\overline{f}(s) = \frac{3s+1}{(s-1)(s^2+1)}$$

Here $P(s) = 3s+1$ and $Q(s) = (s-1)(s^2+1)$. Therefore the roots of $Q(s)$ are $\alpha_1 = -1, \alpha_2 = i$ and $\alpha_3 = -i$. But

$$\begin{aligned} Q(s) &= (s-1)\left(s^2+1\right) \\ &= s^3 - s^2 + s - 1 \\ \Rightarrow Q'(s) &= 3s^2 - 2s + 1 \end{aligned}$$

With this, we now have

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{3} \frac{P(s)}{Q'(s)} e^{\alpha_i t}$$

$$= \frac{P(1)}{Q'(1)}e^t + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it}$$

$$= \left[\frac{3(1)+1}{3s^2-2s+1}\right]e^t + \left[\frac{3(i)+1}{3s^2-2s+1}\right]e^{it} + \left[\frac{3(-i)+1}{3s^2-2s+1}\right]e^{-it}$$

$$= \frac{4}{2}e^t + \frac{3i+1}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

But $e^{it} = \cos t + i\sin t$

$$\Rightarrow L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = 2e^t + \frac{3i+1}{-2-2i}(\cos t + i\sin t) + \frac{-3i+1}{-2+2i}(\cos t - i\sin t)$$

$$\begin{aligned} \frac{3i+1}{-2-2i}\left(\frac{-2+2i}{-2+2i}\right) &= \frac{6i^2 - 6i + 2i - 2}{4 - (4i^2)} \\ &= \frac{-8-4i}{8} \\ &= -1 - \frac{1}{2}i \end{aligned}$$

$$\frac{-3i+1}{-2+2i}\left(\frac{-2-2i}{-2-2i}\right) = \frac{6i^2+6i-2i-2}{4-(4i^2)}$$
$$= \frac{-8+4i}{8}$$
$$= -1+\frac{1}{2}i$$

$$\therefore L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = 2e^t + \left(-1-\frac{1}{2}i\right)(\cos t + i\sin t) + \left(-1+\frac{1}{2}i\right)(\cos t - i\sin t)$$
$$= 2e^t - \cos t - i\sin t - \frac{1}{2}i\cos t - \frac{1}{2}i^2\sin t - \cos t + i\sin t + \frac{1}{2}i\cos t - \frac{1}{2}i^2\sin t$$
$$= 2e^t - 2\cos t + \sin t$$

3.

$$\overline{f}(s) = \frac{s^2-10s+13}{(s-1)(s-2)(s-3)}$$

Here $P(s) = s^2-10s+13$ and $Q(s) = (s-1)(s-2)(s-3)$. Therefore the roots of $Q(s)$ are $\alpha_1 = 1, \alpha_2 = 2$ and $\alpha_3 = 3$. But

$$Q(s) = (s-1)(s-2)(s-3)$$
$$= (s-1)\left(s^2-5s+6\right)$$
$$= s^3-6s^2+11s-6$$
$$\Rightarrow Q'(s) = 3s^2-12s+11$$

We thus have

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{3}\frac{P(s)}{Q'(s)}e^{\alpha_i t}$$

$$= \frac{P(1)}{Q'(1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= \left[\frac{(1)^2-10(1)+13}{3(1)^2-12(1)+11}\right]e^t + \left[\frac{(2)^2-10(2)+13}{3(2)^2-12(2)+11}\right]e^{2t} + \left[\frac{(3)^2-10(3)+13}{3(3)^2-12(3)+11}\right]e^{3t}$$

$$= \frac{4}{2}e^t + \frac{-3}{-1}e^{2t} + \frac{-8}{2}e^{3t}$$

$$= 2e^t + 3e^{2t} - 4e^{3t}$$

4.

$$\overline{f}(s) = \frac{6s}{s^2 - 16}$$

In this case $P(s) = 6s$ and $Q(s) = s^2 - 16 \Rightarrow Q'(s) = 2s$. We can factorize $Q(s)$ as $(s-4)(s+4)$, thus the roots of $Q(s)$ are $\alpha_1 = 4$ and $\alpha_2 = -4$. Also $Q'(\alpha_1) = 2(4) = 8$ and $Q'(\alpha_1) = 2(-4) = -8$.

We thus have

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{2} \frac{P(\alpha_i)}{Q'(\alpha_i)} e^{\alpha_i t}$$

$$= \frac{P(4)}{Q'(4)}e^{4t} + \frac{P(-4)}{Q'(-4)}e^{-4t}$$

$$= \left[\frac{6(4)}{8}\right] e^{4t} + \left[\frac{6(-4)}{-8}\right] e^{-4t}$$

$$= 6\left(\frac{e^{4t} + e^{-4t}}{2}\right)$$

$$= 6\cosh(4t)$$

5.

$$\overline{f}(s) = \frac{s}{s^2 + 2}$$

In this case $P(s) = s$ and $Q(s) = s^2 + 4 \Rightarrow Q'(s) = 2s$. It follows that the roots of $Q(s)$ are $\alpha_1 = \sqrt{2}i$ and $\alpha_2 = -\sqrt{2}i$. Also $Q'(\alpha_1) = 2\left(\sqrt{2}i\right)$ and $Q'(\alpha_1) = 2\left(-\sqrt{2}i\right)$

We thus have

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{2}\frac{P(\alpha_i)}{Q'(\alpha_i)}e^{\alpha_i t}$$

$$= \frac{P\left(\sqrt{2}i\right)}{Q'\left(\sqrt{2}i\right)}e^{\sqrt{2}i} + \frac{P(-\sqrt{2}i)}{Q'\left(-\sqrt{2}i\right)}e^{-\sqrt{2}i}$$

$$= \left[\frac{\sqrt{2}i}{2\sqrt{2}i}\right]e^{\sqrt{2}it} + \left[\frac{-\sqrt{2}i}{-2\sqrt{2}i}\right]e^{-\sqrt{2}it}$$

$$= \left(\frac{e^{\sqrt{2}it} + e^{-\sqrt{2}it}}{2}\right)$$

$$= \frac{1}{2}\left[\cos\left(\sqrt{2}t\right) + i\sin\left(\sqrt{2}t\right) + \cos\left(\sqrt{2}t\right) - i\sin\left(\sqrt{2}t\right)\right]$$

$$= \cos\left(\sqrt{2}t\right)$$

h) Use of tables.

This method involves the use of the table of transform pairs in the determination of the inverse laplace transform of a given function.

Chapter 4

Laplace Transforms of Probability Distributions

4.1 Introduction

In this chapter, we define the Laplace transform of a pdf and show how it can be used to obtain the moment. First the Laplace transforms of distributions are obtained in explicit forms. The distributions considered include uniform, exponential, gamma and the kth order statistic. Secondly the Laplace transforms of other distributions are derived in terms of the modified Bessel function of the third kind. The distributions considered in this case include the inverse gamma, inverse Gaussian and generalized inverse Gaussian. The chapter also focuses on both fixed and random sums of independent and identically distributed random variables. In addition the use of Laplace transform in mixed probability distributions is discussed.The mixed distributions considered are the Poisson, exponential and normal mixtures

The Laplace transform of a non-negative random variable $X \geq 0$ with the probability density function $f(x)$ denoted by $L_X(s)$ is defined as

$$
\begin{aligned}
L_X(s) &= \int_0^\infty e^{-sx} f(x) dx \\
&= E\left[e^{-sx}\right]
\end{aligned}
$$

Using Laplace transform to calculate moments of a distribution

By definition

$$L_X\left(s\right) = E\left[e^{-sx}\right]$$

Differentiating both sides with respect to s we have

$$\begin{aligned}
L_X^{'}\left(s\right) &= \frac{d}{dx}E\left[e^{-sX}\right] \\
&= E\left[\frac{d}{dx}e^{-sX}\right] \\
&= E\left[-Xe^{-sX}\right]
\end{aligned}$$

$$\begin{aligned}
L_X^{''}\left(s\right) &= \frac{d^2}{dx^2}E\left[e^{-sX}\right] \\
&= E\left[\frac{d^2}{dx^2}e^{-sX}\right] \\
&= E\left[-(1)^2X^2e^{-sX}\right]
\end{aligned}$$

$$\begin{aligned}
L_X^{'''}\left(s\right) &= \frac{d^3}{dx^3}E\left[e^{-sX}\right] \\
&= E\left[\frac{d^3}{dx^3}e^{-sX}\right] \\
&= E\left[-(1)^3X^3e^{-sX}\right]
\end{aligned}$$

By mathematical induction we assume that

$$L_X^{(n-1)}\left(s\right) = E\left[-(1)^{n-1}X^{n-1}e^{-sX}\right]$$

We thus have

$$
\begin{aligned}
L_X^{(n)}(s) &= \frac{d}{ds} L_X^{(n-1)}(s) \\
&= \frac{d}{ds} E\left[-(1)^{n-1} X^{n-1} e^{-sX}\right] \\
&= E\left[-(1)^{n-1} X^{n-1} \frac{d}{ds} e^{-sX}\right] \\
&= E\left[-(1)^{n} X^{n} e^{-sX}\right]
\end{aligned}
$$

Evaluating the results as $s \to 0$ we obtain the moments as

$$
\begin{aligned}
&E[X] = L_X^{'}(0) \\
&E\left[X^2\right] = (-1)^2 L_X^{''}(0) = L_X^{''}(0) \\
&E\left[X^3\right] = (-1)^3 L_X^{'''}(0) \\
&\vdots \\
&E[X^r] = (-1)^r L_X^{(r)}(0)
\end{aligned}
$$

4.2 Laplace Transform in explicit forms

4.2.1 Uniform Distribution

Let Xhave uniform distribution in (a, b), the density function of X is

$$
f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{elsewhere} \end{cases}
$$

The Laplace transform $L_X(s)$ of the random variable X is given by

$$
\begin{aligned}
L_X(s) &= E\left[e^{-sX}\right] \\
&= \int_a^b \frac{1}{b-a} e^{-sX} dx \\
&= \frac{1}{b-a} \int_a^b e^{-sX} dx
\end{aligned}
$$

$$= \left[\left. \frac{e^{-sx}}{-s} \right|_a^b \right]$$

$$= \left[\frac{e^{-sa} - e^{-sb}}{s} \right]$$

Moments

$$L_X\left(s\right) = \left[\frac{e^{-sa} - e^{-sb}}{s} \right]$$

Obtaining the first derivative, using quotient rule let

$$U = e^{-sa} - e^{-sb} \qquad \Rightarrow U^{'} = -ae^{-sa} + be^{-sb}$$
$$V = s \qquad \Rightarrow V^{'} = 1$$

This implies that

$$L_X^{'}\left(s\right) = \frac{1}{b-a} \left[\frac{s\left(-ae^{-sa} + be^{-sb}\right) - \left(e^{-sa} - e^{-sb}\right)}{s^2} \right]$$

$$= \frac{1}{b-a} \left[\frac{-ase^{-sa} + bse^{-sb} - e^{-sa} + e^{-sb}}{s^2} \right]$$

$$= \frac{1}{b-a} \left[\frac{\left(bs+1\right)e^{-sb} - \left(as+1\right)e^{-sa}}{s^2} \right]$$

Therefore

$$L_X^{'}\left(0\right) = \lim_{s \to 0} \frac{1}{b-a} \left[\frac{\left(bs+1\right)e^{-sb} - \left(as+1\right)e^{-sa}}{s^2} \right]$$

By L'Hospital's rule we have

$$\begin{aligned}
L_X^{'}(0) &= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{-b(bs+1)\,e^{-sb}+be^{-sb}+\,a(as+1)\,e^{-sa}-ae^{-sa}}{2s}\right]\\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{-b^2se^{-sb}-be^{-sb}+be^{-sb}+a^2se^{-sa}+ae^{-sa}-ae^{-sa}}{2s}\right]\\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{a^2se^{-sa}-b^2se^{-sb}}{2s}\right]\\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{-a^3se^{-sa}+a^2e^{-sa}+b^3se^{-sb}-b^2e^{-sb}}{2}\right]\\
&= \frac{1}{b-a}\left(\frac{a^2-b^2}{2}\right)\\
&= \frac{1}{b-a}\left[\frac{(a-b)\,(a+b)}{2}\right]\\
&= \frac{1}{b-a}\left[\frac{-(b-a)\,(a+b)}{2}\right]\\
&= -\frac{a+b}{2}
\end{aligned}$$

Similarly the second derivative is obtained as follows

$$L_X^{'}(s) = \frac{1}{b-a}\left[\frac{(bs+1)\,e^{-sb}-\,(as+1)\,e^{-sa}}{s^2}\right]$$

Using quotient rule let

$$\begin{aligned}
U = \,(bs+1)\,e^{-sb}-\,(as+1)\,e^{-sa} \quad \Rightarrow U^{'} &= be^{-sb}-b\,(bs+1)\,e^{-sb}-ae^{-sa}+a\,(as+1)\,e^{-sa}\\
&= be^{-sb}-b^2se^{-sb}-be^{-sb}-ae^{-sa}+a^2se^{-sa}+ae^{-sa}\\
&= a^2se^{-sa}-be^{-sb}
\end{aligned}$$

$$V = s^2 \qquad \Rightarrow V^{'} = 2s$$

This implies that

$$
\begin{aligned}
L_X''(s) &= \frac{1}{b-a}\left[\frac{\left(a^2se^{-sa} - b^2se^{-sb}\right)s^2 - 2s\left[(bs+1)e^{-sb} - (as+1)e^{-sa}\right]}{s^4}\right] \\
&= \frac{1}{b-a}\left[\frac{\left(a^2se^{-sa} - b^2se^{-sb}\right)s^2 - 2s\,(bs+1)e^{-sb} + 2s\,(as+1)e^{-sa}}{s^4}\right] \\
&= \frac{1}{b-a}\left[\frac{a^2s^3e^{-sa} - b^2s^3e^{-sb} - 2bs^2e^{-sb} - 2se^{-sb} + 2as^2e^{-sa} + 2se^{-sa}}{s^4}\right] \\
&= \frac{1}{b-a}\left[\frac{e^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) - e^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right)}{s^4}\right]
\end{aligned}
$$

Therefore

$$
L_X''(0) = \lim_{s\to 0}\frac{1}{b-a}\left[\frac{e^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) - e^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right)}{s^4}\right]
$$

Using L'Hospital's rule we have

$$
\begin{aligned}
L_X''(0) &= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{\begin{array}{r}-ae^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) + e^{-sa}\left(3a^2s^2 + 4as + 2\right) + \\ be^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right) - e^{-sb}\left(3b^2s^2 + 4bs + s\right)\end{array}}{4s^3}\right] \\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{\begin{array}{r}e^{-sa}\left(3a^2s^2 + 4as + 2\right) - ae^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) - \\ e^{-sb}\left(3b^2s^2 + 4bs + s\right) + be^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right)\end{array}}{4s^3}\right] \\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{\begin{array}{r}e^{-sa}\left(3a^2s^2 + 4as + 2 - a^3s^3 - 2a^2s^2 - 2as\right) - \\ e^{-sb}\left(3b^2s^2 + 4bs + 2 - b^3s^3 - 2b^2s^2 - 2bs\right)\end{array}}{4s^3}\right] \\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{e^{-sa}\left(a^2s^2 + 2as + 2 - a^3s^3\right) - e^{-sb}\left(b^2s^2 + 2bs + 2 - b^3s^3\right)}{4s^3}\right] \\
&= \lim_{s\to 0}\frac{1}{b-a}\left[\frac{\begin{array}{r}-ae^{-sa}\left(a^2s^2 + 2as + 2 - a^3s^3\right) + e^{-sa}\left(2a^2s + 2a - 3a^3s^2\right) + \\ be^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right) - e^{-sb}\left(2b^2s + 2b - 3b^3s^2\right)\end{array}}{12s^2}\right]
\end{aligned}
$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{\begin{array}{r} e^{-sa}\left(2a^2s+2a-3a^3s^2\right)-ae^{-sa}\left(a^2s^2+2as+2-a^3s^3\right)- \\ e^{-sb}\left(2b^2s+2b-3b^3s^2\right)+be^{-sb}\left(b^2s^3+2bs^2+2s\right)\end{array}}{12s^2}\right]$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{\begin{array}{r} e^{-sa}\left(2a^2s+2a-3a^3s^2-a^3s^2-2a^2s-2a+a^4s^3\right)- \\ e^{-sb}\left(2b^2s+2b-3b^3s^2-b^3s^2-2b^2s-2b+b^4s^3\right)\end{array}}{12s^2}\right]$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{e^{-sa}\left(a^4s^3-4a^3s^2\right)-e^{-sb}\left(b^4s^3-4b^3s^2\right)}{12s^2}\right]$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{\begin{array}{r} -ae^{-sa}\left(a^4s^3-4a^3s\right)+e^{-sa}\left(3a^4s^2-8a^3s^2\right)+ \\ be^{-sb}\left(b^4s^3-4b^3s^2\right)-e^{-sb}\left(3b^4s^2-8b^3s\right)\end{array}}{24s}\right]$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{e^{-sa}\left(3a^4s^2-8a^3s-a^5s^3+4a^4s^2\right)-e^{-sb}\left(3b^4s^2-8b^3s-b^5s^3+4b^4s^2\right)}{24s}\right]$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{e^{-sa}\left(7a^4s^2-8a^3s-a^5s^3\right)-e^{-sb}\left(7b^4s^2-8b^3s-b^5s^3\right)}{24s}\right]$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left[\frac{\begin{array}{r} -ae^{-sa}\left(7a^4s^2-8a^3s-a^5s^3\right)+e^{-sa}\left(14a^4s-8a^3-3a^5s^2\right)+ \\ be^{-sb}\left(7b^4s^2-8b^3s-b^5s^3\right)-e^{-sb}\left(14b^4s-8b^3-3b^5s^2\right)\end{array}}{24}\right]$$

$$= \frac{1}{b-a}\left(\frac{-8a^3+8b^3}{24}\right)$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$= \frac{1}{3}\left(b^2+ba+a^2\right)$$

$$\therefore L_X''(0) = \frac{1}{3}\left(b^2+ba+a^2\right)$$

Extending this to the third derivative, we have

$$L_x''(s) = \frac{1}{b-a}\left[\frac{e^{-sa}\left(a^2s^3+2as^2+2s\right)-e^{-sb}\left(b^2s^3+2bs^2+2s\right)}{s^4}\right]$$

Using quotient rule of differentiation, we let

$$U = e^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) - e^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right) \text{ and } \quad V = s^4 \Rightarrow V' = 4s^3$$

$$\begin{aligned} U' &= -ae^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) + e^{-sa}\left(3a^2s^2 + 4as + 2\right) + \\ &\quad be^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right) - e^{-sb}\left(3b^2s^2 + 4bs + 2\right) \\ &= e^{-sa}\left(3a^2s^2 + 4as + 2 - 2a^3s^3 - 2a^2s^2 - 2as\right) - e^{-sb}\left(3b^2s^2 + 4bs + 2 - 2b^3s^3 - 2b^2s^2 - 2bs\right) \\ &= e^{-sa}\left(a^2s^2 + 2as + 2 - a^3s^3\right) - e^{-sb}\left(b^2s^2 + 2bs + 2 - b^3s^3\right) \end{aligned}$$

By product rule we have

$$\begin{aligned} L_x'''(s) &= \frac{U'V - V'U}{V^2} \\ &= \frac{1}{b-a}\left(\frac{1}{s^8}\right)\left\{ \begin{matrix} \begin{bmatrix} e^{-sa}\left(a^2s^2 + 2as + 2 - a^3s^3\right) - \\ e^{-sb}\left(b^2s^2 + 2bs + 2 - b^3s^3\right) \end{bmatrix} s^4 \\ -4s^3 \begin{bmatrix} e^{-sa}\left(a^2s^3 + 2as^2 + 2s\right) - \\ e^{-sb}\left(b^2s^3 + 2bs^2 + 2s\right) \end{bmatrix} \end{matrix} \right\} \end{aligned}$$

$$= \frac{1}{b-a}\left(\frac{1}{s^8}\right)\left\{ \begin{matrix} \left[e^{-sa}\left(a^2s^6 + 2as^5 + 2s^4 - a^3s^7\right) - e^{-sa}\left(4a^2s^6 + 8as^5 + 8s^4\right) - \right] \\ \left[-e^{-sb}\left(4b^2s^6 + 8bs^5 + 8s^4\right) + e^{-sb}\left(b^2s^6 + 2bs^5 + 2s^4 - b^3s^7\right)\right] \end{matrix} \right\}$$

$$= \frac{1}{b-a}\left\{\frac{e^{-sa}\left(-3a^2s^6 - 6as^5 - 6s^4 - a^3s^7\right) - e^{-sb}\left(-3b^2s^6 - 6bs^5 - 6s^4 - b^3s^7\right)}{s^8}\right\}$$

$$\therefore L_X'''(0) = \lim_{s\to 0}\frac{1}{b-a}\left\{\frac{e^{-sa}\left(-3a^2s^6 - 6as^5 - 6s^4 - a^3s^7\right) - e^{-sb}\left(-3b^2s^6 - 6bs^5 - 6s^4 - b^3s^7\right)}{s^8}\right\}$$

Using L Hospitals rule we have

$$L'''_X(0) = \lim_{s\to 0}\frac{1}{b-a}\left(\frac{1}{8s^7}\right)\left\{\begin{array}{c}\left[\begin{array}{l}-ae^{-sa}\left(-3a^2s^6-6as^5-6s^4-a^3s^7\right)+\\ \\ e^{-sa}\left(-18a^2s^5-30as^4-24s^3-7a^3s^6\right)\end{array}\right]-\\ \left[\begin{array}{l}-be^{-sb}\left(-3b^2s^6-6bs^5-6s^4-b^3s^7\right)+\\ \\ e^{-sb}\left(-18b^2s^5-30bs^4-24s^3-7b^3s^6\right)\end{array}\right]\end{array}\right\}$$

$$= \lim_{s\to 0}\frac{1}{b-a}\left(\frac{1}{8s^7}\right)\left\{\begin{array}{l}e^{-sa}\left[\begin{array}{c}-18a^2s^5-30as^4-24s^3-7a^3s^6+3a^3s^6\\ \\ +6a^2s^5+6as^4+a^4s^7\end{array}\right]\\ \\ e^{-sb}\left[\begin{array}{c}-18b^2s^5-30bs^4-24s^3-7b^3s^6+3b^3s^6\\ \\ +6b^2s^5+6bs^4+b^4s^7\end{array}\right]\end{array}\right\}$$

$$= \lim_{s\to 0}\frac{1}{b-a}\left(\frac{1}{8s^7}\right)\left\{\begin{array}{c}e^{-sa}\left(-12a^2s^5-24as^4-24s^3-4a^3s^6+a^4s^7\right)-\\ \\ e^{-sb}\left(-12b^2s^5-24bs^4-24s^3-4b^3s^6+b^4s^7\right)\end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{56s^6}\right)\left\{\begin{array}{l} \left[\begin{array}{l} -ae^{-sa}\left(-12a^2s^5 - 24as^4 - 24s^3 - 4a^3s^6 + a^4s^7\right) + \\ \\ e^{-sa}\left(-60a^2s^4 - 96as^3 - 72s^2 - 24a^3s^5 + 7a^4s^6\right) - \end{array}\right] - \\ \\ \quad \left[\begin{array}{l} -be^{-sb}\left(-12b^2s^5 - 24bs^4 - 24s^3 - 4b^3s^6 + b^4s^7\right) + \\ \\ \quad e^{-sb}\left(-60b^2s^4 - 96bs^3 - 72s^2 - 24b^3s^5 + 7b^4s^6\right) \end{array}\right] \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{56s^6}\right)\left\{\begin{array}{l} e^{-sa}\left[\begin{array}{l} -60a^2s^4 - 96as^3 - 72s^2 - 24a^3s^5 + 7a^4s^6 \\ \\ +12a^3s^5 + 24a^2s^4 + 24as^3 + 4a^4s^6 - a^5s^7 \end{array}\right] - \\ \\ \quad e^{-sb}\left[\begin{array}{l} -60b^2s^4 - 96bs^3 - 72s^2 - 24b^3s^5 + 7b^4s^6 \\ \\ +12b^3s^5 + 24b^2s^4 + 24bs^3 + 4b^4s^6 - b^5s^7 \end{array}\right] \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{56s^6}\right)\left\{\begin{array}{l} e^{-sa}\left[-36a^2s^4 - 72as^3 - 72s^2 - 12a^3s^5 + 11a^4s^6 - a^5s^7\right] - \\ \\ \quad e^{-sb}\left[-36b^2s^4 - 72bs^3 - 72s^2 - 12b^3s^5 + 11b^4s^6 - b^5s^7\right] \end{array}\right\}$$

$$
= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{336s^5}\right)\left\{\begin{array}{l}\left[\begin{array}{l} e^{-sa}\begin{pmatrix}-144a^2s^3-216as^2-144s- \\ \\ 60a^3s^4+66a^4s^5-7a^5s^6\end{pmatrix} \\ \\ -ae^{-sa}\begin{pmatrix}-36a^2s^4-72as^3-72s^2- \\ \\ 12a^3s^5+11a^4s^6-a^5s^7\end{pmatrix}\end{array}\right] - \\ \\ \left[\begin{array}{l} e^{-sb}\begin{pmatrix}-144b^2s^3-216bs^2-144s- \\ \\ 60b^3s^4+66b^4s^5-7b^5s^6\end{pmatrix} \\ \\ -be^{-sb}\begin{pmatrix}-36b^2s^4-72bs^3-72s^2- \\ 12b^3s^5+11b^4s^6-b^5s^7\end{pmatrix}\end{array}\right]\end{array}\right\}
$$

$$= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{336 s^5} \right) \left\{ \begin{array}{l} \left[\begin{array}{l} e^{-sa} \begin{pmatrix} -144a^2s^3 - 216as^2 - 144s- \\ \\ 60a^3s^4 + 66a^4s^5 - 7a^5s^6 \end{pmatrix} \\ \\ -ae^{-sa} \begin{pmatrix} -36a^2s^4 - 72as^3 - 72s^2- \\ \\ 12a^3s^5 + 11a^4s^6 - a^5s^7 \end{pmatrix} \end{array} \right] - \\ \\ \left[\begin{array}{l} e^{-sb} \begin{pmatrix} -144b^2s^3 - 216bs^2 - 144s \\ \\ -60b^3s^4 + 66b^4s^5 - 7b^5s^6 \end{pmatrix} \\ \\ -be^{-sb} \begin{pmatrix} -36b^2s^4 - 72bs^3 - 72s^2- \\ \\ 12b^3s^5 + 11b^4s^6 - b^5s^7 \end{pmatrix} \end{array} \right] \end{array} \right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{336s^5}\right)\left\{\begin{array}{c} e^{-sa}\begin{pmatrix} -144a^2s^3 - 216as^2 - 144s- \\ \\ 60a^3s^4 + 66a^4s^5 - 7a^5s^6 + \\ \\ 36a^3s^4 + 72a^2s^3 + 72as^2 \\ \\ +12a^4s^5 - 11a^5s^6 + a^6s^7 \end{pmatrix} - \\ e^{-sb}\begin{pmatrix} -144b^2s^3 - 216bs^2 - 144s- \\ \\ 60b^3s^4 + 66b^4s^5 - 7b^5s^6 + \\ \\ 36b^3s^4 + 72b^2s^3 + 72bs^2 + \\ \\ 12b^4s^5 - 11b^5s^6 + b^6s^7 \end{pmatrix} \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{336s^5}\right)\left\{\begin{array}{c} e^{-sa}\begin{bmatrix} -72a^2s^3 - 144as^2 - 144s - 24a^3s^4 \\ \\ +78a^4s^5 - 18a^5s^6 + a^6s^7 \end{bmatrix} \\ \\ -e^{-sb}\begin{bmatrix} -72b^2s^3 - 144bs^2 - 144s - 24b^3s^4 \\ \\ +78b^4s^5 - 18b^5s^6 + b^6s^7 \end{bmatrix} \end{array}\right\}$$

$$
= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{1680s^4}\right)\left\{\begin{array}{c}
e^{-sa}\begin{bmatrix}-216a^2s^2-288as-144-96a^3s^3\\ \\ +390a^4s^4-108a^5s^5+7a^6s^6\end{bmatrix}\\ \\
-ae^{-sa}\begin{bmatrix}-72a^2s^3-144as^2-144s-\\ \\ 24a^3s^4+78a^4s^5-18a^5s^6+a^6s^7\end{bmatrix}\\ \\
e^{-sb}\begin{bmatrix}-216b^2s^2-288bs-144-96b^3s^3\\ \\ +390b^4s^4-108b^5s^5+7b^6s^6\end{bmatrix}\\ \\
-be^{-sb}\begin{bmatrix}-72b^2s^3-144bs^2-144s-24b^3s^4\\ \\ +78b^4s^5-18b^5s^6+b^6s^7\end{bmatrix}
\end{array}\right\}
$$

$$
= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{1680 s^4} \right) \left\{ \begin{array}{c} e^{-sa} \begin{bmatrix} -216a^2s^2 - 288as - 144 - 96a^3s^3 \\ \\ +390a^4s^4 - 108a^5s^5 + 7a^6s^6 \end{bmatrix} \\ \\ +e^{-sa} \begin{bmatrix} 72a^3s^3 + 144a^2s^2 + 144as + 24a^4s^4 \\ \\ -78a^5s^5 + 18a^6s^6 - a^6s^7 \end{bmatrix} \\ \\ e^{-sb} \begin{bmatrix} -216b^2s^2 - 288bs - 144 - 96b^3s^3 \\ \\ +390b^4s^4 - 108b^5s^5 + 7b^6s^6 \end{bmatrix} \\ \\ +e^{-sb} \begin{bmatrix} 72b^3s^3 + 144b^2s^2 + 144bs + 24b^4s^4 \\ \\ -78b^5s^5 + 18b^6s^6 - b^7s^7 \end{bmatrix} \end{array} \right\}
$$

$$
= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{1680 s^4} \right) \left\{ \begin{array}{c} e^{-sa} \begin{bmatrix} -216a^2s^2 - 288as - 144 - 96a^3s^3 + 390a^4s^4 \\ \\ -108a^5s^5 + 7a^6s^6 + 72a^3s^3 + 144a^2s^2 + 144as \\ \\ +24a^4s^4 - 78a^5s^5 + 18a^6s^6 - a^6s^7 \end{bmatrix} \\ \\ e^{-sb} \begin{bmatrix} -216b^2s^2 - 288bs - 144 - 96b^3s^3 + 390b^4s^4 \\ \\ -108b^5s^5 + 7b^6s^6 + 72b^3s^3 + 144b^2s^2 + 144bs \\ \\ +24b^4s^4 - 78b^5s^5 + 18b^6s^6 - b^7s^7 \end{bmatrix} \end{array} \right\}
$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{1680s^4}\right)\left\{\begin{array}{l} e^{-sa}\begin{bmatrix} -72a^2s^2-144as-144-24a^3s^3+ \\ \\ 414a^4s^4-186a^5s^5+25a^6s^6-a^7s^7 \end{bmatrix} \\ \\ -e^{-sb}\begin{bmatrix} -72b^2s^2-144bs-144-24b^3s^3+ \\ \\ 414b^4s^4-186b^5s^5+25b^6s^6-b^7s^7 \end{bmatrix} \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{6720s^3}\right)\left\{\begin{array}{l} e^{-sa}\begin{bmatrix} -144a^2s-144a-72a^3s^2+1456a^4s^3 \\ \\ -930a^5s^4+150a^6s^5-7a^7s^6 \end{bmatrix} \\ \\ -ae^{-sa}\begin{bmatrix} -72a^2s^2-144as-144-24a^3s^3+ \\ \\ 414a^4s^4-186a^5s^5+25a^6s^6-a^7s^7 \end{bmatrix} \\ \\ e^{-sb}\begin{bmatrix} -144b^2s-144b-72b^3s^2+1456b^4s^3 \\ \\ -930b^5s^4+150b^6s^5-7b^7s^6 \end{bmatrix} \\ \\ +be^{-sb}\begin{bmatrix} -72b^2s^2-144bs-144-24b^3s^3+ \\ \\ 414b^4s^4-186b^5s^5+25b^6s^6-b^7s^7 \end{bmatrix} \end{array}\right\}$$

$$= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{6720 s^3} \right) \left\{ \begin{array}{c} e^{-sa} \left[\begin{array}{c} -144a^2 s - 144a - 72a^3 s^2 + 1456a^4 s^3 \\ \\ -930a^5 s^4 + 150a^6 s^5 - 7a^7 s^6 + 72a^3 s^2 \\ \\ +144a^2 s + 144a + 24a^4 s^3 - 414a^5 s^4 \\ \\ +186a^6 s^5 - 25a^7 s^6 - a^8 s^7 \end{array} \right] \\ -e^{-sb} \left[\begin{array}{c} -144b^2 s - 144b - 72b^3 s^2 + 1456b^4 s^3 \\ \\ -930b^5 s^4 + 150b^6 s^5 - 7b^7 s^6 + 72b^3 s^2 \\ \\ +144b^2 s + 144b + 24b^4 s^3 - 414b^4 s^4 \\ \\ +186b^5 s^5 - 25b^6 s^6 + b^7 s^7 \end{array} \right] \end{array} \right\}$$

$$= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{6720 s^3} \right) \left\{ \begin{array}{c} e^{-sa} \left[1680a^4 s^3 - 1344a^5 s^4 + 336a^6 s^5 - 32a^7 s^6 + a^8 s^7 \right] \\ \\ -e^{-sb} \left[1680b^4 s^3 - 1344b^5 s^4 + 336b^6 s^5 - 32b^7 s^6 + b^8 s^7 \right] \end{array} \right\}$$

$$= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{20160 s^2} \right) \left\{ \begin{array}{c} e^{-sa} \left[5040a^4 s^2 - 5376a^5 s^3 + 1680a^6 s^4 - 192a^7 s^5 + 7a^8 s^6 \right] \\ \\ -ae^{-sa} \left[1680a^4 s^3 - 1344a^5 s^4 + 336a^6 s^5 - 32a^7 s^6 + a^8 s^7 \right] \\ \\ -e^{-sb} \left[5040b^4 s^2 - 5376b^5 s^3 + 1680b^6 s^4 - 192b^7 s^5 + 7b^8 s^6 \right] \\ \\ e^{-sb} \left[1680b^4 s^3 - 1344b^5 s^4 + 336b^6 s^5 - 32b^7 s^6 + b^8 s^7 \right] \end{array} \right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{20160s^2}\right)\left\{\begin{array}{l} e^{-sa}\begin{bmatrix} 5040a^4s^2 - 5376a^5s^3 + 1680a^6s^4 - 192a^7s^5 + 7a^8s^6 \\ \\ -1680a^5s^3 + 1344a^6s^4 - 336a^7s^5 + 32a^8s^6 - a^9s^7 \end{bmatrix} \\ \\ -e^{-sb}\begin{bmatrix} 5040b^4s^2 - 5376b^5s^3 + 1680b^6s^4 - 192b^7s^5 + 7b^8s^6 \\ \\ -1680b^5s^3 + 1344b^6s^4 - 336b^7s^5 + 32b^8s^6 - b^9s^7 \end{bmatrix} \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{20160s^2}\right)\left\{\begin{array}{l} e^{-sa}\left[5040a^4s^2 - 7056a^5s^3 + 3024a^6s^4 - 528a^7s^5 + 39a^8s^6 - a^9s^7\right] \\ \\ -e^{-sb}\left[5040b^4s^2 - 7056b^5s^3 + 3024b^6s^4 - 528b^7s^5 + 39b^8s^6 - b^9s^7\right] \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{40320s}\right)\left\{\begin{array}{l} e^{-sa}\left[10080a^4s - 21168a^5s^2 + 12096a^6s^3 - 2640a^7s^4 + 234a^8s^5 - 7a^9s^6\right] \\ \\ -ae^{-sa}\left[5040a^4s^2 - 7056a^5s^3 + 3024a^6s^4 - 528a^7s^5 + 39a^8s^6 - a^9s^7\right] \\ \\ -e^{-sb}\left[10080b^4s - 21168b^5s^2 + 12096b^6s^3 - 2640b^7s^4 + 234b^8s^5 - 7b^9s^6\right] \\ \\ +be^{-sb}\left[5040b^4s^2 - 7056b^5s^3 + 3024b^6s^4 - 528b^7s^5 + 39b^8s^6 - b^9s^7\right] \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{40320s}\right)\left\{\begin{array}{l} e^{-sa}\begin{bmatrix} 10080a^4s - 21168a^5s^2 + 12096a^6s^3 - 2640a^7s^4 + 234a^8s^5 \\ \\ -7a^9s^6 - 5040a^5s^2 + 7056a^6s^3 - 3024a^7s^4 + 528a^8s^5 \\ \\ -39a^9s^6 + a^{10}s^7 \end{bmatrix} \\ \\ -e^{-sb}\begin{bmatrix} 10080b^4s - 21168b^5s^2 + 12096b^6s^3 - 2640b^7s^4 + 234b^8s^5 \\ \\ -7b^9s^6 - 5040b^5s^2 - 7056b^6s^3 - 3024b^7s^4 + 528b^8s^5 \\ \\ -39b^9s^6 + b^{10}s^7 \end{bmatrix} \end{array}\right\}$$

$$= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{40320s} \right) \left\{ \begin{array}{c} e^{-sa} \left[\begin{array}{c} 10080a^4s - 26208a^5s^2 + 19152a^6s^3 - 5664a^7s^4 + 762a^8s^5 \\ \\ -46a^9s^6 + a^{10}s^7 \end{array} \right] \\ \\ -e^{-sb} \left[\begin{array}{c} 10080b^4s - 26208b^5s^2 + 19152b^6s^3 - 5664b^7s^4 + 762b^8s^5 \\ \\ -46b^9s^6 + b^{10}s^7 \end{array} \right] \end{array} \right\}$$

$$= \lim_{s \to 0} \frac{1}{b-a} \left(\frac{1}{40320} \right) \left\{ \begin{array}{c} e^{-sa} \left[\begin{array}{c} 10080a^4 - 52416a^5s + 57456a^6s^2 - 22656a^7s^3 + 3810a^8s^4 \\ \\ -276a^9s^5 + 7a^{10}s^6 \end{array} \right] \\ \\ -ae^{-sa} \left[\begin{array}{c} 10080a^4s - 26208a^5s^2 + 19152a^6s^3 - 5664a^7s^4 + 762a^8s^5 \\ \\ -46a^9s^6 + a^{10}s^7 \end{array} \right] \\ \\ -e^{-sb} \left[\begin{array}{c} 10080b^4 - 52416b^5s + 57456b^6s^2 - 22656b^7s^3 + 3810a^8s^4 \\ \\ -276b^9s^5 + 7b^{10}s^6 \end{array} \right] \\ \\ +be^{-sb} \left[\begin{array}{c} 10080b^4s - 26208b^5s^2 + 19152b^6s^3 - 5664b^7s^4 + 762b^8s^5 \\ \\ -46b^9s^6 + b^{10}s^7 \end{array} \right] \end{array} \right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{40320}\right)\left\{\begin{array}{c} e^{-sa}\left[\begin{array}{c} 10080a^4 - 52416a^5 s + 57456a^6 s^2 - 22656a^7 s^3 + 3810a^8 s^4 \\ -276a^9 s^5 + 7a^{10} s^6 - 10080a^5 s + 26208a^6 s^2 - 19152a^7 s^3 \\ +5664a^8 s^4 - 762a^9 s^5 + 46a^{10} s^6 - a^{11} s^7 \end{array}\right] \\ -e^{-sb}\left[\begin{array}{c} 10080b^4 - 52416b^5 s + 57456b^6 s^2 - 22656b^7 s^3 + 3810a^8 s^4 \\ -276b^9 s^5 + 7b^{10} s^6 - 10080b^5 s + 26208b^6 s^2 - 19152b^7 s^3 \\ +5664b^8 s^4 - 762b^9 s^5 + 46b^{10} s^6 - b^{11} s^7 \end{array}\right] \end{array}\right\}$$

$$= \lim_{s\to 0} \frac{1}{b-a}\left(\frac{1}{40320}\right)\left\{\begin{array}{c} e^{-sa}\left[\begin{array}{c} 10080a^4 - 62496a^5 s + 83664a^6 s^2 - 41808a^7 s^3 + 9474a^8 s^4 \\ -1038a^9 s^5 + 53a^{10} s^6 - a^{11} s^7 \end{array}\right] \\ -e^{-sb}\left[\begin{array}{c} 10080b^4 - 62496b^5 s + 83664b^6 s^2 - 41808b^7 s^3 + 9474b^8 s^4 \\ -1038b^9 s^5 + 53b^{10} s^6 - b^{11} s^7 \end{array}\right] \end{array}\right\}$$

$$\begin{aligned} &= \frac{1}{b-a}\left(\frac{1}{40320}\right)\left\{10080a^4 - 10080b^4\right\} \\ &= \frac{1}{b-a}\left(\frac{1}{40320}\right)10080\left\{a^4 - b^4\right\} \\ &= \frac{1}{4(b-a)}(-1)\left[b^4 - a^4\right] \end{aligned}$$

Thus in general

$$L_X^r(0) = \frac{1}{r+1}\sum_{i=0}^{r} b^{r-i}a^i$$

Mean

$$\begin{aligned} E\left[X\right] &= -L_X^{'}\left(0\right) \\ &= \frac{a+b}{2} \end{aligned}$$

Variance

$$\begin{aligned} Var\left(X\right) &= L_X^{''}\left(0\right) - \left[L_X^{'}\left(0\right)\right]^2 \\ &= \frac{b^2+ba+a^2}{3} - \left(-\frac{a+b}{2}\right)^2 \\ &= \frac{b^2+ba+a^2}{3} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{4\left(b^2+ba+a^2\right) - 3\left(a^2+2ab+b^2\right)}{12} \\ &= \frac{b^2-2ba+a^2}{12} \\ &= \frac{\left(b-a\right)^2}{12} \end{aligned}$$

In Particular if $X \sim U\left(0,1\right) \implies a = 0, b = 1$,we have

$$\begin{aligned} L_X(s) &= \frac{1-e^{-s}}{s} \\ E[X] &= \frac{1}{2} \\ Var[X] &= \frac{1}{2} \end{aligned}$$

4.2.2 Exponential Distribution

Suppose X is exponentially distributed with parameter λ. Then

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0, b > 0 \\ 0 & elsewhere \end{cases}$$

Therefore $L_X(s)$ is given by

$$L_X(s) = \int_0^{\infty} e^{-sx} \lambda e^{-\lambda x} dx$$

$$
\begin{aligned}
&= \lambda \int_0^\infty e^{-(s+\lambda)x} dx \\
&= \lambda \left[\frac{e^{-(s+\lambda)x}}{-(s+\lambda)} \Big|_0^\infty \right] \\
&= -\frac{\lambda}{s+\lambda} \left[e^{-(s+\lambda)x} \Big|_0^\infty \right] \\
&= -\frac{\lambda}{s+\lambda} [0-(-1)] \\
&= \frac{\lambda}{s+\lambda}
\end{aligned}
$$

Moments

$$
\begin{aligned}
L_X(s) &= \frac{\lambda}{s+\lambda} \\
&= \lambda(s+\lambda)^{-1} \\
L_X^{'}(s) &= -\lambda(s+\lambda)^{-2} \\
&= -\frac{\lambda}{(s+\lambda)^2} \\
L_X^{''}(s) &= (-1)^2 2\lambda(s+\lambda)^{-3} \\
&= (-1)^2 \frac{2\,!\lambda}{(s+\lambda)^3} \\
L_X^{'''}(s) &= (-1)^3 6\lambda(s+\lambda)^{-4} \\
&= (-1)^3 \frac{3\,!\lambda}{(s+\lambda)^4}
\end{aligned}
$$

In general it can be shown that

$$
\begin{aligned}
L_X^{(r)}(s) &= (-1)^r \frac{r\,!\lambda}{(s+\lambda)^{r+1}} \\
\therefore E[X]^r &= (-1)^r L_X^r(0) \\
&= (-1)^r (-1)^r \frac{r\,!\lambda}{(0+\lambda)^{r+1}} \\
&= \frac{r\,!}{\lambda^r}
\end{aligned}
$$

Mean

$$
E[X] = \frac{1}{\lambda}
$$

Variance

$$E[X^2] = \frac{2!}{\lambda^2} \\ = \frac{2}{\lambda^2}$$

Therefore

$$Var(X) = L''_X(0) - \left[L'_X(0)\right]^2 \\ = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\ = \frac{1}{\lambda^2}$$

4.2.3 Gamma with one parameter

if X has a gamma distribution with parameter α. Then

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha}x^{\alpha-1}e^{-x} & x > 0, \alpha > 0 \\ 0 & elsewhere \end{cases}$$

Therefore $L_X(s)$ is given by

$$L_X(s) = E\left[e^{-sX}\right] \\ = \int_0^\infty e^{-sx}\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx \\ = \frac{1}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(s+1)x}dx$$

Making the substitution $U = (s+1)x$, we have

$$x = \frac{u}{s+1} \Rightarrow dx = \frac{du}{s+1}$$

The limits remain unchanged. Thus

$$L_X(s) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{s+1}\right)^{\alpha-1} e^{-u} \frac{du}{s+1}$$

$$= \frac{1}{\Gamma(\alpha)} \left(\frac{1}{s+1}\right)^{\alpha} \underbrace{\int_0^\infty u^{\alpha-1} e^{-u} du}_{\Gamma(\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)} \left(\frac{1}{s+1}\right)^{\alpha} \Gamma(\alpha)$$

$$= \left(\frac{1}{s+1}\right)^{\alpha}$$

Moments

$$L_X(s) = (s+1)^{-\alpha} \, L'_X(s)$$

$$= -\alpha \, (s+1)^{-(\alpha+1)}$$

$$= (-1) \frac{\alpha}{(s+1)^{\alpha+1}} L''_X(s)$$

$$= (-1)^2 \alpha (\alpha+1) (s+1)^{-(\alpha+2)}$$

$$= (-1)^2 \frac{\alpha(\alpha+1)}{(s+1)^{\alpha+2}} L'''_X(s)$$

$$= (-1)^3 \alpha(\alpha+1)(\alpha+2)(s+1)^{-(\alpha+3)}$$

$$= (-1)^3 \frac{\alpha(\alpha+1)(\alpha+2)}{(s+1)^{\alpha+3}}$$

In general it can be shown that

$$L_X^{(r)}(s) = (-1)^r \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+r-1)}{(s+1)^{\alpha+r}}$$

$$= (-1)^r \frac{(\alpha+r-1)!}{(\alpha-1)!\,(s+1)^{\alpha+r}}$$

$$\Rightarrow E[X^r] = (-1)^r L_X^{(r)}(0)$$

$$= (-1)^r (-1)^r \frac{(\alpha+r-1)!}{(\alpha-1)!\,(0+1)^{\alpha+r}}$$

$$= \frac{(\alpha+r-1)!}{(\alpha-1)!}$$

Mean

$$\begin{aligned} E[X] &= \frac{(\alpha+1-1)!}{(\alpha-1)!} \\ &= \frac{(\alpha)!}{(\alpha-1)!} \\ &= \alpha \end{aligned}$$

Variance

$$\begin{aligned} E\left[X^2\right] &= \frac{(\alpha+2-1)!}{(\alpha-1)!} \\ &= \frac{(\alpha+1)!}{(\alpha-1)!} \\ &= \alpha(\alpha+1) \end{aligned}$$

Therefore

$$\begin{aligned} \text{var}(X) &= L''_X(0) - \left[L'_X(0)\right]^2 \\ &= \alpha(\alpha+1) - (\alpha)^2 \\ &= \alpha \end{aligned}$$

4.2.4 Gamma with two parameters

If if X has a gamma distribution with parameter α and β. Then

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha} x^{\alpha-1} e^{-x\beta} & x > 0, \alpha > 0 \\ 0 & elsewhere \end{cases}$$

And $L_X(s)$ is given by

$$\begin{aligned} L_X(s) &= E\left[e^{-sX}\right] \\ &= \int_0^\infty e^{-sx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(s+\beta)x} dx \end{aligned}$$

Making the substitution $u = (s + \beta)\, x$ we have

$$x = \frac{u}{s+\beta} \Rightarrow dx = \frac{du}{s+\beta}$$

The limits remain unchanged. Thus

$$\begin{aligned}
L_X(s) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{s+\beta}\right)^{\alpha-1} e^{-u} \frac{du}{s+\beta} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{s+\beta}\right)^{\alpha} \underbrace{\int_0^\infty u^{\alpha-1} e^{-u} du}_{\Gamma(\alpha)} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{s+1}\right)^{\alpha} \Gamma(\alpha) \\
&= \left(\frac{\beta}{s+1}\right)^{\alpha}
\end{aligned}$$

Moments

$$\begin{aligned}
L_X(s) &= \beta^\alpha (s+\beta)^{-\alpha} \\
L'_X(s) &= -\alpha\beta^\alpha (s+\beta)^{-(\alpha+1)} \\
&= (-1) \frac{\alpha\beta^\alpha}{(s+\beta)^{\alpha+1}} \\
L''_X(s) &= (-1)^2 \alpha\beta^\alpha (\alpha+1) (s+\beta)^{-(\alpha+2)} \\
&= (-1)^2 \frac{\alpha\beta^\alpha (\alpha+1)}{(s+\beta)^{\alpha+2}} \\
L'''_X(s) &= (-1)^3 \alpha\beta^\alpha (\alpha+1)(\alpha+2) (s+\beta)^{-(\alpha+3)} \\
&= (-1)^3 \frac{\alpha\beta^\alpha (\alpha+1)(\alpha+2)}{(s+\beta)^{\alpha+3}}
\end{aligned}$$

In general it can be shown that

$$
\begin{aligned}
L_X^{(r)}(s) &= (-1)^r \frac{\alpha\beta^\alpha (\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+r-1)}{(s+\beta)^{\alpha+r}} \\
&= (-1)^r \frac{\beta^\alpha (\alpha+r-1)!}{(\alpha-1)!\,(s+\beta)^{\alpha+r}}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E[X^r] &= (-1)^r L_X^{(r)}(0) \\
&= (-1)^r (-1)^r \frac{\beta^\alpha (\alpha+r-1)!}{(\alpha-1)!\,(0+\beta)^{\alpha+r}} \\
&= \frac{(\alpha+r-1)!}{(\alpha-1)!\beta^r}
\end{aligned}
$$

Mean

$$
\begin{aligned}
E[X] &= \frac{(\alpha+1-1)!}{(\alpha-1)!\beta} \\
&= \frac{\alpha}{\beta}
\end{aligned}
$$

Variance

$$
\begin{aligned}
E\left[X^2\right] &= \frac{(\alpha+2-1)!}{(\alpha-1)!\beta^2} \\
&= \frac{(\alpha+1)!}{(\alpha-1)!\beta^2} \\
&= \frac{\alpha(\alpha+1)}{\beta^2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\operatorname{var}(X) &= L_X''(0) - [L_X'(0)]^2 \\
&= \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 \\
&= \frac{\alpha}{\beta^2}
\end{aligned}
$$

A special case of this distribution is when α is a positive integer, the distribution in that case is refered to as Erlang $(\alpha,\ \beta)$

4.2.5 Rayleigh Distribution

Laplace transform of $\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$ for $x > 0$

$$f(x) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$$

Let

$$f_1(x) = e^{-\frac{x^2}{2\sigma^2}}$$
$$\Rightarrow f(x) = \frac{x}{\sigma^2}f_1(x)$$

First of all, We are going to find the Laplace transform of transform of $f_1(x)$.

$$L\{f_1(x)\} = \overline{f}_1(s)$$
$$\overline{f}_1(s) = \int_0^\infty f_1(x)e^{-sx}dx$$
$$= \int_0^\infty e^{-\frac{x^2}{2\sigma^2}}e^{-sx}dx$$
$$= \int_0^\infty e^{-\frac{x^2}{2\sigma^2}-sx}dx$$
$$= \int_0^\infty e^{-\left(\frac{x^2}{2\sigma^2}+sx\right)}dx$$
$$= \int_0^\infty e^{-\left(\frac{x^2+sx}{2\sigma^2}\right)}dx$$
$$= \int_0^\infty e^{-\frac{1}{2\sigma^2}\left(x^2+2\sigma^2 sx\right)}dx$$
$$= \int_0^\infty e^{-\frac{1}{2\sigma^2}\left(x^2+2\sigma^2\left(x+\sigma^2 s\right)^2-\left(\sigma^2 s\right)^2\right)}dx$$

$$= \int_0^\infty e^{-\frac{1}{2\sigma^2}\left(\left[x+\sigma^2 s\right]^2-\left(\sigma^2 s\right)^2\right)}dx$$

$$= \int_0^\infty e^{-\frac{1}{2\sigma^2}\left[x+\sigma^2 s\right]^2} e^{\frac{1}{2\sigma^2}\left(\sigma^2 s\right)^2}dx$$

$$= \int_0^\infty e^{-\frac{1}{2\sigma^2}\left[x+\sigma^2 s\right]^2} e^{\frac{\sigma^2 s^2}{2}}dx$$

Let

$$\frac{x+\sigma^2 s}{\sqrt{2}\sigma} = p$$

$$\Rightarrow dx = \sqrt{2}\sigma dp$$

Changing limits of integration yields

$$x = 0 \Rightarrow p = \frac{\sigma s}{\sqrt{2}}$$

$$x = \infty \Rightarrow p = \infty$$

Hence

$$\overline{f}_1(s) = \int_0^\infty e^{-\frac{1}{2\sigma^2}\left[x+\sigma^2 s\right]^2} e^{\frac{\sigma^2 s^2}{2}}dx$$

$$= \int_{\frac{\sigma s}{\sqrt{2}}}^\infty e^{-p^2} e^{\frac{\sigma^2 s^2}{2}}\left(\sqrt{2}\sigma\right)dp$$

$$= e^{\frac{\sigma^2 s^2}{2}}\int_{\frac{\sigma s}{\sqrt{2}}}^\infty e^{-p^2}\left(\sqrt{2}\sigma\right)dp$$

But since

$$\overline{f}_1(s) = \int_k^\infty e^{-x^2}dx = erfc(k)$$

Hence the expression for

$$
\begin{aligned}
\overline{f}_1(s) &= \left(\sqrt{2}\sigma\right) e^{\frac{\sigma^2 s^2}{2}} \int\limits_{\frac{\sigma s}{\sqrt{2}}}^{\infty} e^{-p^2} dp \\
&= \frac{\sqrt{\pi}}{2} \left(\sqrt{2}\sigma\right) e^{\frac{\sigma^2 s^2}{2}} \left[\frac{2}{\sqrt{\pi}} \int\limits_{\frac{\sigma s}{\sqrt{2}}}^{\infty} e^{-p^2} dp \right] \\
&= \sqrt{\frac{\pi}{2}} \sigma e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)
\end{aligned}
$$

Now for finding the Laplace transform that if

$$
\begin{aligned}
L\left\{g(x)\right\} &= \overline{g}(s) \\
L\left\{xg(x)\right\} &= -\frac{d}{ds}\overline{g}(s)
\end{aligned}
$$

Hence

$$
L\left\{f(x)\right\} = L\left\{\frac{x}{\sigma^2} f_1(x)\right\} = \frac{1}{\sigma^2} L\left\{x f_1(x)\right\}
$$

But

$$
\begin{aligned}
L\left\{x f_1(x)\right\} &= -\frac{d}{ds}\overline{f}_1(s) \\
&= -\frac{d}{ds}\left[\sqrt{\frac{\pi}{2}} \sigma e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)\right] \\
&= \left(-\sqrt{\frac{\pi}{2}}\sigma\right) \frac{d}{ds}\left[e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)\right] \\
&= \left(-\sqrt{\frac{\pi}{2}}\sigma\right) \left[e^{\frac{\sigma^2 s^2}{2}} \frac{d}{ds} erfc\left(\frac{\sigma s}{\sqrt{2}}\right) + erfc\left(\frac{\sigma s}{\sqrt{2}}\right) \frac{d}{ds} e^{\frac{\sigma^2 s^2}{2}}\right]
\end{aligned}
$$

Computing

$$
\frac{d}{ds} e^{\frac{\sigma^2 s^2}{2}} = \sigma^2 s e^{\frac{\sigma^2 s^2}{2}}
$$

And $\frac{d}{ds} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)$ will be Let

$$U = \frac{\sigma s}{\sqrt{2}} \Rightarrow dU = \frac{\sigma}{\sqrt{2}} ds$$
$$ds = \frac{\sqrt{2}}{\sigma} dU$$

Hence

$$\begin{aligned}
\frac{d}{ds} erfc\left(\frac{\sigma s}{\sqrt{2}}\right) &= \frac{\sigma}{\sqrt{2}\sigma} \frac{d}{dU} erfc\,(U) \\
&= \frac{\sigma}{\sqrt{2}} \left[-\frac{2}{\sqrt{\pi}} e^{-\frac{1}{2}(\sigma s)^2}\right] \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} (-\sigma)\, e^{-\frac{1}{2}(\sigma s)^2}
\end{aligned}$$

Using this expressions yields

$$\begin{aligned}
L\{xf_1(x)\} &= \left(-\sqrt{\frac{\pi}{2}}\sigma\right) \left[e^{\frac{\sigma^2 s^2}{2}} \frac{d}{ds} erfc\left(\frac{\sigma s}{\sqrt{2}}\right) + erfc\left(\frac{\sigma s}{\sqrt{2}}\right) \sigma^2 s e^{\frac{\sigma^2 s^2}{2}}\right] \\
&= \left(-\sqrt{\frac{\pi}{2}}\sigma\right) \left[e^{\frac{\sigma^2 s^2}{2}} \left(\sqrt{\frac{2}{\pi}} (-\sigma)\, e^{-\frac{1}{2}(\sigma s)^2}\right) + erfc\left(\frac{\sigma s}{\sqrt{2}}\right) \sigma^2 s e^{\frac{\sigma^2 s^2}{2}}\right] \\
&= \left(-\sqrt{\frac{\pi}{2}}\sigma\right) \left[e^{\frac{\sigma^2 s^2}{2}} \left(\sqrt{\frac{2}{\pi}} (-\sigma)\, e^{-\frac{1}{2}(\sigma s)^2}\right) + erfc\left(\frac{\sigma s}{\sqrt{2}}\right) \sigma^2 s e^{\frac{\sigma^2 s^2}{2}}\right] \\
&= \sigma^2 e^0 - \sqrt{\frac{\pi}{2}}\sigma^3 s e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right) \\
&= \sigma^2 \left[1 - \sqrt{\frac{\pi}{2}}\sigma s e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)\right]
\end{aligned}$$

Therefore

$$\begin{aligned}
L\{f(x)\} &= \frac{1}{\sigma^2} L\{xf_1(x)\} \\
&= \left(\frac{1}{\sigma^2}\right) \sigma^2 \left[1 - \sqrt{\frac{\pi}{2}}\sigma s e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)\right] \\
&= 1 - \sqrt{\frac{\pi}{2}}\sigma s e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)
\end{aligned}$$

Recall that

$$\frac{d}{dz}erfc\left(f\left(z\right)\right)=-\frac{2}{\sqrt{\pi}}e^{-\left[f(z)\right]^2}f'\left(z\right)$$

Using product rule of differentiation We obtain $L'_X\left(s\right)$ as follows

Let

$$U=se^{\frac{\sigma^2s^2}{2}}\Rightarrow U'=e^{\frac{\sigma^2s^2}{2}}+s^2\sigma^2e^{\frac{\sigma^2s^2}{2}}$$
$$V=erfc\left(\frac{\sigma^2s}{\sqrt{2}}\right)\Rightarrow V'=-\sigma\sqrt{\frac{2}{\pi}}e^{-\frac{\sigma^2s^2}{2}}$$

Thus

$$\begin{aligned}L'_X\left(s\right)&=\frac{d}{ds}\left[1-\sqrt{\frac{\pi}{2}}\sigma se^{\frac{\sigma^2s^2}{2}}erfc\left(\frac{\sigma s}{\sqrt{2}}\right)\right]\\&=-\sigma\sqrt{\frac{\pi}{2}}\frac{d}{ds}\left[se^{\frac{\sigma^2s^2}{2}}erfc\left(\frac{\sigma s}{\sqrt{2}}\right)\right]\\&=-\sigma\sqrt{\frac{\pi}{2}}\frac{d}{ds}\left[U'V+V'U\right]\\&=-\sigma\sqrt{\frac{\pi}{2}}\left[\left(e^{\frac{\sigma^2s^2}{2}}+s^2\sigma^2e^{\frac{\sigma^2s^2}{2}}\right)erfc\left(\frac{\sigma s}{\sqrt{2}}\right)-\sigma\sqrt{\frac{2}{\pi}}e^{-\frac{\sigma^2s^2}{2}}se^{\frac{\sigma^2s^2}{2}}\right]\\&-\sigma\sqrt{\frac{\pi}{2}}\left[\left(e^{\frac{\sigma^2s^2}{2}}+s^2\sigma^2e^{\frac{\sigma^2s^2}{2}}\right)erfc\left(\frac{\sigma s}{\sqrt{2}}\right)-s\sigma\sqrt{\frac{2}{\pi}}\right]\end{aligned}$$

Similarly

$$\begin{aligned}L''_X\left(s\right)&=\frac{d}{ds}L'_X\left(s\right)\\&=-\sigma\sqrt{\frac{\pi}{2}}\left[\left(s\sigma^2e^{\frac{\sigma^2s^2}{2}}+s^3\sigma^4e^{\frac{\sigma^2s^2}{2}}\right)erfc\left(\frac{\sigma s}{\sqrt{2}}\right)-\sigma\sqrt{\frac{2}{\pi}}e^{-\frac{\sigma^2s^2}{2}}\left(e^{\frac{\sigma^2s^2}{2}}+s^2\sigma^2e^{\frac{\sigma^2s^2}{2}}\right)-\sigma\sqrt{\frac{2}{\pi}}\right]\\&=-\sigma\sqrt{\frac{\pi}{2}}\left[\left(s\sigma^2e^{\frac{\sigma^2s^2}{2}}+s^3\sigma^4e^{\frac{\sigma^2s^2}{2}}\right)erfc\left(\frac{\sigma s}{\sqrt{2}}\right)-\sigma\sqrt{\frac{2}{\pi}}-s^2\sigma^3\sqrt{\frac{2}{\pi}}-\sigma\sqrt{\frac{2}{\pi}}\right]\\&=-\sigma\sqrt{\frac{\pi}{2}}\left[\left(s\sigma^2e^{\frac{\sigma^2s^2}{2}}+s^3\sigma^4e^{\frac{\sigma^2s^2}{2}}\right)erfc\left(\frac{\sigma s}{\sqrt{2}}\right)-2\sigma\sqrt{\frac{2}{\pi}}-s^2\sigma^3\sqrt{\frac{2}{\pi}}\right]\end{aligned}$$

Therefore

$$\begin{aligned} L'_X(0) &= -\sigma\sqrt{\frac{\pi}{2}}\left[(1+0)\,erf(0)-0\right] \\ &= -\sigma\sqrt{\frac{\pi}{2}} \end{aligned}$$

Since $erf(0) = 1$

$$\begin{aligned} L''_X(0) &= \sigma\sqrt{\frac{\pi}{2}}\left[(0+0)\,erf(0)-0-2\sigma\sqrt{\frac{2}{\pi}}\right] \\ &= -\sigma\sqrt{\frac{\pi}{2}}\left(-2\sigma\sqrt{\frac{2}{\pi}}\right) \\ &= 2\sigma^2 \end{aligned}$$

With this we have

Mean

$$\begin{aligned} E(X) &= -L'_X(0) \\ &= -\left(-\sigma\sqrt{\frac{\pi}{2}}\right) \\ &= \sigma\sqrt{\frac{\pi}{2}} \end{aligned}$$

Variance

$$\begin{aligned} Var(X) &= L''_X(0) - \left[L'_X(0)\right]^2 \\ &= 2\sigma^2 - \left(-\sigma\sqrt{\frac{\pi}{2}}\right)^2 \\ &= 2\sigma^2 - \sigma^2\frac{\pi}{2} \\ &= \sigma^2\left(2-\frac{\pi}{2}\right) \\ &= \sigma^2\frac{4-\pi}{2} \end{aligned}$$

4.3 Laplace Transforms in terms of Bessel Functions

4.3.1 Modified Bessel Function of the third kind

It is defined as

$$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1}e^{-\frac{w}{2}\left(x+\frac{1}{x}\right)}dx$$

where v is called the index/ order. The function satisfies the following $K_v(w) = K_{-v}(w)$

This is known as the symmetry property.

Proof

By definition

$$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1}e^{-\frac{w}{2}\left(x+\frac{1}{x}\right)}dx$$

Letting

$$t = \frac{1}{x} \Rightarrow x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$$

we have

$$\begin{aligned}
K_v(w) &= \frac{1}{2}\int_\infty^0 \left(\frac{1}{t}\right)^{v-1} e^{-\frac{w}{2}\left(\frac{1}{t}+t\right)}\left(-\frac{dt}{t^2}\right) \\
&= \frac{1}{2}\int_\infty^0 t^{-v-1}e^{-\frac{w}{2}\left(t+\frac{1}{t}\right)}dt \\
&= K_{-v}(w)
\end{aligned}$$

4.3.2 Inverse Gamma

Suppose Y has the inverse gamma distribution with parameters α and β , then

The Laplace transform of Y is obtained as follows,

$$\begin{aligned} L_Y(s) &= E\left[e^{-sY}\right] \\ &= \int_0^\infty e^{-sy}\frac{\beta^\alpha}{\Gamma\alpha}e^{-\frac{\beta}{y}}y^{-\alpha-1}dy \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-sy}e^{-\frac{\beta}{y}}y^{-\alpha-1}dy \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty y^{-\alpha-1}e^{-\left[sy+\frac{\beta}{y}\right]}dy \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty y^{-\alpha-1}e^{-s\left[y+\frac{\beta}{y}\right]}dy \end{aligned}$$

let

$$y = \sqrt{\frac{\beta}{s}}\ z \Rightarrow\ dy = \sqrt{\frac{\beta}{s}}\ dz$$

$$\begin{aligned} \therefore L_y(s) &= \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty \left(\sqrt{\frac{\beta}{s}}\ z\right)^{-\alpha-1} e^{-s\left[\sqrt{\frac{\beta}{s}}\ z+{}^{\beta}\!/_{s}\left(\sqrt{\frac{\beta}{s}}z\right)\right]}\sqrt{\frac{\beta}{s}}dz \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\left(\sqrt{\frac{\beta}{s}}\right)^{-\alpha}\int_0^\infty z^{-\alpha-1}e^{-s\left[\sqrt{\frac{\beta}{s}}\ z+\frac{\sqrt{\frac{\beta}{s}}}{z}\right]}dz \\ &= \frac{\beta^\alpha}{\Gamma\alpha}\left(\sqrt{\frac{\beta}{s}}\right)^{-\alpha}\int_0^\infty z^{-\alpha-1}e^{-s\sqrt{\frac{\beta}{s}}\left[\ z+\frac{1}{z}\right]}dz \\ &= \frac{2\beta^\alpha}{\Gamma\alpha}\left(\sqrt{\frac{\beta}{s}}\right)^{-\alpha}\underbrace{\frac{1}{2}\int_0^\infty z^{-\alpha-1}e^{-\frac{2\sqrt{\beta s}}{2}\left[\ z+\frac{1}{z}\right]}dz}_{K_{-\alpha}\left(2\sqrt{\beta s}\right)} \\ &= \frac{2\beta^\alpha}{\Gamma\alpha}\left(\sqrt{\frac{\beta}{s}}\right)^{-\alpha}K_{-\alpha}\left(2\sqrt{\beta s}\right) \end{aligned}$$

$$= \frac{2\beta^{\alpha}}{\Gamma\alpha}\left(\sqrt{\frac{s}{\beta}}\right)^{\alpha} K_{-\alpha}\left(2\sqrt{\beta s}\right)$$

$$= \frac{2}{\Gamma\alpha}\beta^{\alpha} s^{\alpha/2}\beta^{-\alpha/2} K_{-\alpha}\left(2\sqrt{\beta s}\right)$$

$$= \frac{2}{\Gamma\alpha}\beta^{\alpha/2} s^{\alpha/2} K_{-\alpha}\left(2\sqrt{\beta s}\right)$$

$$= \frac{2}{\Gamma\alpha}\left(\sqrt{\beta s}\right)^{\alpha} K_{-\alpha}\left(2\sqrt{\beta s}\right)$$

$$= \frac{2}{\Gamma\alpha}\left(\sqrt{\beta s}\right)^{\alpha} K_{\alpha}\left(2\sqrt{\beta s}\right)$$

4.3.3 Inverse Gaussian

Suppose Y has the inverse gamma distribution with parameters μ and ϕ , then

The Laplace transform of Y is obtained as follows

$$L_Y(s) = E\left[e^{-sY}\right]$$

$$= \int_0^{\infty} e^{-sy}\left(\frac{\phi}{2\pi y^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi(y-\mu)^2}{2y\mu^2}\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^{\infty} y^{-\frac{3}{2}} \exp\left\{-sy - \frac{\phi(y-\mu)^2}{2y\mu^2}\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^{\infty} y^{-\frac{3}{2}} \exp\left\{-sy - \frac{\phi(y^2-2\mu y+\mu^2)}{2y\mu^2}\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty y^{-\frac{3}{2}} \exp\left\{-sy - \frac{\phi y}{2\mu^2} + \frac{\phi}{\mu} - \frac{\phi}{2y}\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty y^{-\frac{3}{2}} \exp\left\{-sy - \frac{\phi y}{2\mu^2} - \frac{\phi}{2y}\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty y^{-\frac{3}{2}} \exp\left\{-\left[\left(s + \frac{\phi}{2\mu^2}\right) y + \frac{\phi}{2y}\right]\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty y^{-\frac{3}{2}} \exp\left\{-\left[\left(\frac{2s\mu^2 + \phi}{2\mu^2}\right) y + \frac{\phi}{2\lambda}\right]\right\} dy$$

$$\therefore L_Y(s) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty y^{-\frac{3}{2}} \exp\left\{-\left(\frac{2s\mu^2 + \phi}{2\mu^2}\right)\left[y + \frac{\phi}{2}\left(\frac{2\mu^2}{2s\mu^2 + \phi}\right)\frac{1}{y}\right]\right\} dy$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty y^{-\frac{3}{2}} \exp\left\{-\left(\frac{2s\mu^2 + \phi}{2\mu^2}\right)\left[y + \frac{\phi\mu^2}{2s\mu^2 + \phi}\left(\frac{1}{y}\right)\right]\right\} dy$$

Letting

$$y = \left(\sqrt{\frac{\phi\mu^2}{\phi + 2s\mu^2}}\right) z \Rightarrow dy = \left(\sqrt{\frac{\phi\mu^2}{\phi + 2s\mu^2}}\right) dz$$

$$L_Y(s) = \left\{ \begin{array}{c} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int\limits_0^\infty \left(\sqrt{\frac{\phi\mu^2}{\varphi+2s\mu^2}} z\right)^{-\frac{3}{2}} * \\ \exp\left\{-\left(\frac{2s\mu^2+\phi}{2\mu^2}\right)\left[\sqrt{\frac{\phi\mu^2}{\varphi+2s\mu^2}} z + \frac{\phi\mu^2}{2s\mu^2+\phi}\left(\frac{1}{\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}} z}\right)\right]\right\} \sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}} dz \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}}\right)^{-\frac{1}{2}} * \\ \int\limits_0^\infty z^{-\frac{3}{2}} \exp\left\{-\left(\frac{2s\mu^2+\phi}{2\mu^2}\right) \left[\left(\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}}\right) z + \left(\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}}\right)\left(\frac{1}{z}\right)\right]\right\} dz \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi+2s\mu^2}{\phi\mu^2}}\right)^{\frac{1}{2}} * \\ \int\limits_0^\infty z^{-\frac{1}{2}-1} \exp\left\{-\left(\frac{2s\mu^2+\phi}{2\mu^2}\right) \left(\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}}\right) \left[z + \frac{1}{z}\right]\right\} dz \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi+2s\mu^2}{\phi\mu^2}}\right)^{\frac{1}{2}} * \\ \int\limits_0^\infty z^{-\frac{1}{2}-1} \exp\left\{-\left(\frac{2s\mu^2+\varphi}{2\mu^2}\right) \left(\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}}\right) \left[z + \frac{1}{z}\right]\right\} dz \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi+2s\mu^2}{\phi\mu^2}}\right)^{\frac{1}{2}} * \\ \underbrace{\int\limits_0^\infty z^{-\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\frac{\phi\left(\phi+2s\mu^2\right)}{\mu^2}} \left(\sqrt{\frac{\phi\mu^2}{\phi+2s\mu^2}}\right) \left[z + \frac{1}{z}\right]\right\} dz}_{2K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi(\phi+2s\mu^2)}{\mu^2}}\right)} \end{array} \right\}$$

$$\therefore L_Y(s) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi + 2s\mu^2}{\phi\mu^2}}\right)^{\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi(\phi+2s\mu^2)}{\mu^2}}\right)$$

Using willmots notations, let

$$\phi = \frac{\mu^2}{\beta}$$

$$\begin{aligned}
\Rightarrow L_Y(s) &= \left(\frac{\mu^2}{2\pi\beta}\right)^{\frac{1}{2}} e^{\frac{\mu^2}{\beta\mu}} \left(\sqrt{\frac{\frac{\mu^2}{\beta} + 2s\mu^2}{\frac{\mu^2}{\beta}\mu^2}}\right)^{\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\frac{\mu^2}{\beta}\left(\frac{\mu^2}{\beta} + 2s\mu^2\right)}{\mu^2}}\right) \\
&= \left(\frac{\mu^2}{2\pi\beta}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \left(\sqrt{\frac{\frac{\mu^2}{\beta}(1+2s\beta)}{\frac{\mu^4}{\beta}}}\right)^{\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\left(\frac{\mu^2}{\beta} + 2s\mu^2\right)}{\beta}}\right) \\
&= \frac{\mu}{\sqrt{2\pi\beta}} e^{\frac{\mu}{\beta}} \frac{1}{\sqrt{\mu}} \left(\sqrt{1+2s\beta}\right)^{\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\mu^2}{\beta^2}(1+2\beta s)}\right) \\
&= \frac{\sqrt{\mu}}{\sqrt{2\pi\beta}} e^{\frac{\mu}{\beta}} \left(\sqrt{1+2s\beta}\right)^{\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{1+2\beta s}\right) \\
&= e^{\frac{\mu}{\beta}} \left(\frac{\mu}{2\pi\beta}\sqrt{1+2s\beta}\right)^{\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{1+2\beta s}\right) \\
&= e^{\frac{\mu}{\beta}} \left(\frac{2\mu}{\pi\beta}\sqrt{1+2s\beta}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{1+2\beta s}\right) \\
&= e^{\frac{\mu}{\beta}} \left(\frac{2\mu}{\pi\beta}\sqrt{1+2s\beta}\right)^{\frac{1}{2}} \sqrt{\frac{\pi}{\frac{2\mu}{\beta}\sqrt{1+2\beta s}}}\, e^{-\frac{\mu}{\beta}\sqrt{1+2\beta s}} \\
&= e^{\frac{\mu}{\beta}} e^{-\frac{\mu}{\beta}\sqrt{1+2\beta s}} \\
&= \exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s} - 1\right)\right\}
\end{aligned}$$

Obtaining the first and second derivatives of $L_Y(s)$ with respect to s we have

$$\begin{aligned}
L'_Y(s) &= -\frac{\mu}{\beta}\left(\frac{1}{2}\right)(1+2\beta s)^{-\frac{1}{2}}\,2\beta\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\} \\
&= -\frac{\mu}{\sqrt{1+2\beta s}}\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\} \\
\therefore L'_Y(0) &= -\frac{\mu}{\sqrt{1+2\beta(0)}}\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta(0)}-1\right)\right\} \\
&= -\mu
\end{aligned}$$

Using the product rule of differentiation we let

$$U = -\frac{\mu}{\sqrt{1+2\beta s}} = -\mu(1+2\beta s)^{-1/2} \Rightarrow U' = -\mu\left(-\frac{1}{2}\right)(1+2\beta s)^{-3/2}\,2\beta = \frac{\mu\beta}{\left(\sqrt{1+2\beta s}\right)^3}$$

$$V = \exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\} \Rightarrow V' = -\frac{\mu}{\sqrt{1+2\beta s}}\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\}$$

Therefore

$$L''_Y(s) = \frac{\mu\beta}{\left(\sqrt{1+2\beta s}\right)^3}\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\} + \left(\frac{\mu}{\sqrt{1+2\beta s}}\right)^2\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\}$$

$$\therefore L''_Y(0) = \left\{\begin{matrix} \frac{\mu\beta}{\left(\sqrt{1+2\beta(0)}\right)^3}\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta(0)}-1\right)\right\} + \\ \\ \left(\frac{\mu}{\sqrt{1+2\beta(0)}}\right)^2\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta(0)}-1\right)\right\} \end{matrix}\right\}$$

$$= \mu\beta + \mu^2$$

Mean

$$\begin{aligned}
E[Y] &= -L_X(0) \\
&= \mu
\end{aligned}$$

Variance

$$\begin{aligned} Var\left(Y\right) &= L_Y''\left(0\right) - \left[L_Y'\left(0\right)\right]^2 \\ &= \mu\beta + \mu^2 - \left(\mu\right)^2 \\ &= \mu\beta \end{aligned}$$

But by Willnots notation

$$\phi = \frac{\mu^2}{\beta} \Rightarrow \beta = \frac{\mu^2}{\phi}$$

Thus

$$\begin{aligned} Var\left(Y\right) &= \mu\left(\frac{\mu^2}{\phi}\right) \\ &= \frac{\mu^3}{\phi} \end{aligned}$$

4.3.4 Generalized Inverse Gaussian

Suppose Y has the generalized inverse Gaussian distribution with parameters λ, χ and ψ , then

$$f(y, \lambda, \chi, \psi) = \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} y^{\lambda-1}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \exp\left\{-\frac{1}{2}\left(\psi y + \frac{\chi}{y}\right)\right\} \quad y > 0$$

The Laplace transform of Y is given by

$$\begin{aligned} L_Y(s) &= E\left[e^{-sY}\right] \\ &= \int_0^\infty e^{-sy} \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} y^{\lambda-1}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} e^{-\frac{1}{2}\left(\psi y + \frac{\chi}{y}\right)} dy \\ &= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty y^{\lambda-1} e^{-sy} e^{-\frac{1}{2}\left(\psi y + \frac{\chi}{y}\right)} dy \end{aligned}$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty y^{\lambda-1} e^{-sy-\frac{1}{2}\left(\psi y+\frac{\chi}{y}\right)} dy$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty y^{\lambda-1} \exp\left\{-sy - \frac{\psi y}{2} - \frac{\chi}{2y}\right\} dy$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty y^{\lambda-1} \exp\left\{-y\left(s+\frac{\psi}{2}\right) - \frac{\chi}{2y}\right\} dy$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty y^{\lambda-1} \exp\left\{-\left(s+\frac{\psi}{2}\right)\left[y+\frac{\chi}{2y\left(s+\frac{\psi}{2}\right)}\right]\right\} dy$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty y^{\lambda-1} \exp\left\{-\left(\frac{2s+\psi}{2}\right)\left[y+\frac{\chi}{y\left(2s+\psi\right)}\right]\right\} dy$$

Let

$$y = \sqrt{\frac{\chi}{2s+\psi}}\, z \;\Rightarrow\; dy = \sqrt{\frac{\chi}{2s+\psi}}\, dz$$

$$L_y(s) = \left\{ \begin{array}{c} \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int\limits_0^\infty \left(\sqrt{\frac{\chi}{2s+\psi}}\ z\right)^{\lambda-1} * \\ \exp\left\{-\left(\frac{2s+\psi}{2}\right)\left[\sqrt{\frac{\chi}{2s+\psi}}\ z + \frac{\chi}{(2s+\psi)\sqrt{\frac{\chi}{2s+\psi}}\ z}\right]\right\}\sqrt{\frac{\chi}{2s+\psi}}\ dz \end{array} \right\}$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)}\left(\sqrt{\frac{\chi}{2s+\psi}}\right)^{\lambda}\int\limits_0^\infty z^{\lambda-1}\exp\left\{-\left(\frac{2s+\psi}{2}\right)\left[\sqrt{\frac{\chi}{2s+\psi}}\ z + \sqrt{\frac{\chi}{2s+\psi}} \times \frac{1}{z}\right]\right\}dz$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)}\left(\sqrt{\frac{\chi}{2s+\psi}}\right)^{\lambda}\int\limits_0^\infty z^{\lambda-1}\exp\left\{-\left(\frac{2s+\psi}{2}\right)\sqrt{\frac{\chi}{2s+\psi}}\left[z+\frac{1}{z}\right]\right\}dz$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)}\left(\sqrt{\frac{\chi}{2s+\psi}}\right)^{\lambda}\underbrace{\int\limits_0^\infty z^{\lambda-1}\exp\left\{-\left(\frac{\sqrt{\chi\left(2s+\psi\right)}}{2}\right)\left[z+\frac{1}{z}\right]\right\}dz}_{2K_\lambda\left(\sqrt{\chi(2s+\psi)}\right)}$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)}\left(\sqrt{\frac{\chi}{2s+\psi}}\right)^{\lambda}2K_\lambda\left(\sqrt{\chi\left(2s+\psi\right)}\right)$$

$$= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}\left(\frac{\chi}{2s+\psi}\right)^{\frac{\lambda}{2}}\frac{K_\lambda\left(\sqrt{\chi\left(2s+\psi\right)}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)}$$

$$= \left(\frac{\psi}{2s+\psi}\right)^{\frac{\lambda}{2}}\frac{K_\lambda\left(\sqrt{\chi\left(2s+\psi\right)}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)}$$

4.4 A fixed sum of Independent Random variables

Suppose X and Y are two independent non negative integer valued random variables. Let $L_X(s)$ and $L_Y(s)$ denote the Laplace transform of X and Y respectively. Futher suppose that $Z = X + Y$, we wish to determine the Laplace transform of Z denoted by $L_Z(s)$ .In doing so, we consider both the expectation and convolution approaches.

4.4.1 Expectation approach

By the definition of Laplace transform of a random variable Z

$$L_Z(s) = E\left(e^{-sZ}\right)$$

But $Z = X + Y$ implying that

$$\begin{aligned} L_Z(s) &= E\left(e^{-s(X+Y)}\right) \\ &= E\left(e^{-sX}e^{-sY}\right) \\ &= E\left(e^{-sX}\right)E\left(e^{-sY}\right) \\ &= L_X(s)\,L_Y(s) \end{aligned}$$

Since X and Y are independent. Further if they are identically distributed then

$$\begin{aligned} L_Z(s) &= [L_X(s)]^2 \\ &= [L_Y(s)]^2 \end{aligned}$$

This can be extended to a fixed sum of n random variables. In a similar fashion, we let $Z = X_1 + X_2 + X_3 + ... + X_n$ where the $X_i's$ are independent random variables and n is a fixed positive integer. Further if $L_{X_i}(s)$ is the Laplace transform of X_i , we determine the Laplace transform of Z as follows.
By definition

$$
\begin{aligned}
L_Z(s) &= E\left(e^{-sZ}\right) \\
&= E\left[e^{-s(X_1+X_2+X_3+...+X_n)}\right] \\
&= E\left[e^{-sX_1}e^{-sX_2}e^{-sX_3}...e^{-sX_n}\right] \\
&= E\left(e^{-sX_1}\right)E\left(e^{-sX_2}\right)E\left(e^{-sX_3}\right)...E\left(e^{-sX_n}\right) \\
&= \prod_{i=1}^{n} E\left(e^{-sX_i}\right)
\end{aligned}
$$

If $X_i's$ are identically distributed then
$L_Z(s) = [L_X(s)]^n$

4.4.2 Convolution approach

Consider three sets of sequences namely $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$. If

$$
\begin{aligned}
\{c_k\} &= a_0b_k + a_1b_{k-1} + a_2b_{k-2} + ... + a_kb_0 \\
&= \sum_{r=0}^{k} a_rb_{k-r}
\end{aligned}
$$

Then we say that the sequence $\{c_k\}$ is a convolution of the sequences $\{a_k\}$ and $\{b_k\}$
Notationwise we write

$$\{c_k\} = \{a_k\} * \{b_k\}$$

Consider three random variables X, Y and Z . If

$$
\begin{aligned}
\{a_k\} &= \Pr ob\,(X = k) \\
\{b_k\} &= \Pr ob\,(Y = k) \\
\{c_k\} &= \Pr ob\,(Z = k)
\end{aligned}
$$

Let $L_X(s)$, $L_Y(s)$ and $L_Z(s)$ denote the Laplace transforms of X, Y and Z respectively, then

$$\begin{aligned} L_X(s) &= E\left(e^{-sX}\right) \\ &= \sum_{k=0}^{\infty} P(X=k)\, e^{-sk} \\ &= \sum_{k=0}^{\infty} a_k e^{-sk} \end{aligned}$$

$$\begin{aligned} L_Y(s) &= E\left(e^{-sY}\right) \\ &= \sum_{k=0}^{\infty} P(Y=k)\, e^{-sk} \\ &= \sum_{k=0}^{\infty} b_k e^{-sk} \end{aligned}$$

$$\begin{aligned} L_Z(s) &= E\left(e^{-sZ}\right) \\ &= \sum_{k=0}^{\infty} P(Z=k)\, e^{-sk} \\ &= \sum_{k=0}^{\infty} c_k e^{-sk} \end{aligned}$$

Then $L_Z(s) = L_X(s) \times L_Y(s)$. To Prove this, we start with the right hand side

$$\begin{aligned} L_X(s) \times L_Y(s) &= \left(\sum_{k=0}^{\infty} a_k e^{-sk}\right)\left(\sum_{k=0}^{\infty} b_k e^{-sk}\right) \\ &= \left(a_0 + a_1 e^{-s} + a_2 e^{-2s} + a_3 e^{-3s} + ...\right)\left(b_0 + b_1 e^{-s} + b_2 e^{-2s} + b_3 e^{-3s} + ...\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, e^{-s} + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, e^{-2s} + \\ &\quad (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)\, e^{-3s} + ... \end{aligned}$$

but

$$
\begin{aligned}
c_k &= \sum_{k=0}^{k} a_r b_{r-k} \\
\Rightarrow c_0 &= a_0 b_0 \\
c_1 &= a_0 b_1 + a_1 b_0 \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_2
\end{aligned}
$$

$$
\begin{aligned}
\therefore L_X(s) \times L_Y(s) &= \sum_{k=0}^{\infty} c_k e^{-sk} \\
&= L_Z(s)
\end{aligned}
$$

Let X and Y be two independent non negative integer-valued random variables with probability $P(X = k) = a_k$ and $P(Y = j) = b_j$. The sum $Z = X + Y$ is also a random variable, Now the event $Z = r$ can happen in the following mutually exclusive ways with the corresponding probabilities. $Z = r$ if

$X = 0$ and $Y = r$ with probability $a_0 b_r$ or

$X = 1$ and $Y = r - 1$ with probability $a_1 b_{r-1}$ or

$X = 2$ and $Y = r - 2$ with probability $a_2 b_{r-2}$ or

$X = 3$ and $Y = r - 3$ with probability $a_3 b_{r-3}$ or

.

.

.

$X = r$ and $Y = 0$ with probability $a_r b_0$

The probability distribution of Z is thus given by

$$
\begin{aligned}
c_r &= P(Z = r) \\
&= \Pr ob\,(X = 0 \ and\ Y = r)\, or \Pr ob\,(X = 1 \ and\ Y = r-1)\, or... \\
&\quad or \Pr ob\,(X = 2 \ and\ Y = r-2)
\end{aligned}
$$

$$
\begin{aligned}
&= \Pr ob\,(X = 0 \ and\ Y = r) + \Pr ob\,(X = 1 \ and\ Y = r-1) + ... \\
&\quad + \Pr ob\,(X = 2 \ and\ Y = r-2)
\end{aligned}
$$

$$
= a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + a_3 b_{r-3} + ... + a_r b_0
$$

$$
\therefore c_r = \sum_{m=0}^{r} a_m b_{r-m}
$$

This means that c_r is a convolution of $\{a_k\}$ and $\{b_j\}$. From above it follows that

$$
L_Z(s) = L_X(s)\, L_Y(s)
$$

Similarly if $Z = X_1 + X_2 + X_3 + X_4 + ... + X_n$ where n is a fixed positive integer, it can be shown that

$$
\begin{aligned}
L_Z(s) &= L_{X_1}(s)\, L_{X_2}(s)\, L_{X_3}(s) ... L_{X_n}(s) \\
&= \prod_{i=1}^{n} L_{X_i}(s)
\end{aligned}
$$

Further if the $X_i's$ are independent then $L_Z(s) = [L_X(s)]^n$

Examples

Let $Z = X_1 + X_2 + X_3 + X_4 + ... + X_n$ where the $X_i's$ are independent and identically distributed and n is a fixed positive integer.

1. If $X_i's$ are from uniform $(0,1)$, then From above if $X \sim U(0,\ 1)$ then

$$\begin{aligned} L_X(s) &= \frac{1-e^{-s}}{s} \\ \Rightarrow L_Z(s) &= [L_X(s)]^n \\ &= \left(\frac{1-e^{-s}}{s}\right)^n \end{aligned}$$

We now use binomial expansion to determine the distribution of Z

$$\begin{aligned} L_Z(s) &= s^{-n}\left(1-e^{-s}\right)^n \\ &= s^{-n}\sum_{k=0}^{n}\binom{n}{k}\left(-e^{-s}\right)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-sk}s^{-n} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-sk}s^{-n} \end{aligned}$$

The Laplace transform of $\frac{x^n}{n!}$ is

$$\begin{aligned} L\left\{\frac{x^n}{n!}\right\} &= \int_0^\infty e^{-sx}\frac{x^n}{n!}dx \\ &= \frac{1}{n!}\frac{\Gamma(n+1)}{s^{n+1}} \\ &= \frac{1}{s^{n+1}} \\ \Rightarrow L\left\{\frac{x^{n-1}}{(n-1)!}\right\} &= \frac{1}{s^n} \\ &= s^{-n} \end{aligned}$$

And

$$L\left\{\frac{(x-a)^{n-1}}{(n-1)!}\right\} = \int_0^\infty e^{-sx}\frac{(x-a)^{n-1}}{(n-1)!}dx$$
$$= \frac{1}{(n-1)!}\int_0^\infty e^{-sx}(x-a)^{n-1}\,dx$$

Let $y = x - a \Rightarrow dy = dx$
Then

$$\therefore L\left\{\frac{(x-a)^{n-1}}{(n-1)!}\right\} = \frac{1}{(n-1)!}\int_0^\infty e^{-s(y+a)}y^{n-1}dy$$
$$= \frac{e^{-sa}}{(n-1)!}\int_{-a}^\infty e^{-sy}y^{n-1}dy$$
$$= \frac{e^{-sa}}{(n-1)!}\int_0^\infty e^{-sy}y^{n-1}dy$$
$$= \frac{e^{-sa}}{(n-1)!}\frac{\Gamma n}{s^n}$$
$$= \frac{e^{-sa}}{s^n}$$
$$\therefore L\left\{\frac{(x-a)^{n-1}}{(n-1)!}\right\} = e^{-sa}s^{-n}$$

Thus

$$L_Z(s) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}L\left\{\frac{(x-a)^{n-1}}{(n-1)!}\right\}$$
$$f(Z) = L^{-1}\{L_Z(s)\}$$
$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(x-a)^{n-1}}{(n-1)!} \qquad x \geqslant k$$

Thus if $X_i's \sim iid \;\; unif\,(0,1)$ and $Z = \sum\limits_{i=1}^{n} x_i$ where n is fixed then the pdf of Z is given by

$$f\,(Z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(t-k)^{n-1}}{(n-1)!} \qquad t \geqslant k$$

2. If $X_i's$ are exponentially distributed with parameter λ
From above

$$\begin{aligned} L_X\,(s) &= \frac{\lambda}{\lambda+s} \\ \Rightarrow L_Z\,(s) &= [L_X\,(s)]^n \\ &= \left(\frac{\lambda}{\lambda+s}\right)^n \end{aligned}$$

But this is the Laplace transform of a Gamma distribution with parameters n and λ. Therefore $Z \sim \text{Gamma}(n,\lambda)$

3. If $X_i's$ are from Gamma $(\alpha,\ \beta)$
From above if $X \sim \text{Gamma}\,(\alpha,\ \beta)$ then

$$L_X\,(s) = \left(\frac{\beta}{\beta+s}\right)^\alpha$$

$$\begin{aligned} \Rightarrow L_Z\,(s) &= [L_X\,(s)]^n \\ &= \left[\left(\frac{\beta}{\beta+s}\right)^\alpha\right]^n \\ &= \left(\frac{\beta}{\beta+s}\right)^{\alpha n} \end{aligned}$$

which is the Laplace transform of Gamma distribution with parameters αn and β . Thus $Z \sim \text{Gamma}(\alpha n, \beta)$

4.4.3 Reciprocal of Inverse Gaussian

Let $\lambda = \frac{1}{Y}$ where Y is inverse Gaussian i.e

Using the change of variable technique we have

$$|J| = \left|\frac{dy}{d\lambda}\right| = \left|\frac{1}{-\lambda^2}\right| = \frac{1}{\lambda^2}$$

Therefore the pdf of λ is given by

$$\begin{aligned}
g(\lambda) &= h(y)\frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}\left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} y^{-\frac{3}{2}} e^{\frac{\varphi}{\mu}} \exp\left\{-\frac{\varphi}{2\mu^2}\left(y+\frac{\mu^2}{y}\right)\right\} \\
&= \frac{1}{\lambda^2}\left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\lambda}\right)^{-\frac{3}{2}} e^{\frac{\varphi}{\mu}} \exp\left\{-\frac{\varphi}{2\mu^2}\left(\frac{1}{\lambda}+\mu^2\lambda\right)\right\} \\
&= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} (\lambda)^{\frac{3}{2}-2} e^{\frac{\varphi}{\mu}} \exp\left\{-\frac{\varphi}{2\mu^2}\left(\frac{1}{\lambda}+\mu^2\lambda\right)\right\} \\
&= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} \lambda^{-\frac{1}{2}} e^{\frac{\varphi}{\mu}} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2}+\lambda\right)\right\} \\
&= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} \lambda^{\frac{1}{2}-1} e^{\frac{\varphi}{\mu}} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2}+\lambda\right)\right\} \qquad \lambda > 0
\end{aligned}$$

We now obtain the Laplace transform of λ as follows

$$\begin{aligned}
L_\lambda(s) &= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty e^{-s\lambda}\lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2}+\lambda\right)\right\} d\lambda \\
&= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2}+\lambda\right) - s\lambda\right\} d\lambda
\end{aligned}$$

$$= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2} + \lambda + \frac{2}{\varphi}s\lambda\right)\right\} d\lambda$$

$$= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2} + \left[1 + \frac{2}{\varphi}s\right]\lambda\right)\right\} d\lambda$$

$$= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2}\left(\frac{1}{\lambda\mu^2} + \left[1 + \frac{2}{\varphi}s\right]\lambda\right)\right\} d\lambda$$

$$= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2}\left[1 + \frac{2}{\varphi}s\right]\left(\frac{1}{\lambda\left[1 + \frac{2}{\varphi}s\right]\mu^2} + \lambda\right)\right\} d\lambda$$

let

$$\lambda = \frac{1}{\sqrt{\left(1 + \frac{2}{\varphi}s\right)\mu^2}} z$$

$$\Rightarrow L_\lambda(s) = \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \int_0^\infty \left[\frac{1}{\sqrt{\left(1 + \frac{2}{\varphi}s\right)\mu^2}} z\right]^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2\mu}\sqrt{1 + \frac{2}{\varphi}s}\left(z + \frac{1}{z}\right)\right\} \frac{1}{\sqrt{\left(1 + \frac{2}{\varphi}s\right)\mu^2}} dz$$

$$= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} \frac{e^{\frac{\varphi}{\mu}}}{\left(\sqrt{\left(1 + \frac{2}{\varphi}s\right)\mu^2}\right)^{\frac{1}{2}}} \underbrace{\int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\frac{\varphi}{2\mu}\sqrt{1 + \frac{2}{\varphi}s}\left(z + \frac{1}{z}\right)\right\} dz}_{2K_{\frac{1}{2}}\left(\frac{\varphi}{2}\sqrt{1 + \frac{2}{\varphi}s}\right)}$$

$$= \left(\frac{\varphi}{2\pi}\right)^{\frac{1}{2}} \frac{e^{\frac{\varphi}{\mu}}}{\left(\sqrt{\left(1 + \frac{2}{\varphi}s\right)\mu^2}\right)^{\frac{1}{2}}} 2K_{\frac{1}{2}}\left(\frac{\varphi}{\mu}\sqrt{1 + \frac{2}{\varphi}s}\right)$$

$$= \left(\frac{2\varphi}{\mu\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \frac{1}{\left(\sqrt{1 + \frac{2}{\varphi}s}\right)^{\frac{1}{2}}} K_{\frac{1}{2}}\left(\frac{\varphi}{\mu}\sqrt{1 + \frac{2}{\varphi}s}\right)$$

$$= \left(\frac{2\varphi}{\mu\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \frac{1}{\left(\sqrt{1+\frac{2}{\varphi}s}\right)^{\frac{1}{2}}} \times \sqrt{\frac{\pi}{2\left(\frac{\varphi}{\mu}\right)\sqrt{1+\frac{2}{\varphi}s}}} \exp\left(-\frac{\varphi}{\mu}\sqrt{1+\frac{2}{\varphi}s}\right)$$

$$= \left(\frac{2\varphi}{\mu\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \frac{1}{\left(\sqrt{1+\frac{2}{\varphi}s}\right)^{\frac{1}{2}}} \times \sqrt{\frac{\pi\mu}{2\varphi}\left(\frac{1}{\sqrt{1+\frac{2}{\varphi}s}}\right)} \exp\left(-\frac{\varphi}{\mu}\sqrt{1+\frac{2}{\varphi}s}\right)$$

$$= \left(\frac{2\varphi}{\mu\pi}\right)^{\frac{1}{2}} e^{\frac{\varphi}{\mu}} \frac{1}{\left(\sqrt{1+\frac{2}{\varphi}s}\right)^{\frac{1}{2}}} \times \left(\frac{\pi\mu}{2\varphi}\right)^{1/2} \left(\frac{1}{\sqrt{1+\frac{2}{\varphi}s}}\right)^{1/2} \exp\left(-\frac{\varphi}{\mu}\sqrt{1+\frac{2}{\varphi}s}\right)$$

$$= e^{\frac{\varphi}{\mu}} \frac{1}{\sqrt{1+\frac{2}{\varphi}s}} \exp\left(-\frac{\varphi}{\mu}\sqrt{1+\frac{2}{\varphi}s}\right)$$

$$\therefore L_\lambda(s) = \left(\sqrt{1+\frac{2}{\varphi}s}\right)^{-1} \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

$$= \left(1+\frac{2}{\varphi}s\right)^{-1/2} \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

We now obtain the first and second derivatives , using product rule

$$U = (1+\frac{2}{\varphi}s)^{-1/2}$$

$$\Rightarrow U' = -\frac{1}{2}\left(1+\frac{2}{\varphi}s\right)^{-3/2} \times \frac{2}{\varphi}$$

$$= -\frac{1}{\varphi}\left(1+\frac{2}{\varphi}s\right)^{-3/2}$$

$$V = \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

$$\begin{aligned}\Rightarrow V' &= -\frac{\varphi}{2\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2}\times\frac{2}{\varphi}\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}\\ &= -\frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2}\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}\end{aligned}$$

Therefore

$$L'_\lambda(s) = \left\{\begin{array}{c} -\frac{1}{\varphi}\left(1+\frac{2}{\varphi}s\right)^{-3/2}\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\} - \\ \frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2}\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}\left(1+\frac{2}{\varphi}s\right)^{-1/2}\end{array}\right\}$$

$$= \left[-\frac{1}{\varphi}\left(1+\frac{2}{\varphi}s\right)^{-3/2} - \frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1}\right]\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

$$\begin{aligned}\therefore L'_\lambda(0) &= \left[-\frac{1}{\varphi}\left(1+\frac{2}{\varphi}(0)\right)^{-3/2} - \frac{1}{\mu}\left(1+\frac{2}{\varphi}(0)\right)^{-1}\right]\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}(0)}-1\right)\right\}\\ &= -\left(\frac{1}{\varphi}+\frac{1}{\mu}\right)\end{aligned}$$

Similarly from $L'_\lambda(s)$ we let

$$U = -\frac{1}{\varphi}\left(1+\frac{2}{\varphi}s\right)^{-3/2} - \frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1}$$

$$\Rightarrow U' = \frac{3}{2\varphi}\left(1+\frac{2}{\varphi}s\right)^{-5/2}\left(\frac{2}{\varphi}\right)+\frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-2}\left(\frac{2}{\varphi}\right)$$
$$= \frac{3}{\varphi^2}\left(1+\frac{2}{\varphi}s\right)^{-5/2}+\frac{2}{\mu\varphi}\left(1+\frac{2}{\varphi}s\right)$$

and

$$V = \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$
$$\Rightarrow V' = -\frac{\varphi}{2\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2}\times\frac{2}{\varphi}\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$
$$= -\frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2}\exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

Thus

$$L_\lambda(s) = \left\{\begin{array}{c} \left[\frac{3}{\varphi^2}\left(1+\frac{2}{\varphi}s\right)^{-5/2}+\frac{2}{\mu\varphi}\left(1+\frac{2}{\varphi}s\right)^{-2}\right]* \\ \\ \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}\ +\frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2}* \\ \\ \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}\left[\frac{1}{\varphi}\left(1+\frac{2}{\varphi}s\right)^{-3/2}-\frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1}\right] \end{array}\right\}$$

$$= \left\{ \begin{array}{c} \frac{3}{\varphi^2}\left(1+\frac{2}{\varphi}s\right)^{-5/2} + \frac{2}{\mu\varphi}\left(1+\frac{2}{\varphi}s\right)^{-2} + \\ \\ \frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1/2} * \\ \\ \left[\frac{1}{\varphi}\left(1+\frac{2}{\varphi}s\right)^{-3/2} - \frac{1}{\mu}\left(1+\frac{2}{\varphi}s\right)^{-1}\right] \end{array} \right\} \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

$$= \left\{ \begin{array}{c} \frac{3}{\varphi^2}\left(1+\frac{2}{\varphi}s\right)^{-5/2} + \frac{2}{\mu\varphi}\left(1+\frac{2}{\varphi}s\right)^{-2} + \\ \\ \frac{1}{\mu\varphi}\left(1+\frac{2}{\varphi}s\right)^{-2} - \frac{1}{\mu^2}\left(1+\frac{2}{\varphi}s\right)^{-3/2} \end{array} \right\} \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}s}-1\right)\right\}$$

$$\therefore L_\lambda(0) = \left\{ \begin{array}{c} \frac{3}{\varphi^2}\left(1+\frac{2}{\varphi}(0)\right)^{-5/2} + \frac{2}{\mu\varphi}\left(1+\frac{2}{\varphi}(0)\right)^{-2} + \\ \\ \frac{1}{\mu\varphi}\left(1+\frac{2}{\varphi}(0)\right)^{-2} - \frac{1}{\mu^2}\left(1+\frac{2}{\varphi}(0)\right)^{-3/2} \end{array} \right\} \exp\left\{-\frac{\varphi}{\mu}\left(\sqrt{1+\frac{2}{\varphi}(0)}-1\right)\right\}$$

$$\begin{aligned} &= \frac{3}{\varphi^2} + \frac{2}{\mu\varphi} + \frac{1}{\mu\varphi} - \frac{1}{\mu^2} \\ &= \frac{3}{\varphi^2} + \frac{3}{\mu\varphi} - \frac{1}{\mu^2} \end{aligned}$$

Mean

$$\begin{aligned} E[\lambda] &= -L'_\lambda(0) \\ &= \frac{1}{\varphi} + \frac{1}{\mu} \end{aligned}$$

Variance

$$\begin{aligned}
\text{var}(\lambda) &= L''_{\lambda}(0) - [L'_{\lambda}(0)]^2 \\
&= \frac{3}{\varphi^2} + \frac{3}{\mu\varphi} - \frac{1}{\mu^2} - \left(\frac{1}{\varphi} + \frac{1}{\mu}\right)^2 \\
&= \frac{3}{\varphi^2} + \frac{3}{\mu\varphi} - \frac{1}{\mu^2} - \left(\frac{1}{\varphi^2} + \frac{2}{\varphi\mu} + \frac{1}{\mu^2}\right) \\
&= \frac{2}{\varphi^2} + \frac{1}{\mu\varphi}
\end{aligned}$$

4.4.4 Non Central Chi-Squared Distribution

Let

$$Y = X_1^2 + X_2^2 + X_3^2 + ... + X_n^2$$

Where X_i s are independent random variables and $X_i \sim N(\mu_i, 1)$ for $i = 1, 2, 3, ..., n$ Then the distribution of Y is said to be non-central chi-squared distribution with n degrees of freedom and non-central parameter

$$\theta = \sum_{i=1}^{n} \mu_i$$

Derivation of mgf of Y. if $X \sim N(\mu, 1)$, then the mgf of X^2 is given by

$$\begin{aligned}
M_{X^2}(t) &= E\left[e^{tX^2}\right] \\
&= \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx^2 - \frac{1}{2}\left[x^2 - 2x\mu + \mu^2\right]} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left[t - \frac{1}{2}\right]x^2 + x\mu - \frac{1}{2}\mu^2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left[\frac{2t-1}{2}\right]x^2 + x\mu - \frac{1}{2}\mu^2} dx
\end{aligned}$$

$$= \exp\left(-\frac{\mu^2}{2}\right) \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{\left[\frac{2t-1}{2}\right] x^2 + x\mu\right\} dx$$

$$= \exp\left(-\frac{\mu^2}{2}\right) \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[1-2t\right] x^2 + x\mu\right\} dx$$

$$= \exp\left(-\frac{\mu^2}{2}\right) \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[(1-2t)\, x^2 + 2x\mu\right]\right\} dx$$

$$= \exp\left(-\frac{\mu^2}{2}\right) \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-2t)}{2}\left[x^2 + 2\left(\frac{\mu}{1-2t}\right) x\right]\right\} dx$$

$$= \exp\left(-\frac{\mu^2}{2}\right) \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-2t)}{2}\left[\left(x - \frac{\mu}{1-2t}\right)^2 - \left(\frac{\mu}{1-2t}\right)^2\right]\right\} dx$$

$$= \exp\left[-\frac{\mu^2}{2} + \frac{1-2t}{2}\left(\frac{\mu}{1-2t}\right)^2\right] \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-2t)}{2}\left(x - \frac{\mu}{1-2t}\right)^2\right\} dx$$

$$= \exp\left[-\frac{\mu^2}{2} + \frac{\mu^2}{2\,(1-2t)}\right] \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-2t)}{2}\left(x - \frac{\mu}{1-2t}\right)^2\right\} dx$$

$$= \exp\left[-\frac{\mu^2}{2} + \frac{\mu^2}{2\,(1-2t)}\right] \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\left(x - \frac{\mu}{1-2t}\right)^2}{\frac{1}{1-2t}}\right\} dx$$

$$= \exp\left[-\frac{\mu^2}{2} + \frac{1-2t}{2}\left(\frac{\mu}{1-2t}\right)^2\right] \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-2t)}{2}\left(x - \frac{\mu}{1-2t}\right)^2\right\} dx$$

$$= \frac{1}{\sqrt{1-2t}} \exp\left[-\frac{\mu^2}{2} + \frac{\mu^2}{2\,(1-2t)}\right] \underbrace{\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{\frac{2\pi}{1-2t}}} \exp\left\{-\frac{1}{2}\frac{\left(x - \frac{\mu}{1-2t}\right)^2}{\frac{1}{1-2t}}\right\} dx}_{1}$$

$$= \frac{1}{\sqrt{1-2t}} \exp\left[-\frac{\mu^2}{2} + \frac{\mu^2}{2\,(1-2t)}\right]$$

$$= \frac{1}{\sqrt{1-2t}} \exp\left\{-\frac{\mu^2}{2}\left[1 - \frac{1}{1-2t}\right]\right\}$$

$$= \frac{1}{(1-2t)^2} \exp\left\{-\frac{\mu^2}{2}\left[\frac{1-2t-1}{1-2t}\right]\right\}$$

$$\therefore M_{X^2}(t) = \frac{1}{(1-2t)^{\frac{1}{2}}} \exp\left\{-\frac{\mu^2}{2} * \frac{-2t}{1-2t}\right\}$$

$$= \frac{1}{(1-2t)^{\frac{1}{2}}} \exp\left\{\frac{t\mu^2}{1-2t}\right\} \text{for} \quad 1-2t > 0 \Rightarrow t < \frac{1}{2}$$

Now since $X_i's$ are independent and $X_i \sim N(\mu_i, 1)$ for $i = 1, 2, ..., n$

Then mgf of $Y = \sum\limits_{i=1}^{n} X_i^2$ is

$$M_Y(t) = E\left[e^{tY}\right]$$

$$= M_{X_1^2}(t)\, M_{X_2^2}(t) ... M_{X_n^2}(t)$$

$$= \frac{1}{(1-2t)^{\frac{n}{2}}} \exp\left\{\frac{t}{1-2t} \sum_{i=1}^{n} \mu_i^2\right\}$$

Therefore the Laplace transform is

$$L_Y(t) = \frac{1}{(1+2t)^{\frac{n}{2}}} \exp\left\{\frac{-t}{1+2t} \sum_{i=1}^{n} \mu_i^2\right\}$$

$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \exp\left\{\frac{-2t}{1+2t} * \frac{\theta}{2}\right\}$$

$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \exp\left\{-\frac{\theta}{2}\left[\frac{1+2t-1}{1+2t}\right]\right\}$$

$$\therefore L_Y(t) = \frac{1}{(1+2t)^{\frac{n}{2}}} \exp\left\{-\frac{\theta}{2}\left[\frac{1+2t-1}{1+2t}\right]\right\}$$

$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \exp\left\{-\frac{\theta}{2}\left[1 - \frac{1}{1+2t}\right]\right\}$$

$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t} - 1\right]\right\}$$

$$\therefore L_Y'(t) = \left\{\begin{array}{c} \frac{1}{(1+2t)^{\frac{n}{2}}} * \frac{\theta}{2}\frac{d}{dt}(1+2t)^{-1}\exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t}-1\right]\right\} \\ \\ +\frac{d}{dt}(1+2t)^{-\frac{n}{2}}\exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t}-1\right]\right\} \end{array}\right\}$$

$$= \left\{\frac{\frac{\theta}{2}\left[-2(1+2t)^{-2}\right]}{(1+2t)^{\frac{n}{2}}} - \frac{n}{2}(1+2t)^{-\frac{n}{2}-1}2\right\}\exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t} - 1\right]\right\}$$

$$= \left[\frac{-\theta}{(1+2t)^{\frac{n}{2}+2}} - \frac{n}{(1+2t)^{\frac{n}{2}+1}}\right]\exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t} - 1\right]\right\}$$

$$= -\frac{1}{(1+2t)^{\frac{n}{2}+2}}\left\{\frac{\theta}{1+2t} + n\right\}\exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t} - 1\right]\right\}$$
$$= -\frac{1}{(1+2t)^{\frac{n}{2}+1}}\left\{\frac{\theta}{1+2t} + n\right\}\exp\left\{\frac{\theta}{2}\left[\frac{1}{1+2t} - 1\right]\right\}$$

$$h(t) = \frac{-L_Y'(t)}{L_Y(t)}$$

$$= \left[\frac{\theta}{(1+2t)^{\frac{n}{2}+2}} + \frac{n}{(1+2t)^{\frac{n}{2}+1}}\right](1+2t)^{\frac{n}{2}}$$

$$\begin{aligned} h(t) &= \frac{\theta}{(1+2t)^2} + \frac{n}{(1+2t)} \\ &= \frac{n}{(1+2t)^{1-0}} + \frac{\theta}{(1+2t)^{1-(-1)}} \end{aligned}$$

since

$$\begin{aligned} &a = 1-\alpha \\ &1 = 1-\alpha \Rightarrow \alpha = 0 \\ &2 = 1-\alpha \Rightarrow \alpha = -1 \end{aligned}$$

This is the sum of hazard functions of (exponential-gamma) and exponential Hougaard distribution. This sum gives rise to convolution of Poisson-gamma and Poisson- Hougaard distribution whose pgf is given by

$$H(s,t) = \left[\frac{1}{1-2(s-1)}\right]^{\frac{m}{2}} \exp\left\{\frac{\theta}{2}\left[\frac{1}{1-2(s-1)}\right] - \frac{\theta}{2}\right\}$$

4.5 A Random Sum of Independent Random variables

4.5.1 Compound Distributions

If a probability distribution is altered by allowing one of its parameters to behave as a random variable, the resulting distribution is said to be compound. An important compound distribution is that of the sum of random variables.

Let $X_1, X_2, X_3, ..., X_N$ be independently and identically distributed random variables. Let N be also a random variable independent of the $X_i's$.

If $Z_N = X_1 + X_2 + X_3 + ... + X_N$ then now N and Z_N are two random variables to be studied. Given the distribution of $X_i's$ and N , the problem is to find the Laplace transform, probability distribution, mean and variance of Z_N .

The Laplace transform technique

Let $\{X_i\}$ be a sequence of independent and identically distributed random variables with common Laplace transform

$$L_{X_i}(s) = E(e^{-sX_i})$$

and let

$$Z_N = X_1 + X_2 + X_3 + ... + X_N$$

where N is a random variable independent of the $X_i's$ with pgf

$$F(s) = E\left[S^n\right]$$

We denote the Laplace transform of Z_N by $L_{Z_N}(s)$ and it is given by

$$\begin{aligned}
L_{Z_N}(s) &= E\left[e^{-sZ_N}\right] \\
&= EE\left[e^{-sZ_N|N=n}\right] \\
&= EE\left[e^{-s(X_1+X_2+X_3+...+X_n)}\right] \\
&= E\prod_{i=1}^{n} E\left[e^{-sX_i}\right], \text{because of independence of } X_i's \\
&= E\left[E\left[e^{-sX_i}\right]\right]^n, \text{ since } X_i's \text{ are identical} \\
&= E\left[L_X(s)\right]^n
\end{aligned}$$

Therefore

$$L_{Z_N}(s) = F\left[L_X(s)\right]$$

Therefore to obtain mean and variance using Laplace transform technique, we find the first and second derivatives of $L_{Z_N}(s)$ with respect to s to obtain

$$L'_{Z_N}(s) = F'\left[L_X(s)\right] L'_X(s)$$

and

$$L''_{Z_N}(s) = F''\left[L_X(s)\right]\left[L'_X(s)\right]^2 + L''_X(s)\, F'\left[L_X(s)\right]$$

Setting $s = 0$, we have

$$\begin{aligned}
L''_{Z_N}(0) &= F''\left[L_X(0)\right]\left[L'_X(0)\right]^2 + L''_X(0)\, F'\left[L_X(0)\right] \\
&= F''[1]\left[L'_X(0)\right]^2 + L''_X(0)\, F'[1]
\end{aligned}$$

Therefore

$$
\begin{aligned}
E\left[Z_N\right] &= -L_Z'\left(0\right) \\
&= -F'\left[1\right]L_X'\left(0\right) \\
&= F'\left[1\right]\left[-L_X'\left(0\right)\right] \\
&= E(N)E(X_i)
\end{aligned}
$$

$$
\begin{aligned}
Var(Z_N) &= L_{Z_N}''\left(0\right) - \left[L_Z'\left(0\right)\right]^2 \\
&= F''\left[1\right]\left[L_X'\left(0\right)\right]^2 + L_X''\left(0\right)F'\left[1\right] - \left[F'\left[1\right]L_X'\left(0\right)\right]^2 \\
&= F''\left[1\right]\left[L_X'\left(0\right)\right]^2 + L_X''\left(0\right)F'\left[1\right] - \left[F'\left[1\right]\right]^2\left[L_X'\left(0\right)\right]^2 \\
&= \left[L_X'\left(0\right)\right]^2\left\{F''\left[1\right] - \left[F'\left[1\right]\right]^2\right\} + L_X''\left(0\right)F'\left[1\right] \\
&= \left[L_X'\left(0\right)\right]^2\left\{F''\left[1\right] - \left[F'\left[1\right]\right]^2 + F'\left[1\right]\right\} + L_X''\left(0\right)F'\left[1\right] - F'\left[1\right]\left[L_X'\left(0\right)\right]^2 \\
&= \left[L_X'\left(0\right)\right]^2\left\{F''\left[1\right] - \left[F'\left[1\right]\right]^2 + F'\left[1\right]\right\} + F'\left[1\right]\left\{L_X''\left(0\right) - \left[L_X'\left(0\right)\right]^2\right\} \\
&= \left[E\left(X\right)\right]^2 Var\left(N\right) + E\left(N\right)Var\left(X\right)
\end{aligned}
$$

4.5.2 Compound Poisson Distribution

Suppose $Z_N = X_1 + X_2 + X_3 + ... + X_N$ where $X_i's$ are independent random variables with N being a Poisson random variable. Then Z_N is said to have a Compound Poisson Distribution. Suppose N is Poisson with parameter λ . Then the pgf of N is given by,

$$F(s) = e^{\lambda(s-1)}$$

and the Laplace transform of Z_N is given by,

$$
\begin{aligned}
L_Z\left(s\right) &= F\left[L_X(s)\right] \\
&= e^{\lambda\left[L_X(s)-1\right]}
\end{aligned}
$$

Where $L_{X_i}\left(s\right)$ is the Laplace transform of X_i

Hence, the mean and the variance of Z_N is given by

$$\begin{aligned} E\left(Z_N\right) &= E\left(N\right)E\left(X_i\right) \\ &= \lambda E(X_i) \end{aligned}$$

and

$$\begin{aligned} Var\left[Z_N\right] &= E\left[N\right]Var\left(X_i\right) + \left[E\left(X_i\right)\right]^2 VarN \\ &= \lambda Var\left(X_i\right) + \lambda\left[E\left(X_i\right)\right]^2 \\ &= \lambda\left[Var\left(X_i\right) + \left[E\left(X_i\right)\right]^2\right] \end{aligned}$$

Case 1: If the $X_i's$ are uniformly distributed over the interval $(0,1)$ Then

$$L_{X_i}\left(s\right) = \frac{1-e^{-s}}{s}$$

$$L_{Z_N}\left(s\right) = e^{\lambda\left(\frac{1-e^{-s}}{s}-1\right)}$$

$$\begin{aligned} E\left(Z_N\right) &= E\left(N\right)E\left(X_i\right) \\ &= \lambda\left(\frac{1}{2}\right) \\ &= \frac{\lambda}{2} \end{aligned}$$

$$\begin{aligned} Var\left(Z_N\right) &= \lambda\left[Var\left(X_i\right) + E\left(X_i\right)^2\right] \\ &= \lambda\left[\frac{1}{12} + \left(\frac{1}{2}\right)^2\right] \\ &= \frac{\lambda}{3} \end{aligned}$$

Case 2:

If the $X_i's$ are exponentially distributed with parameter λ^*

Then

$$L_{X_i}(s) = \frac{\lambda^*}{\lambda^* + s}$$

$$L_{Z_N}(s) = e^{\lambda\left(\frac{\lambda^*}{\lambda^*+s}-1\right)}$$

$$\begin{aligned} E(Z_N) &= E(N)\,E(X_i) \\ &= \lambda\left(\frac{1}{\lambda^*}\right) \\ &= \frac{\lambda}{\lambda^*} \end{aligned}$$

$$\begin{aligned} Var(Z_N) &= \lambda\left[Var(X_i) + E(X_i)^2\right] \\ &= \lambda\left[\left(\frac{1}{\lambda^*}\right)^2 + \left(\frac{1}{\lambda^*}\right)^2\right] \\ &= \frac{2\lambda}{(\lambda^*)^2} \end{aligned}$$

Case 3: If the $X_i's$ are from gamma distribution with parameter α and β
Then

$$\begin{aligned} L_{X_i}(s) &= \left(\frac{\beta}{s+\beta}\right)^{\alpha} \\ L_{Z_N}(s) &= e^{\lambda\left[\left(\frac{\beta}{s+\beta}\right)^{\alpha}-1\right]} \\ E(Z_N) &= E(N)\,E(X_i) \\ &= \lambda\left(\frac{\alpha}{\beta}\right) \\ &= \frac{\lambda\alpha}{\beta} \end{aligned}$$

$$\begin{aligned} Var(Z_N) &= \lambda\left[Var(X_i) + E(X_i)^2\right] \\ &= \lambda\left[\frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2\right] \\ &= \frac{\lambda\alpha}{\beta^2}(1+\alpha) \end{aligned}$$

4.5.3 Compound Binomial Distribution

Suppose $Z_N = X_1 + X_2 + X_3 + ... + X_N$ where $X_i's$ are continuous independent random variables, with N being a Binomial random variable. Then Z_N is said to have a Compound Binomial Distribution. Suppose N is Binomial with parameters n and p . Then the pgf of N is given by

$$F(s) = [1 - p + ps]^n$$

And the Laplace transform of Z_N is given by

$$\begin{aligned} L_{Z_N}(s) &= F[L_X(s)] \\ &= [1 - p + p[L_X(s)]]^n \end{aligned}$$

Where $L_{X_i}(s)$ is the Laplace transform of X_i
Hence the mean and the variance of Z_N is given by

$$\begin{aligned} E(Z_N) &= E(N)E(X_i) \\ &= npE(X_i) \end{aligned}$$

and

$$\begin{aligned} Var(Z_N) &= E(N)VarX_i + [E(X)]^2 VarN \\ &= npVarX_i + np(1-p)[E(X_i)]^2 \\ &= np\left[VarX_i + (1-p)[E(X_i)]^2\right] \end{aligned}$$

Case 1: If the $X_i's$ are uniformly distributed over the interval $(0,\ 1)$

Then

$$L_{X_i}(s) = \frac{1-e^{-s}}{s}$$

$$L_{Z_N}(s) = \left[1-p+p\left(\frac{1-e^{-s}}{s}\right)\right]^n$$

$$\begin{aligned} E(Z_N) &= E(N)\,E(X_i) \\ &= np\left(\frac{1}{2}\right) \\ &= \frac{np}{2} \end{aligned}$$

$$\begin{aligned} Var(Z_N) &= np\left[Var(X_i) + (1-p)\,E(X_i)^2\right] \\ &= np\left[\frac{1}{12} + (1-p)\left(\frac{1}{2}\right)^2\right] \\ &= \frac{np\,(1+3\,(1-p))}{12} \\ &= \frac{np\,(4-3p)}{12} \end{aligned}$$

Case 2: If the $X_i's$ are exponentially distributed with parameter λ^*

Then

$$L_{X_i}(s) = \frac{\lambda^*}{\lambda^*+s}$$

$$L_{Z_N}(s) = \left[1-p+p\left(\frac{\lambda^*}{\lambda^*+s}\right)\right]^n$$

$$\begin{aligned} E(Z_N) &= E(N)\,E(X_i) \\ &= np\left(\frac{1}{\lambda^*}\right) \\ &= \frac{np}{\lambda^*} \end{aligned}$$

$$
\begin{aligned}
Var\left(Z_N\right) &= np\left[Var\left(X_i\right) + (1-p)\, E\left(X_i\right)^2\right] \\
&= np\left[\left(\frac{1}{\lambda^*}\right)^2 + (1-p)\left(\frac{1}{\lambda^*}\right)^2\right] \\
&= \frac{np\,(2-p)}{\left(\lambda^*\right)^2}
\end{aligned}
$$

Case 3: If the $X_i's$ are from gamma distribution with parameterα and β. Then

$$L_{X_i}(s) = \left(\frac{\beta}{s+\beta}\right)^{\alpha}$$

$$L_{Z_N}(s) = \left[1 - p + p\left(\frac{\beta}{s+\beta}\right)^{\alpha}\right]^n$$

$$
\begin{aligned}
E\left(Z_N\right) &= E\left(N\right) E\left(X_i\right) \\
&= np\left(\frac{\alpha}{\beta}\right) \\
&= \frac{np\alpha}{\beta}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= np\left[Var\left(X_i\right) + (1-p)\, E\left(X_i\right)^2\right] \\
&= np\left[\frac{\alpha}{\beta^2} + (1-p)\left(\frac{\alpha}{\beta}\right)^2\right] \\
&= \frac{np\alpha\left[1 + \alpha\,(1-p)\right]}{\beta^2}
\end{aligned}
$$

4.5.4 Compound Negative Binomial Distribution

Suppose $Z_N = X_1 + X_2 + X_3 + ... + X_N$ where $X_i'\ s$ are continuous independent random variables with N being a Negative Binomial random variable. Then Z_N is said to have a Compound Negative Binomial distribution. Suppose N is Negative Binomial with parameters α and p then the pgf of N is given by

$$F(s) = \left(\frac{p}{1-(1-p)s}\right)^{\alpha}$$

and thus the Laplace transform of is given by

$$\begin{aligned} L_Z(s) &= F\left[L_X(s)\right] \\ &= \left(\frac{p}{1-(1-p)L_X(s)}\right)^{\alpha} \end{aligned}$$

Where $L_{X_i}(s)$ is the Laplace transform of X_i

Hence the mean and the variance of Z_N is given by

$$\begin{aligned} E\left[Z_N\right] &= E(N)E(X_i) \\ &= \frac{\alpha(1-p)}{p}E(X_i) \end{aligned}$$

and

$$\begin{aligned} Var(Z_N) &= E(N)Var\left(X_i\right) + \left[E(X_i)\right]^2 VarN \\ &= \frac{\alpha(1-p)}{p}Var\left(X_i\right) + \frac{\alpha(1-p)}{p^2}\left[E(X_i)\right]^2 \\ &= \frac{\alpha(1-p)}{p}\left[Var\left(X_i\right) + \frac{1}{p}\left[E(X_i)\right]^2\right] \end{aligned}$$

Case 1: If the $X_i's$ are uniformly distributed over the interval $(0,\ 1)$

Then

$$L_{X_i}(s) = \frac{1-e^{-s}}{s}$$

$$
\begin{aligned}
E\left(Z_N\right) &= E\left(N\right) E\left(X_i\right) \\
&= \frac{\alpha\left(1-p\right)}{p}\left(\frac{1}{2}\right) \\
&= \frac{\alpha\left(1-p\right)}{2p}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{\alpha(1-p)}{p}\left[Var\left(X_i\right)+\frac{1}{p}\left[E(X_i)\right]^2\right] \\
&= \frac{\alpha(1-p)}{p}\left[\frac{1}{12}+\frac{1}{p}\left(\frac{1}{2}\right)^2\right] \\
&= \frac{\alpha(1-p)}{p}\left[\frac{p+3}{12p}\right] \\
&= \frac{\alpha(1-p)\left(p+3\right)}{12p^2}
\end{aligned}
$$

Case 2: If the $X_i's$ are exponentially distributed with parameter λ^*

Then

$$
L_{X_i}\left(s\right) = \frac{\lambda^*}{\lambda^*+s}
$$

$$
L_{Z_N}\left(s\right) = \left(\frac{p}{1-\left(1-p\right)\left(\frac{\lambda^*}{\lambda^*+s}\right)}\right)^{\alpha}
$$

$$
\begin{aligned}
E\left(Z_N\right) &= E\left(N\right) E\left(X_i\right) \\
&= \frac{\alpha\left(1-p\right)}{p}\left(\frac{1}{\lambda^*}\right) \\
&= \frac{\alpha\left(1-p\right)}{p\lambda^*}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{\alpha(1-p)}{p}\left[Var\left(X_i\right)+\frac{1}{p}\left[E(X_i)\right]^2\right] \\
&= \frac{\alpha(1-p)}{p}\left[\left(\frac{1}{\lambda^*}\right)^2+\frac{1}{p}\left(\frac{1}{\lambda^*}\right)^2\right] \\
&= \frac{\alpha(1-p)}{p}\left[\frac{p+1}{p\left(\lambda^*\right)^2}\right] \\
&= \frac{\alpha(1-p)\left(p+1\right)}{\left(p\lambda^*\right)^2}
\end{aligned}
$$

Case 3: If the $X_i's$ are from gamma distribution with parameter α^* and β

Then

$$
L_{X_i}\left(s\right) = \left(\frac{\beta}{s+\beta}\right)^{\alpha^*}
$$

$$
L_{Z_N}\left(s\right) = \left(\frac{p}{1-\left(1-p\right)\left(\frac{\beta}{s+\beta}\right)^{\alpha^*}}\right)^{\alpha}
$$

$$
\begin{aligned}
E\left(Z_N\right) &= E\left(N\right)E\left(X_i\right) \\
&= \frac{\alpha\left(1-p\right)}{p}\left(\frac{\alpha^*}{\beta}\right) \\
&= \frac{\alpha\alpha^*\left(1-p\right)}{\beta p}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{\alpha(1-p)}{p}\left[Var\left(X_i\right)+\frac{1}{p}\left[E(X_i)\right]^2\right] \\
&= \frac{\alpha(1-p)}{p}\left[\frac{\alpha^*}{\beta^2}+\frac{1}{p}\left(\frac{\alpha^*}{\beta}\right)^2\right] \\
&= \frac{\alpha(1-p)}{p}\left[\frac{p\alpha^*+\beta\left(\alpha^*\right)^2}{\beta^2 p}\right] \\
&= \frac{\alpha\alpha^*(1-p)\left[p+\beta\alpha^*\right]}{\beta^2 p^2}
\end{aligned}
$$

4.5.5 Compound Geometric Distribution

Suppose $Z_N = X_1 + X_2 + X_3 + ... + X_N$ where X_i' are continuous independent random variables with N being a Geometric random variable. Then Z_N is said to have a Compound Geometric Distribution. Suppose N is Geometric with parameter p , then the pgf of N is given by

$$F(s) = \frac{p}{[1 - (1 - p)s]}$$

And thus the Laplace transform of Z_N is given by

$$\begin{aligned} L_{Z_N}(s) &= F\left[L_X(s)\right] \\ &= \frac{p}{[1 - (1 - p)L_X(s)]} \end{aligned}$$

Hence the mean and the variance of Z_N is given by

$$\begin{aligned} E(Z_N) &= E(N)\,E(X_i) \\ &= \frac{1 - p}{p} E(X_i) \end{aligned}$$

$$\begin{aligned} Var(Z_N) &= E(N)Var(X_i) + \left[E(X_i)\right]^2 Var(N) \\ &= \frac{(1 - p)}{p} Var(X_i) + \frac{(1 - p)}{p^2}\left[E(X_i)\right]^2 \\ &= \frac{(1 - p)}{p}\left[Var(X_i) + \frac{1}{p}\left[E(X_i)\right]^2\right] \end{aligned}$$

Case 1: If the $X_i's$ are uniformly distributed over the interval $(0,\ 1)$

Then

$$L_{X_i}(s) = \frac{1 - e^{-s}}{s}$$

$$L_{Z_N}(s) = \frac{p}{1-(1-p)\left(\frac{1-e^{-s}}{s}\right)}$$

$$\begin{aligned} E(Z_N) &= E(N)\,E(X_i) \\ &= \frac{1-p}{p}\left(\frac{1}{2}\right) \\ &= \frac{1-p}{2p} \end{aligned}$$

$$\begin{aligned} Var(Z_N) &= \frac{1-p}{p}\left[Var(X_i) + \frac{1}{p}\left[E(X_i)\right]^2\right] \\ &= \frac{1-p}{p}\left[\frac{1}{12} + \frac{1}{p}\left(\frac{1}{2}\right)^2\right] \\ &= \frac{1-p}{p}\frac{(p+3)}{12p} \\ &= \frac{(1-p)(p+3)}{12p^2} \end{aligned}$$

Case 2: If the $X_i's$ are exponentially distributed with parameter λ^*

Then

$$\begin{aligned} L_{X_i}(s) &= \frac{\lambda^*}{\lambda^*+s} \\ L_{Z_N}(s) &= \frac{p}{1-(1-p)\left(\frac{\lambda^*}{\lambda^*+s}\right)} \end{aligned}$$

$$\begin{aligned} E(Z_N) &= E(N)\,E(X_i) \\ &= \frac{1-p}{p}\left(\frac{1}{\lambda^*}\right) \\ &= \frac{1-p}{p\lambda^*} \end{aligned}$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{1-p}{p}\left[Var\left(X_i\right)+\frac{1}{p}\left[E\left(X_i\right)\right]^2\right] \\
&= \frac{1-p}{p}\left[\left(\frac{1}{\lambda^*}\right)^2+\frac{1}{p}\left(\frac{1}{\lambda^*}\right)^2\right] \\
&= \frac{1-p}{p}\left[\frac{p+1}{p\left(\lambda^*\right)^2}\right] \\
&= \frac{(1-p)(1+p)}{\left(p\lambda^*\right)^2} \\
&= \frac{1-p^2}{\left(p\lambda^*\right)^2}
\end{aligned}
$$

Case 3: If the $X_i's$ are from gamma distribution with parameter α and β Then

$$
\begin{aligned}
L_{X_i}(s) &= \left(\frac{\beta}{s+\beta}\right)^\alpha \\
L_{Z_N}(s) &= \frac{p}{1-(1-p)\left(\frac{\beta}{s+\beta}\right)^\alpha} \\
E\left(Z_N\right) &= \frac{1-p}{p}E\left(X_i\right) \\
&= \frac{1-p}{p}\left(\frac{\alpha}{\beta}\right) \\
&= \frac{\alpha(1-p)}{\beta p}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{1-p}{p}\left[Var\left(X_i\right)+\frac{1}{p}\left[E\left(X_i\right)\right]^2\right] \\
&= \frac{1-p}{p}\left[\frac{\alpha}{\beta^2}+\frac{1}{p}\left(\frac{\alpha}{\beta}\right)^2\right] \\
&= \frac{1-p}{p}\left[\frac{\alpha p+\alpha^2}{\beta^2 p}\right] \\
&= \frac{\alpha(1-p)(\alpha+p)}{\beta^2}
\end{aligned}
$$

4.6 Compound Logarithmic Series Distribution

Suppose $Z_N = X_1 + X_2 + X_3 + ... + X_N$ where are continuous independent random variables with N being a Logarithmic Series random variable. Then Z_N is said to have a Compound Logarithmic Series Distribution. Suppose N is Logarithmic Series Distribution with parameter π, then the pgf of N is given by

$$F(s) = \frac{\ln(1-ps)}{\ln(1-p)}$$

and the Laplace transform of Z_N is given by

$$\begin{aligned} L_{Z_N}(s) &= F\left[L_X(s)\right] \\ &= \frac{\ln\left[1-pL_X(s)\right]}{\ln(1-p)} \end{aligned}$$

Hence the mean and the variance of z_N is given by

$$\begin{aligned} E(Z_N) &= E(N)E(X_i) \\ &= \frac{p}{-(1-p)\log_e(1-p)}E(X_i) \end{aligned}$$

$$\begin{aligned} Var(Z_N) &= E(N)VarX_i + \left[E(X)\right]^2 VarN \\ &= \frac{p}{-(1-p)\log_e(1-p)}VarX_i - \frac{\left[p+\log_e(1-p)\right]}{\left[(1-p)\log_e(1-p)\right]^2}\left[E(X_i)\right]^2 \\ &= \frac{p}{-(1-p)\log_e(1-p)}\left[VarX_i + \frac{\left[p+\log_e(1-p)\right]}{\left[(1-p)\log_e(1-p)\right]}\left[E(X_i)\right]^2\right] \end{aligned}$$

Case 1: If the $X_i's$ are uniformly distributed over the interval $(0,\ 1)$

Then

$$L_{X_i}(s) = \frac{1-e^{-s}}{s}$$

$$L_{Z_N}(s) = \frac{\ln\left[1-p\left(\frac{1-e^{-s}}{s}\right)\right]}{\ln(1-p)}$$

$$
\begin{aligned}
E\left(Z_N\right) &= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} E\left(X_i\right) \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left(\frac{1}{2}\right) \\
&= \frac{p}{-2\left(1-p\right)\log_e\left(1-p\right)}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left[Var\left(X_i\right) + \frac{p+\log_e\left(1-p\right)}{\left(1-p\right)\log_e\left(1-p\right)} \left[E\left(X_i\right)\right]^2\right] \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left[\frac{1}{12} + \frac{p+\log_e\left(1-p\right)}{\left(1-p\right)\log_e\left(1-p\right)} \left(\frac{1}{2}\right)^2\right] \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left[\frac{\left(1-p\right)\log_e\left(1-p\right) + 3\left[p+\log_e\left(1-p\right)\right]}{12\left(1-p\right)\log_e\left(1-p\right)}\right] \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left[\frac{\log_e\left(1-p\right) - p\log_e\left(1-p\right) + 3p + 3\log_e\left(1-p\right)}{12\left(1-p\right)\log_e\left(1-p\right)}\right] \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left[\frac{3p + 4\log_e\left(1-p\right) - p\log_e\left(1-p\right)}{12\left(1-p\right)\log_e\left(1-p\right)}\right] \\
&= \frac{3p^2 + 4p\log_e\left(1-p\right) - p^2\log_e\left(1-p\right)}{-12\left[\left(1-p\right)\log_e\left(1-p\right)\right]^2}
\end{aligned}
$$

Case 2: If the $X_i's$ are exponentially distributed with parameter λ^*. Then

$$
\begin{aligned}
L_{X_i}\left(s\right) &= \frac{\lambda^*}{\lambda^* + s} \\
L_{Z_N}\left(s\right) &= \frac{\ln\left[1 - p\left(\frac{\lambda^*}{\lambda^*+s}\right)\right]}{\ln(1-p)}
\end{aligned}
$$

$$
\begin{aligned}
E\left(Z_N\right) &= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} E\left(X_i\right) \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)} \left(\frac{1}{\lambda^*}\right) \\
&= \frac{p}{-\lambda^*\left(1-p\right)\log_e\left(1-p\right)}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{p}{-(1-p)\log_e(1-p)}\left[VarX_i - \frac{[p+\log_e(1-p)]}{(1-p)\log_e(1-p)}\left[E(X_i)\right]^2\right] \\
&= \frac{p}{-(1-p)\log_e(1-p)}\left[\left(\frac{1}{\lambda^*}\right)^2 + \frac{[p+\log_e(1-p)]}{(1-p)\log_e(1-p)}\left(\frac{1}{\lambda^*}\right)^2\right] \\
&= \frac{p}{-(1-p)\log_e(1-p)}\left[\frac{(1-p)\log_e(1-p)+[p+\log_e(1-p)]}{\left(\lambda^*\right)^2(1-p)\log_e(1-p)}\right] \\
&= \frac{p}{-(1-p)\log_e(1-p)}\left[\frac{\log_e(1-p)-p\log_e(1-p)+p+\log_e(1-p)}{\left(\lambda^*\right)^2(1-p)\log_e(1-p)}\right] \\
&= \frac{p\left[p+2\log_e(1-p)-p\log_e(1-p)\right]}{-\left(\lambda^*\right)^2\left[(1-p)\log_e(1-p)\right]^2}
\end{aligned}
$$

Case 3: If the $X_i's$ are from gamma distribution with parameter α and β. Then

$$
\begin{aligned}
L_{X_i}\left(s\right) &= \left(\frac{\beta}{s+\beta}\right)^{\alpha} \\
L_{Z_N}\left(s\right) &= \frac{\ln\left[1-p\left(\frac{\beta}{s+\beta}\right)^{\alpha}\right]}{\ln(1-p)}
\end{aligned}
$$

$$
\begin{aligned}
E\left(Z_N\right) &= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)}E\left(X_i\right) \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)}\left(\frac{\alpha}{\beta}\right) \\
&= \frac{\alpha p}{-\beta\left(1-p\right)\log_e\left(1-p\right)}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(Z_N\right) &= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)}\left[Var\left(X_i\right) + \frac{p+\log_e\left(1-p\right)}{\left(1-p\right)\log_e\left(1-p\right)}E\left(X_i\right)^2\right] \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)}\left[\frac{\alpha}{\beta^2} + \frac{p+\log_e\left(1-p\right)}{\left(1-p\right)\log_e\left(1-p\right)}\left(\frac{\alpha}{\beta}\right)^2\right] \\
&= \frac{p}{-\left(1-p\right)\log_e\left(1-p\right)}\left[\frac{\alpha\left(1-p\right)\log_e\left(1-p\right)+\alpha^2\left[p+\log_e\left(1-p\right)\right]}{\beta^2\left(1-p\right)\log_e\left(1-p\right)}\right] \\
&= \frac{\alpha p}{-\beta^2\left[\left(1-p\right)\log_e\left(1-p\right)\right]^2}\left[\left(1-p\right)\log_e\left(1-p\right)+\alpha\left[p+\log_e\left(1-p\right)\right]\right]
\end{aligned}
$$

Chapter 5

Laplace Transform in Pure Birth Processes

5.1 Introduction

In this chapter, we derive the basic-difference-differential equations for the general birth process. The chapter will focus on specific cases of the pure birth process namely the Poisson, simple birth, simple birth with immigration and Polya processes. Laplace transform method will be employed in solving the basic-difference-differential equations for each of the special cases .

Two approaches will be used, first the basic-difference-differential equations are solved directly using Laplace transform method.This involves determining a general expression for the Laplace transform of $P_n(t)$ by iteration and then use both the complex inversion formula and partial fractions method to obtain the inverse laplace transform. Secondly, the probability generating function method is applied and then Laplace transform is used to solve the resulting ODE/PDE. For this method, two routes have been considered, one involves the Dirac delta function whereas the other one uses the Gauss hyper geometric function.

5.2 The Basic Difference-Differential Equations for General Birth Process

Let $X(t)$ be the population size at time t and $P_n(t) = Prob[X(t) = n]$, therefore

$$P_n(t+h) = Prob[X(t+h) = n]$$

From first principles of differentiation

$$\begin{aligned} f^{'}(x) &= \frac{d}{dx} f(x) \\ &= \lim_{h \to 0} \frac{f(x+h-f(x)}{h} \\ \Rightarrow P_n^{'}(t) &= \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} \end{aligned}$$

Where h is a small change in time.

The problem is to derive the basic-difference-differential equations for a pure birth process and solve them using Laplace transform. Symbolically, we wish to derive $P_n^{'}(t)$ interms of $P_n(t)$ and $P_{n-1}(t)$. We shall make the following assumptions.

Assumptions

- The probability of having a birth in the time interval t and $(t+h)$ when the population size at time t is n, that is $X(t) = n$ is $\lambda_n(h) + O(h)$ where $O(h)$ is of order h (it is a negligible function) and

$$\lim_{h \to 0} \frac{O(h)}{h} = 0$$

- The probability of no birth in the time interval t and $(t+h)$ is $1 - \lambda_n(h) + O(h)$
- The probability of having two or more births in the time interval t and $(t+h)$ is $O(h)$, in other words this probability is negligible.

For the population size at time $t+h$ to be n , there can be three possibilities

- The population size at time t was n and there was no birth within the time interval $(t, t+h)$

- The population size at time t was $n-1$ and there was one birth within the time interval $(t, t+h)$
- The population size at time t was $n-i$ and there were i births within the time interval $(t, t+h)$

Mathematically, this can be put as follows;

$$\begin{aligned}P_n(t+h) &= Prob[X(t+h)=n] \\ &= Prob[X(t+h)=n,\ X(t)=n] \\ &+ Prob[X(t+h)=n,\ X(t)=n-1] \\ &+ \sum_{i=2}^{n} Prob[X(t+h)=n,\ X(t)=n-i]\end{aligned}$$

Therefore

$$\begin{aligned}P_n(t+h) = &\underbrace{Prob\left[X(t+h)=n|X(t)=n\right]}_{no\ birth} Prob\left[X(t)=n\right] \\ &+ \underbrace{Prob\left[X(t+h)=n|X(t)=n-1\right]}_{one\ birth} Prob\left[X(t)=n-1\right] \\ &+ \sum_{i=2}^{n} \underbrace{Prob\left[X(t+h)=n|X(t)=n-1\right]}_{more\ than\ one\ birth} Prob\left[X(t)=n-i\right]\end{aligned}$$

We thus have

$$P_n(t+h) = \left[1-\lambda_n h + O(h)\right] P_n(t) + \left[\lambda_{n-1} h + O(h)\right] P_{n-1}(t) + \sum_{i=2}^{n} O(h) P_{n-i}(t)$$

$$P_n(t+h) - P_n(t) = \left[-\lambda_n h + O(h)\right] P_n(t) + \left[\lambda_{n-1} h + O(h)\right] P_{n-1}(t) + \sum_{i=2}^{n} O(h) P_{n-i}(t)$$

Dividing both sides by h and taking the limit as $h \to 0$ we have;

$$\lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h} = \lim_{h\to 0}\left\{\frac{\left[-\lambda_n h + O(h)\right]P_n(t) + \left[\lambda_{n-1}h + O(h)\right]P_{n-1}(t) + \sum_{i=2}^{n} O(h)P_{n-i}(t)}{h}\right\}$$

$$\Rightarrow P_n'(t) = \left[-\lambda_n + \underbrace{\lim_{h\to 0}\frac{O(h)}{h}}_{0}\right]P_n(t) + \left[\lambda_{n-1} + \underbrace{\lim_{h\to 0}\frac{O(h)}{h}}_{0}\right]P_{n-1}(t)$$
$$+ \underbrace{\lim_{h\to 0}\frac{O(h)}{h}}_{0}\sum_{i=2}^{n} P_{n-i}(t)$$

$$\therefore P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t) \quad n \geqslant 1$$

For the population size at time $t+h$ to be 0, there is only one possibility. The population size at time t was 0 and there was no birth within the time interval $(t,\ t+h)$.
Therefore

$$\begin{aligned} P_0(t+h) &= Prob\left[X(t+h)=0\right] \\ &= Prob\left[X(t+h)=0, X(t)=0\right] \\ &= Prob\underbrace{\left[X(t+h)=0|X(t)=0\right]}_{no\ birth}\Pr ob\left[X(t)=0\right] \\ &= \left[1-\lambda_0 h + O(h)\right]\mathrm{P}_0(t) \end{aligned}$$

$$\Rightarrow P_0(t+h) - \mathrm{P}_0(t) = \left[-\lambda_0 h + O(h)\right]\mathrm{P}_0(t)$$

Dividing both sides by h and taking the limit as $h \to 0$ we have;

$$\lim_{h\to 0}\frac{P_0(t+h)-\mathrm{P}_0(t)}{h} = \lim_{h\to 0}\frac{\left[-\lambda_0 h + O(h)\right]\mathrm{P}_0(t)}{h}$$
$$P_0'(t) = -\lambda_0\,\mathrm{P}_0(t) + \underbrace{\lim_{h\to 0}\frac{O(h)}{h}}_{0}\mathrm{P}_0(t)$$

$$\therefore P_0'(t) = -\lambda_0\,\mathrm{P}_0(t)$$

Thus the basic difference-differential equations for the general birth process are

$$P_0'(t) = -\lambda_0\, \mathrm{P}_0(t) \tag{5.1a}$$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \quad n \geqslant 1 \tag{5.1b}$$

We shall solve the basic difference-differential equation for special cases using Laplace transform.

5.3 Poisson process

When the growth rate for a general birth process is constant i.e. $\lambda_n = \lambda \quad \forall n$. The process is said to be a Poisson process.

The basic difference-differential equations for this process are

$$P_0'(t) = -\lambda\, \mathrm{P}_0(t) \tag{5.2a}$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \;\; n \geqslant 1 \tag{5.2b}$$

We now solve these equations.

Method 1: By Iteration

Using equation (5.2b) we determine the Laplace transform of both sides as follows:

$$\begin{aligned} L\left\{P_n'(t)\right\} &= L\left\{\lambda P_{n-1}(t) - \lambda P_n(t)\right\} \\ &= \lambda L\left\{P_{n-1}(t)\right\} - L\left\{\lambda P_n(t)\right\} \end{aligned}$$

By linearity property (property 1)

$$L\left\{P_n'(t)\right\} = -\lambda L\left\{P_n(t)\right\} + \lambda L\left\{P_{n-1}(t)\right\} \tag{5.3}$$

But by property 5 (Laplace transform of derivatives) we had

$$\begin{aligned} L\left\{f(t)\right\} &= \overline{f}(s) \\ L\left\{f'(t)\right\} &= s\overline{f}(s) - f(0) \end{aligned}$$

This implies that

$$L\left\{P_n'(t)\right\} = sL\left\{P_n(t)\right\} - P_n(0)$$

Substituting this in equation (5.3) we obtain

$$(s+\lambda)L\left\{P_n(t)\right\} - P_n(0) = \lambda L\left\{P_{n-1}(t)\right\} \tag{5.4}$$

Remark

But for the Poisson process $X(0) = 0$ (the number of events at time $t = 0$ is 0). This implies that

$$P_0(0) = 1 \text{ and } P_n(0) = 0 \ \ \forall n \neq 0$$

Therefore solving equation (5.4) iteratively yields,

For $n = 0$ we have

$$\begin{aligned}(s+\lambda)L\left\{P_0(t)\right\} - 1 &= \lambda L\left\{0\right\}\\ (s+\lambda)L\left\{P_0(t)\right\} &= 1\\ \Rightarrow L\left\{P_0(t)\right\} &= \frac{1}{s+\lambda}\end{aligned}$$

Similarly for $n = 1$, equation (5.4) becomes

$$\begin{aligned}(s+\lambda)L\left\{P_1(t)\right\} - P_1(0) &= \lambda L\left\{P_{1-1}(t)\right\}\\ (s+\lambda)L\left\{P_1(t)\right\} - P_1(0) &= \lambda L\left\{P_0(t)\right\}\end{aligned}$$

But
$P_1(0) = 0$ and $L\left\{P_0(t)\right\} = \frac{1}{s+\lambda}$

We therefore have

$$\begin{aligned}(s+\lambda)L\left\{P_1(t)\right\} - 0 &= \lambda\left(\frac{1}{s+\lambda}\right)\\ (s+\lambda)L\left\{P_1(t)\right\} &= \frac{\lambda}{s+\lambda}\\ \Rightarrow L\left\{P_1(t)\right\} &= \frac{\lambda}{(s+\lambda)^2}\end{aligned}$$

Similarly for $n = 2$ equation (5.4) becomes

$$\begin{aligned}(s+\lambda) L\{P_2(t)\} - P_2(0) &= \lambda L\{P_{2-1}(t)\} \\ (s+\lambda) L\{P_2(t)\} &= P_2(0) + \lambda L\{P_1(t)\}\end{aligned}$$

But

$$P_2(0) = 0 \text{ and } L\{P_1(t)\} = \frac{\lambda}{(s+\lambda)^2}$$

Thus we have

$$\begin{aligned}(s+\lambda) L\{P_2(t)\} &= 0 + \lambda \left[\frac{\lambda}{(s+\lambda)^2}\right] \\ (s+\lambda) L\{P_2(t)\} &= \frac{\lambda^2}{(s+\lambda)^2} \\ \Rightarrow L\{P_2(t)\} &= \frac{\lambda^2}{(s+\lambda)^3}\end{aligned}$$

Similarly for $n = 3$ equation (5.4) becomes

$$\begin{aligned}(s+\lambda) L\{P_3(t)\} - P_3(0) &= \lambda L\{P_{3-1}(t)\} \\ (s+\lambda) L\{P_3(t)\} &= P_3(0) + \lambda L\{P_2(t)\}\end{aligned}$$

But

$$P_3(0) = 0 \text{ and } L\{P_2(t)\} = \frac{\lambda^2}{(s+\lambda)^3}$$

Thus we have

$$\begin{aligned}(s+\lambda) L\{P_3(t)\} &= 0 + \lambda \left[\frac{\lambda^2}{(s+\lambda)^3}\right] \\ (s+\lambda) L\{P_3(t)\} &= \frac{\lambda^3}{(s+\lambda)^3} \\ \Rightarrow L\{P_3(t)\} &= \frac{\lambda^3}{(s+\lambda)^4}\end{aligned}$$

Similarly for $n = 4$ equation (5.4) becomes

$$
\begin{aligned}
(s+\lambda) L\{P_4(t)\} - P_4(0) &= \lambda L\{P_{4-1}(t)\} \\
(s+\lambda) L\{P_4(t)\} &= P_4(0) + \lambda L\{P_3(t)\}
\end{aligned}
$$

But

$$P_4(0) = 0 \quad \text{and} \quad L\{P_3(t)\} = \frac{\lambda^3}{(s+\lambda)^4}$$

Thus we have

$$
\begin{aligned}
(s+\lambda) L\{P_4(t)\} &= 0 + \lambda \left[\frac{\lambda^3}{(s+\lambda)^4}\right] \\
(s+\lambda) L\{P_4(t)\} &= \frac{\lambda^4}{(s+\lambda)^4} \\
\Rightarrow L\{P_4(t)\} &= \frac{\lambda^4}{(s+\lambda)^5}
\end{aligned}
$$

For $n = 5$ equation (5.4) becomes

$$
\begin{aligned}
(s+\lambda) L\{P_5(t)\} - P_5(0) &= \lambda L\{P_{5-1}(t)\} \\
(s+\lambda) L\{P_5(t)\} &= P_5(0) + \lambda L\{P_4(t)\}
\end{aligned}
$$

But

$$P_5(0) = 0 \quad \text{and} \quad L\{P_4(t)\} = \frac{\lambda^4}{(s+\lambda)^5}$$

Thus we have

$$
\begin{aligned}
(s+\lambda) L\{P_5(t)\} &= 0 + \lambda \left[\frac{\lambda^4}{(s+\lambda)^5}\right] \\
(s+\lambda) L\{P_5(t)\} &= \frac{\lambda^5}{(s+\lambda)^5} \\
\Rightarrow L\{P_5(t)\} &= \frac{\lambda^5}{(s+\lambda)^6}
\end{aligned}
$$

We now use mathematical induction to determine $L\{P_n(t)\}$. Assume that

$$L\{P_{n-1}(t)\} = \frac{\lambda^{n-1}}{(s+\lambda)^n}$$

By equation (5.4) we have

$$(s+\lambda)\,L\{P_n(t)\} - P_n(0) = \lambda L\{P_{n-1}(t)\}$$

But $P_n(0) = 0$ and

$$L\{P_{n-1}(t)\} = \frac{\lambda^{n-1}}{(s+\lambda)^n}$$

Thus equation (5.4) becomes

$$\begin{aligned}
(s+\lambda)\,L\{P_n(t)\} - 0 &= \lambda\left[\frac{\lambda^{n-1}}{(s+\lambda)^n}\right]\\
(s+\lambda)\,L\{P_n(t)\} &= \frac{\lambda^n}{(s+\lambda)^n}\\
L\{P_n(t)\} &= \frac{\lambda^n}{(s+\lambda)^{n+1}} \qquad (5.5)
\end{aligned}$$

Applying inverse Laplace transform to both sides of equation (5.5) yields

$$\begin{aligned}
L^{-1}\{L\{P_n(t)\}\} &= L^{-1}\left\{\frac{\lambda^n}{(s+\lambda)^{n+1}}\right\}\\
\Rightarrow P_n(t) &= L^{-1}\left\{\frac{\lambda^n}{(s+\lambda)^{n+1}}\right\}\\
&= \lambda^n L^{-1}\left\{\frac{1}{(s+\lambda)^{n+1}}\right\} \qquad (5.6)
\end{aligned}$$

We use both the miscellaneous and complex inversion formula to determine the inverse Laplace transform.

Miscellaneous Method

By the first shifting property

$$L\left\{e^{-at}f(t)\right\} = \overline{f}(s-a)$$

and

$$
\begin{aligned}
L\left\{t^n\right\} &= \frac{\Gamma(n+1)}{s^{n+1}} \\
\Rightarrow \frac{1}{\Gamma(n+1)} L\left\{t^n\right\} &= \frac{1}{s^{n+1}} \\
\Rightarrow \frac{1}{n!} L\left\{t^n\right\} &= \frac{1}{s^{n+1}}
\end{aligned}
$$

Thus it follows that

$$\frac{1}{n!} L\left\{e^{-at} t^n\right\} = \left(\frac{1}{s+a}\right)^{n+1}$$

Letting $a = \lambda$, we have

$$\frac{1}{n!} L\left\{e^{-\lambda t} t^n\right\} = \left(\frac{1}{s+\lambda}\right)^{n+1}$$

Multiplying both sides by λ^n yields

$$
\begin{aligned}
\frac{\lambda^n}{n!} L\left\{e^{-\lambda t} t^n\right\} &= \frac{\lambda^n}{(s+\lambda)^{n+1}} \\
\Rightarrow L\left\{\frac{e^{-\lambda t} (\lambda t)^n}{n!}\right\} &= \frac{\lambda^n}{(s+\lambda)^{n+1}}
\end{aligned}
$$

$$\therefore \; L^{-1}\left\{\frac{\lambda^n}{(s+\lambda)^{n+1}}\right\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Relating this to equation (5.6) it follows that

$$
\begin{aligned}
P_n(t) &= L^{-1}\left\{\frac{\lambda^n}{(s+\lambda)^{n+1}}\right\} \\
&= \frac{e^{-\lambda t} \lambda t^n}{n!} \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

Which is the probability mass function of a Poisson random variable with parameter λt .

Complex inversion formula

By equation (5.6) we had

$$P_n(t) = \lambda^n L^{-1}\left\{\frac{1}{(s+\lambda)^{n+1}}\right\}$$

By the complex inversion formula

$$L^{-1}\left\{\frac{1}{(s+\lambda)^{n+1}}\right\} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\left(\frac{1}{s+\lambda}\right)^{n+1} ds$$

Which is simply the sum of the residues of

$$\frac{e^{st}}{(s+\lambda)^{n+1}}$$

at the poles of

$$\frac{1}{(s+\lambda)^{n+1}}$$

It can be seen that

$$\frac{1}{(s+\lambda)^{n+1}}$$

is analytical at every point except when $s = -\lambda$. Thus $s = -\lambda$ is a pole of order $n+1$

The residues of

$$\frac{e^{st}}{(s+\lambda)^{n+1}}$$

are given by

$$\begin{aligned} a_{-1} &= \lim_{s\to-\lambda}\frac{1}{(n+1-1)!}\frac{d^{n+1-1}}{ds^{n+1-1}}(s+\lambda)^{n+1}\frac{e^{st}}{(s+\lambda)^{n+1}} \\ &= \lim_{s\to-\lambda}\frac{1}{n!}\frac{d^n}{ds^n}e^{st} \end{aligned} \tag{5.7}$$

But

$$\begin{aligned} \frac{d}{ds}e^{st} &= te^{st} \\ \frac{d^2}{ds^2}e^{st} &= t^2e^{st} \\ \frac{d^3}{ds^3}e^{st} &= t^3e^{st} \end{aligned}$$

By Mathematical induction We assume that,

$$\frac{d^{n-1}}{ds^{n-1}}e^{st} = t^{n-1}e^{st}$$

Then

$$\begin{aligned}\frac{d^{n}}{ds^{n}}e^{st} &= \frac{d}{ds}\left[t^{n-1}e^{st}\right]\\ &= t^{n-1}\left[\frac{d}{ds}e^{st}\right]\\ &= t^{n-1}te^{st}\\ &= t^{n}e^{st}\end{aligned}$$

Substituting this in equation (5.7) we obtain

$$\begin{aligned}Res\left[\frac{e^{st}}{(s+\lambda)^{n+1}}\right] &= \lim_{s\to-\lambda}\frac{1}{n!}\frac{d^{n}}{ds^{n}}e^{st}\\ &= \lim_{s\to-\lambda}\frac{1}{n!}t^{n}e^{st}\\ &= \frac{1}{n!}t^{n}e^{-\lambda t}\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{1}{(s+\lambda)^{n+1}}\right\} = \frac{1}{n!}t^{n}e^{-\lambda t}$$

But

$$\begin{aligned}P_{n}(t) &= \lambda^{n}L^{-1}\left\{\frac{1}{(s+\lambda)^{n+1}}\right\}\\ &= \lambda^{n}\left[\frac{t^{n}}{n!}e^{-\lambda t}\right]\end{aligned}$$

$$= \frac{e^{-\lambda t}\lambda t^{n}}{n!} \qquad n = 0, 1, 2, ...$$

Which is a Poisson distribution with parameter λt

Method 2: Using pgf approach

The basic difference-differential equations for the Poisson process as defined by equations (5.2a) and (5.2b) are

$$P_0'(t) = -\lambda P_0(t) \tag{5.2a}$$

$$P_n'(t) = \lambda P_{n-1}(t) - \lambda P_n(t) \quad n \geqslant 1 \tag{5.2b}$$

Multiplying both sides of equation (5.2b) by z^n and summing the result over n, We have

$$\underbrace{\sum_{n=1}^{\infty} P_n'(t) z^n}_{I} = \underbrace{\lambda \sum_{n=1}^{\infty} P_{n-1}(t) z^n}_{II} - \underbrace{\lambda \sum_{n=1}^{\infty} P_n(t) z^n}_{III} \tag{5.8}$$

Let $G(z,t)$ be the probability generating function of $X(t)$ defined as follows;

$$\begin{aligned} G(z,t) &= \sum_{n=0}^{\infty} P_n(t) z^n \\ &= P_0(t) + \sum_{n=1}^{\infty} P_n(t) z^n \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial t} G(z,t) &= \sum_{n=0}^{\infty} P_n'(t) z^n \\ &= P_0'(t) + \sum_{n=1}^{\infty} P_n'(t) z^n \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial z} G(z,t) &= \sum_{n=0}^{\infty} n P_n(t) z^{n-1} \\ &= \sum_{n=1}^{\infty} n P_n(t) z^{n-1} \end{aligned}$$

We now simplify the three parts of equation (5.8)

Part I

$$\sum_{n=1}^{\infty} P_n'(t)z^n = \frac{\partial}{\partial t}G(z,t) - P_0'(t)$$

Part II

$$\begin{aligned}\lambda\sum_{n=1}^{\infty} P_{n-1}(t)z^n &= \lambda z\sum_{n=1}^{\infty} P_{n-1}(t)z^{n-1} \\ &= \lambda z G(z,t)\end{aligned}$$

Part III

$$-\lambda\sum_{n=1}^{\infty} P_n(t)z^n = -\lambda\left[G(z,t) - P_0(t)\right]$$

Substituting the above in equation (5.8) yields

$$\begin{aligned}\frac{\partial}{\partial t}G(z,t) - P_0'(t) &= \lambda z G(z,t) - \lambda\left[G(z,t) - P_0(t)\right] \\ &= \lambda z G(z,t) - \lambda G(z,t) + \lambda P_0(t)\end{aligned}$$

But from equation (5.2a) $P_0'(t) = -\lambda P_0(t)$. Therefore

$$\begin{aligned}\frac{\partial}{\partial t}G(z,t) + \lambda P_0(t) &= \lambda z G(z,t) - \lambda G(z,t) + \lambda P_0(t) \\ \Rightarrow \frac{\partial}{\partial t}G(z,t) &= \lambda z G(z,t) - \lambda G(z,t) \\ &= -\lambda\left(1-z\right)G(z,t)\end{aligned}$$

Implying that

$$\frac{\partial}{\partial t}G(z,t) + \lambda\left(1-z\right)G(z,t) = 0 \tag{5.9}$$

Which is a linear partial differential equation. The next step is to use the Laplace transform to solve equation (5.9) .

By definition the Laplace transform of $G(z,t)$ is

$$L\{G(z,t)\} = \int_0^\infty e^{-st} G(z,t) dt$$
$$= \overline{G}(z,s)$$

and

$$L\left\{\frac{\partial}{\partial t} G(z,t)\right\} = \int_0^\infty e^{-st} \frac{\partial}{\partial t} G(z,t) dt$$
$$= \int_0^\infty e^{-st} dG(z,t)$$

Using integration by parts, let

$$U = e^{-st} \qquad dV = dG(z,t)$$
$$\therefore dU = e^{-st} \qquad V = G(z,t)$$

$$\therefore L\left\{\frac{\partial}{\partial t} G(z,t)\right\} = e^{-st}\ G(z,t)|_0^\infty + s\int_0^\infty e^{-st} G(z,t) dt$$
$$= [0 - G(z,0)] + sL\{G(z,t)\}$$

But

$$G(z,t) = \sum_{n=0}^\infty P_n(t) z^n$$
$$\therefore G(z,0) = \sum_{n=0}^\infty P_n(0) z^n$$
$$= P_0(0) + \sum_{n=1}^\infty P_n(0) z^n$$

And

$$P_0(0) = 1, \quad P_0(0) = 0 \quad \forall\, n \neq 0$$
$$\Rightarrow G(z,0) = 1$$

$$\begin{aligned} \therefore L\left\{\frac{\partial}{\partial t}G(z,t)\right\} &= -1 + sL\left\{G(z,t)\right\} \\ &= sL\left\{G(z,t)\right\} - 1 \\ &= s\overline{G}(z,s) - 1 \end{aligned}$$

$$L\left\{\frac{\partial}{\partial t}G(z,t)\right\} = -\lambda\left(1-z\right)L\left\{G(z,t)\right\}$$

$$\begin{aligned} \Rightarrow s\overline{G}(z,s) - 1 &= L\left\{-\lambda\left(1-z\right)G(z,t)\right\} \\ &= -\lambda\left(1-z\right)L\left\{G(z,t)\right\} \\ &= -\lambda\left(1-z\right)\overline{G}(z,s) \end{aligned}$$

$$\Rightarrow \left[s + \lambda(1-z)\right]\overline{G}(z,s) = 1$$

$$\therefore \overline{G}(z,s) = \frac{1}{s+\lambda(1-z)}$$

We now determine the inverse Laplace transform of both sides as follows

$$L^{-1}\left\{\overline{G}(z,s)\right\} = L^{-1}\left\{\frac{1}{s+\lambda(1-z)}\right\}$$

$$\Rightarrow G(z,t) = L^{-1}\left\{\frac{1}{s+\lambda(1-z)}\right\} \tag{5.10}$$

Various methods have been considered in solving equation (5.10).

Miscellaneous Method

In this method we determine the inverse Laplace transform with the help of the properties of Laplace transforms. We know that

$$\begin{aligned} L\left\{e^{-at}\right\} &= \int_0^\infty e^{-st}e^{-at}dt \\ &= \int_0^\infty e^{-(s+a)t}dt \\ &= \left.\frac{e^{-(s+a)t}}{s+a}\right|_0^\infty \\ &= \frac{1}{s+a} \end{aligned}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

Letting $a = \lambda(1-z)$, We have

$$L^{-1}\left\{\frac{1}{s+\lambda(1-z)}\right\} = e^{-\lambda(1-z)t}$$

Relating this to equation (5.11) it follows that

$$G(z,t) = e^{-\lambda(1-z)t}$$

which is the pgf of a Poisson distribution with parameter λt

$$\Rightarrow P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2, ...$$

Use of Tables

By equation (5.14) the Laplace transform of $G(z,t)$ is of the form

$$L\{f(t)\} = \frac{k}{s-a}$$

Where $a = -(1-z)$
Which from the table is the table of transform pairs yields

$$f(t) = ke^{at}$$

In our case $k = 1 \quad , a = -\lambda(z-1)$

$$\Rightarrow G(z,t) = e^{-\lambda(1-z)t}$$

Which is the probability generating function of a Poisson distribution with parameter λt.

$$\Rightarrow P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2, ...$$

Complex Inversion Formula

By equation (5.11) we had

$$G(z,t) = L^{-1}\left\{\frac{1}{s+\lambda(1-z)}\right\}$$

By the complex inversion formula

$$L^{-1}\left\{\frac{1}{s+\lambda(1-z)}\right\} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\left(\frac{1}{s+\lambda(1-z)}\right) ds$$

Which is simply the sum of the residues of

$$\frac{e^{st}}{s+\lambda(1-z)}$$

at the poles of

$$\frac{1}{s+\lambda(1-z)}$$

It can be seen that

$$\frac{1}{s+\lambda(1-z)}$$

is analytical at every point except when $s=-\lambda(1-z)$.

Hence

$$\frac{1}{s+\lambda(1-z)}$$

has a simple pole at $s=-\lambda(1-z)$

Thus the residue of

$$\frac{e^{st}}{s+\lambda(1-z)}$$

at $s=-\lambda(1-z)$ is given by

$$\begin{aligned} a_{-i} &= \lim_{s\to-\lambda(1-z)} \quad s+\lambda(1-z)\frac{e^{st}}{s+\lambda(1-z)} \\ &= \lim_{s\to-\lambda(1-z)} e^{st} \\ &= e^{-\lambda(1-z)t} \end{aligned}$$

Therefore

$$\begin{aligned} G(z,t) =& L^{-1}\left\{\frac{1}{s+\lambda(1-z)}\right\} \\ &= e^{-\lambda(1-z)t} \end{aligned}$$

By identification, $G(z,t)$ is the pgf of a Poisson distribution with parameter λt. Hence

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n=0,1,2,...$$

Alternatively, $P_n(t)$ is the coefficient of z^k in the expansion of $G(z,t)$. But

$$\begin{aligned} e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+... \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!} \end{aligned}$$

And

$$\begin{aligned}
G(z,t) &= e^{-\lambda(1-z)t} \\
&= e^{-\lambda t+\lambda tz} \\
&= e^{-\lambda t}e^{\lambda tz} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda tz)^n}{n!} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}z^n
\end{aligned}$$

$$\therefore P_n(t) = \frac{e^{-\lambda t}\,(\lambda t)^n}{n!} \qquad n = 0, 1, 2, ...$$

5.4 Simple Birth Process

The basic difference- differential equations for this process are obtained from the basic difference-differential equations for the general pure birth process by letting $\lambda_n = n\lambda \ \forall\, n$ These equations are

$$P_0'(t) = 0 \tag{5.11a}$$

$$P_n'(t) = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t) \qquad n \geqslant 1 \tag{5.11b}$$

We now solve these equations;

Method 1: By Iteration
Taking the Laplace transform of both sides of equation (5.11b) yields

$$L\{P_n(t)\} = L\{(n-1)\lambda P_{n-1}(t) - n\lambda P_n(t)\}$$

But by linearity of Laplace transform (property 1)

$$L\{(n-1)\lambda P_{n-1}(t) - n\lambda P_n(t)\} = (n-1)\lambda L\{P_{n-1}(t)\} - n\lambda L\{P_n(t)\}$$

And by using Laplace transform of derivatives (property 5)

$$L\{f(t)\} = \overline{f}(s)$$
$$L\{f'(t)\} = s\overline{f}(s) - f(0)$$

This implies that

$$L\{P_n'(t)\} = sL\{P_n(t)\} - P_n(0)$$

Using these properties we have

$$L^{-1}\{P_n(t)\} = (n-1)\lambda L\{P_{n-1}(t)\} - n\lambda L\{P_n(t)\}$$
$$sL\{P_n(t)\} - P_n(0) = (n-1)\lambda L\{P_{n-1}(t)\} - n\lambda L\{P_n(t)\}$$

$$(s+n\lambda)L\{P_n(t)\} = P_n(0) + (n-1)\lambda L\{P_{n-1}(t)\} \tag{5.12}$$

Remark

We assume that the initial population at t=0 is n_0 i.e $X(0) = n_0$

$$\Rightarrow P_{n_0}(0) = 1 \text{ and } P_n(0) = 0 \ \ \forall\, n \neq n_0$$

With this assumption, we now solve equation (5.12) iteratively.
When $n = n_0$ we have

$$(s+n_0\lambda)L\{P_{n_0}(t)\} = P_{n_0}(0) + (n_0-1)\lambda L\{P_{n_0-1}(t)\}$$

But since $X(0) = n_0$ and $P_{n_0}(0) = 1$ and $P_{n_0-1}(0) = 0$

Thus

$$L\{P_{n_0-1}(t)\} = L\{0\} = 0$$

Hence we have;

$$
\begin{aligned}
(s+n_0\lambda)L\left\{P_{n_0}(t)\right\} &= 1+(n_0-1)\lambda L\left\{0\right\} \\
(s+n_0\lambda)L\left\{P_{n_0}(t)\right\} &= 1 \\
\Rightarrow L\left\{P_{n_0}(t)\right\} &= \frac{1}{(s+n_0\lambda)}
\end{aligned}
$$

Similarly for $n = n_0 + 1$ we have

$$
\begin{aligned}
(s+n\lambda)L\left\{P_n(t)\right\} &= P_n(0)+(n-1)\lambda L\left\{P_{n-1}(t)\right\} \\
\left[s+(n_0+1)\lambda\right]L\left\{P_{n_0+1}(t)\right\} &= P_{n_0+1}(0)+(n_0+1-1)\lambda L\left\{P_{n_0+1-1}(t)\right\} \\
\left[s+(n_0+1)\lambda\right]L\left\{P_{n_0+1}(t)\right\} &= P_{n_0+1}(0)+n_0\lambda L\left\{P_{n_0}(t)\right\}
\end{aligned}
$$

But since $X(0) = n_0$ and $P_{n_0}(0) = 1$and $P_{n_0+1}(0) = 0$ and

$$
L\left\{P_{n_0}(t)\right\} = \frac{1}{(s+n_0\lambda)}
$$

It follows that;

$$
\begin{aligned}
\left[s+(n_0+1)\lambda\right]L\left\{P_{n_0+1}(t)\right\} &= 0+n_0\lambda\left[\frac{1}{(s+n_0\lambda)}\right] \\
\left[s+(n_0+1)\lambda\right]L\left\{P_{n_0+1}(t)\right\} &= \frac{n_0\lambda}{(s+n_0\lambda)} \\
\Rightarrow L\left\{P_{n_0+1}(t)\right\} &= \frac{n_0\lambda}{(s+n_0\lambda)\left(s+(n_0+1)\lambda\right)}
\end{aligned}
$$

When $n = n_0 + 2$ we have

$$
\begin{aligned}
(s+n\lambda)L\left\{P_n(t)\right\} &= P_n(0)+(n-1)\lambda L\left\{P_{n-1}(t)\right\} \\
\left[s+(n_0+2)\lambda\right]L\left\{P_{n_0+2}(t)\right\} &= P_{n_0+2}(0)+(n_0+2-1)\lambda L\left\{P_{n_0+2-1}(t)\right\} \\
\left[s+(n_0+2)\lambda\right]L\left\{P_{n_0+2}(t)\right\} &= P_{n_0+2}(0)+(n_0+1)\lambda L\left\{P_{n_0+1}(t)\right\}
\end{aligned}
$$

But since $X(0) = n_0$ and $P_{n_0}(0) = 1$ and $P_{n_0+2}(0) = 0$ and

$$L\left\{P_{n_0+1}(t)\right\} = \frac{n_0\lambda}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]}$$

We thus have

$$\left[s + (n_0+2)\lambda\right] L\left\{P_{n_0+2}(t)\right\} = 0 + (n_0+1)\lambda\left[\frac{n_0\lambda}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]}\right]$$

$$\left[s + (n_0+2)\lambda\right] L\left\{P_{n_0+2}(t)\right\} = \frac{n_0\lambda\left[(n_0+1)\lambda\right]}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]}$$

$$\Rightarrow L\left\{P_{n_0+2}(t)\right\} = \frac{\lambda^2 n_0(n_0+1)}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]\left[s + (n_0+2)\lambda\right]}$$

When $n = n_0 + 3$ we have

$$(s + n\lambda)L\left\{P_n(t)\right\} = P_n(0) + (n-1)\lambda L\left\{P_{n-1}(t)\right\}$$

$$\left[s + (n_0+3)\lambda\right] L\left\{P_{n_0+3}(t)\right\} = P_{n_0+3}(0) + (n_0+3-1)\lambda L\left\{P_{n_0+3-1}(t)\right\}$$

$$\left[s + (n_0+3)\lambda\right] L\left\{P_{n_0+3}(t)\right\} = P_{n_0+3}(0) + (n_0+2)\lambda L\left\{P_{n_0+2}(t)\right\}$$

But since $X(0) = n_0$ and $P_{n_0}(0) = 1$ and $P_{n_0+4}(0) = 0$ and

$$L\left\{P_{n_0+3}(t)\right\} = \frac{\lambda^3 n_0(n_0+1)}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]\left[s + (n_0+2)\lambda\right]\left[s + (n_0+3)\lambda\right]}$$

We have;

$$\left[s + (n_0+4)\lambda\right] L\left\{P_{n_0+4}(t)\right\} = (n_0+3)\lambda\left[\frac{\lambda^3 n_0(n_0+1)}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]\left[s + (n_0+2)\lambda\right]\left[s + (n_0+3)\lambda\right]}\right]$$

$$\left[s + (n_0+4)\lambda\right] L\left\{P_{n_0+4}(t)\right\} = \frac{\lambda^4 n_0(n_0+1)(n_0+2)(n_0+3)}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]\left[s + (n_0+2)\lambda\right]\left[s + (n_0+3)\lambda\right]}$$

$$\therefore L\left\{P_{n_0+4}(t)\right\} = \frac{\lambda^4 n_0(n_0+1)(n_0+2)(n_0+3)}{\left[s + n_0\lambda\right]\left[s + (n_0+1)\lambda\right]\left[s + (n_0+2)\lambda\right]\left[s + (n_0+3)\lambda\right]\left[s + (n_0+4)\lambda\right]}$$

The general expression for $L\left\{P_n\left(t\right)\right\}$ can be determined using mathematical induction.

Let us assume that for $n = n_0 + k - 1$ we have

$$L\left\{P_{n_0+k-1}(t)\right\} = \frac{\lambda^{k-1} n_0(n_0+1)(n_0+2)...(n_0+k-3)(n_0+k-2)}{\left[s+n_0\lambda\right]\left[s+(n_0+1)\lambda\right]...\left[s+(n_0+k-2)\lambda\right]\left[s+(n_0+k-1)\lambda\right]}$$

$$= \frac{\lambda^{k-1}\prod_{i=0}^{k-2}(n_0+i)}{\prod_{i=0}^{k-1}\left[s+(n_0+i)\lambda\right]}$$

Thus using equation (5.12) we have

$$(s+n\lambda)L\left\{P_n(t)\right\} = P_n(0) + (n-1)\lambda L\left\{P_{n-1}(t)\right\}$$

Then for $n = n_0 + k$, it follows that;

$$\left[s+(n_0+k)\lambda\right]L\left\{P_{n_0+k}(t)\right\} = P_{n_0+k}(0) + (n_0+k-1)\lambda L\left\{P_{n_0+k-1}(t)\right\}$$

But by the initial condition $X(0) = n_0$ we have $P_{n_0+k-1}(0) = 0$ and from above

$$L\left\{P_{n_0+k-1}(t)\right\} = \frac{\lambda^{k-1}\prod_{i=0}^{k-2}(n_0+i)}{\prod_{i=0}^{k-1}\left[s+(n_0+i)\lambda\right]}$$

Thus

$$\left[s+(n_0+k)\lambda\right]L\left\{P_{n_0+k}(t)\right\} = 0 + (n_0+k-1)\lambda\left[\frac{\lambda^{k-1}\prod_{i=0}^{k-2}(n_0+i)}{\prod_{i=0}^{k-1}\left[s+(n_0+i)\lambda\right]}\right]$$

$$\left[s+(n_0+k)\lambda\right]L\left\{P_{n_0+k}(t)\right\} = \frac{\lambda^{k}\prod_{i=0}^{k-1}(n_0+i)}{\prod_{i=0}^{k-1}\left[s+(n_0+i)\lambda\right]}$$

$$L\{P_{n_0+k}(t)\} = \frac{\lambda^k \prod_{i=0}^{k-1}(n_0+i)}{[s+(n_0+k)\lambda]\prod_{i=0}^{k-1}[s+(n_0+i)\lambda]}$$

$$\therefore L\{P_{n_0+k}(t)\} = \frac{\lambda^k \prod_{i=0}^{k-1}(n_0+i)}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]} \tag{5.13}$$

The problem is now to determine $P_n(t) = P_{n_0+k}(t)$ where $k = 0, 1, 2,$ this implies $n = n_0+1, n_0+2, n_0+3$ and so on.

Using equation (5.13) we apply inverse Laplace transform on both sides

$$L^{-1}\{L\{P_{n_0+k}(t)\}\} = L^{-1}\left\{\frac{\lambda^k \prod_{i=0}^{k-1}(n_0+i)}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]}\right\}$$

$$P_n(t) = P_{n_0+k}(t) = L^{-1}\left\{\frac{\lambda^k \prod_{i=0}^{k-1}(n_0+i)}{\prod_{i=0}^{k}[s+\lambda(n_0+i)]}\right\}$$

Since

$$L^{-1}\left\{a\overline{f}(s)\right\} = aL^{-1}\left\{\overline{f}(s)\right\}$$

We have

$$P_n(t) = \lambda^k \prod_{i=0}^{k-1}(n_0+i) L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}[s+\lambda(n_0+i)]}\right\} \tag{5.14}$$

The following methods have been considered in solving equation (5.14)

- Complex Inversion Formula
- Partial Fractions Method

Complex Inversion Formula

By the complex inversion formula,

If $L\left\{f(t)\right\} = \overline{f}(s)$ then;

$$\begin{aligned} f(t) &= L^{-1}\left\{\overline{f}(s)\right\} \\ &= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\overline{f}(s)ds \end{aligned}$$

Which is simply the sum of the residues of $e^{st}\overline{f}(s)$ at the poles of $\overline{f}(s)$

In this case

$$\overline{f}(s) = \frac{1}{\prod\limits_{i=0}^{k}\left[s + (n_0 + i)\lambda\right]}$$

$$\Rightarrow e^{st}\overline{f}(s) = \frac{e^{st}}{\prod\limits_{i=0}^{k}\left[s + (n_0 + i)\lambda\right]}$$

It can be seen that $\overline{f}(s)$ is analytical at every other point except when

$$s = -\lambda(n_0 + i)\ ,i = 0, 1, 2...k$$

Therefore $\overline{f}(s)$ has $k+1$ simple poles (poles of order 1) at values of s listed below;

$$\begin{aligned} s &= -\lambda n_0 \\ s &= -\lambda(n_0 + 1) \\ s &= -\lambda(n_0 + 2) \\ s &= -\lambda(n_0 + 3) \\ &. \\ &. \\ &. \\ s &= -\lambda(n_0 + k) \end{aligned}$$

Thus we have

$$L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}\left[s+(n_0+i)\,\lambda\right]}\right\} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{\prod_{i=0}^{k}\left[s+(n_0+i)\,\lambda\right]}$$

$$= \sum a_{-i}$$

Where a_{-i} are the residues of

$$\frac{e^{st}}{\prod_{i=0}^{k}\left[s+(n_0+i)\,\lambda\right]}$$

at the poles of

$$\frac{1}{\prod_{i=0}^{k}\left[s+(n_0+i)\,\lambda\right]}$$

We now determine the a_{-i}'s as follows;

In general for a pole of order n say at $s=a$,

$$a_{-i} = \lim_{s\to a}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\,(s-a)^n\,\overline{f}(s)$$

And for a simple pole at $s=a$

$$a_{-i} = \lim_{s\to a}(s-a)\overline{f}(s)$$

Thus residue of

$$\frac{e^{st}}{\prod_{i=0}^{k}\left[s+(n_0+i)\,\lambda\right]}$$

at $s = -\lambda n_0$ is given by

$$a_{-i} = \lim_{s \to -\lambda n_0} (s + \lambda n_0) \frac{e^{st}}{(s + \lambda n_0) \prod_{i=0}^{k} [s + (n_0 + i)\lambda]}$$

$$= \lim_{s \to -\lambda n_0} (s + \lambda n_0) \frac{e^{st}}{(s + \lambda n_0) \prod_{i=1}^{k} [s + (n_0 + i)\lambda]}$$

$$= \lim_{s \to -\lambda n_0} \frac{e^{st}}{\prod_{i=1}^{k} [s + (n_0 + i)\lambda]}$$

$$= \frac{e^{-\lambda n_0 t}}{\prod_{i=1}^{k} [-\lambda n_0 + (n_0 + i)\lambda]}$$

$$= \frac{e^{-\lambda n_0 t}}{\prod_{i=1}^{k} [-\lambda n_0 + \lambda n_0 + i\lambda]}$$

$$= \frac{e^{-\lambda n_0 t}}{\prod_{i=1}^{k} \lambda i}$$

$$= \frac{e^{-\lambda n_0 t}}{\lambda^k \prod_{i=1}^{k} i}$$

$$= \frac{e^{-\lambda n_0 t}}{\lambda^k k!}$$

Similarly the residue at $s = -\lambda(n_0 + 1)$ is given by

$$
\begin{aligned}
a_{-i} &= \lim_{s \to -\lambda(n_0+1)} [s + \lambda(n_0 + 1)] \frac{e^{st}}{\prod_{i=0}^{k} [s + (n_0 + i)\lambda]} \\
&= \lim_{s \to -\lambda(n_0+1)} [s + \lambda(n_0 + 1)] \frac{e^{st}}{(s + \lambda n_0) [s + \lambda(n_0 + 1)] \prod_{i=2}^{k} [s + (n_0 + i)\lambda]} \\
&= \lim_{s \to -\lambda(n_0+1)} [s + \lambda(n_0 + 1)] \frac{e^{st}}{(s + \lambda n_0) [s + \lambda(n_0 + 1)] \prod_{i=2}^{k} [s + (n_0 + i)\lambda]} \\
&= \lim_{s \to -\lambda(n_0+1)} \frac{e^{st}}{(s + \lambda n_0) \prod_{i=2}^{k} [s + (n_0 + i)\lambda]} \\
&= \frac{e^{-\lambda(n_0+1)t}}{(-\lambda n_0 - \lambda + \lambda n_0) \prod_{i=1}^{k} [-\lambda n_0 - \lambda + (n_0 + i)\lambda]} \\
&= \frac{e^{-\lambda(n_0+1)t}}{-\lambda \prod_{i=1}^{k} [-\lambda n_0 - \lambda + \lambda n_0 + i\lambda]} \\
&= \frac{e^{-\lambda(n_0+1)t}}{-\lambda \prod_{i=1}^{k} \lambda i - \lambda} \\
&= \frac{e^{-\lambda(n_0+1)t}}{-\lambda \prod_{i=1}^{k} \lambda(i - 1)} \\
&= \frac{e^{-\lambda(n_0+1)t}}{-\lambda\lambda^{k-1} \prod_{i=1}^{k} (i - 1)} \\
&= \frac{e^{-\lambda(n_0+1)t}}{(-1)\lambda^k (k - 1)!}
\end{aligned}
$$

Similarly the residue at $s = -\lambda(n_0 + 2)$ is given by

$$a_{-i} = \lim_{s \to -\lambda(n_0+2)} [s + \lambda(n_0 + 2)] \frac{e^{st}}{\prod_{i=0}^{k} [s + (n_0 + i)\lambda]}$$

$$= \lim_{s \to -\lambda(n_0+2)} [s + \lambda(n_0 + 2)] \frac{e^{st}}{[s + \lambda(n_0 + 2)] \prod_{i=0}^{1} [s + \lambda(n_0 + i)] \prod_{i=3}^{k} [s + (n_0 + i)\lambda]}$$

$$= \lim_{s \to -\lambda(n_0+2)} \frac{e^{st}}{\prod_{i=0}^{1} [s + \lambda(n_0 + 1)] \prod_{i=3}^{k} [s + (n_0 + i)\lambda]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\prod_{i=0}^{1} [-\lambda(n_0 + 2) + \lambda(n_0 + 1)] \prod_{i=3}^{k} [-\lambda(n_0 + 2) + (n_0 + i)\lambda]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\prod_{i=0}^{1} [-\lambda n_0 - 2\lambda + \lambda n_0 + \lambda i)] \prod_{i=3}^{k} [-\lambda n_0 - 2\lambda + \lambda n_0 + \lambda i]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\prod_{i=0}^{1} (\lambda i - 2\lambda) \prod_{i=3}^{k} (\lambda i - 2\lambda)}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-2\lambda(-\lambda) \prod_{i=3}^{k} \lambda(i - 2)}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{(-1)^2 \lambda^2 \lambda^{k-2} (2) \prod_{i=3}^{k} (i - 2)}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{(-1)^2 \lambda^k (2!) \prod_{i=3}^{k} (i - 2)}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{(-1)^2 \lambda^k (2!) (k - 2)!}$$

Similarly the residue at $s = -\lambda(n_0+3)$ is given by

$$a_{-i} = \lim_{s\to-\lambda(n_0+3)} [s+\lambda(n_0+3)] \frac{e^{st}}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]}$$

$$= \lim_{s\to-\lambda(n_0+3)} [s+\lambda(n_0+3)] \frac{e^{st}}{[s+\lambda(n_0+3)]\prod_{i=0}^{2}[s+\lambda(n_0+i)]\prod_{i=4}^{k}[s+(n_0+i)\lambda]}$$

$$= \lim_{s\to-\lambda(n_0+3)} \frac{e^{st}}{\prod_{i=0}^{2}[s+\lambda(n_0+i)]\prod_{i=4}^{k}[s+(n_0+i)\lambda]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\prod_{i=0}^{2}[-\lambda(n_0+3)+\lambda(n_0+i)]\prod_{i=4}^{k}[-\lambda(n_0+3)+(n_0+i)\lambda]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\prod_{i=0}^{2}[-\lambda n_0-3\lambda+\lambda n_0+\lambda i)]\prod_{i=4}^{k}[-\lambda n_0-3\lambda+\lambda n_0+\lambda i]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\prod_{i=0}^{2}(\lambda i-3\lambda)\prod_{i=4}^{k}(\lambda i-3\lambda)}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{-3\lambda\,(-2\lambda)\,(-\lambda)\prod_{i=4}^{k}\lambda(i-3)}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{(-1)^3\lambda^3\lambda^{k-3}\,(6)\prod_{i=3}^{k}(i-3)}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{(-1)^3\lambda^k\,(3!)\prod_{i=3}^{k}(i-3)}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{(-1)^3\lambda^k\,(3!)\,(k-3)!}$$

Similarly the residue at$s = -\lambda(n_0+4)$ is given by

$$a_{-i} = \lim_{s \to -\lambda(n_0+4)} [s+\lambda(n_0+4)] \frac{e^{st}}{\prod_{i=0}^{k} [s+(n_0+i)\lambda]}$$

$$= \lim_{s \to -\lambda(n_0+4)} [s+\lambda(n_0+4)] \frac{e^{st}}{[s+\lambda(n_0+4)] \prod_{i=0}^{3} [s+\lambda(n_0+i)] \prod_{i=5}^{k} [s+(n_0+i)\lambda]}$$

$$= \lim_{s \to -\lambda(n_0+4)} \frac{e^{st}}{\prod_{i=0}^{3} [s+\lambda(n_0+i)] \prod_{i=5}^{k} [s+(n_0+i)\lambda]}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{\prod_{i=0}^{3} [-\lambda(n_0+4)+\lambda(n_0+i)] \prod_{i=5}^{k} [-\lambda(n_0+4)+(n_0+i)\lambda]}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{\prod_{i=0}^{3} [-\lambda n_0 - 4\lambda + \lambda n_0 + \lambda i)] \prod_{i=5}^{k} [-\lambda n_0 - 4\lambda + \lambda n_0 + \lambda i]}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{\prod_{i=0}^{3} (\lambda i - 4\lambda) \prod_{i=5}^{k} (\lambda i - 4\lambda)}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{-4\lambda\,(-3\lambda)\,(-2\lambda)\,(-\lambda) \prod_{i=5}^{k} \lambda(i-4)}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{(-1)^4\lambda^4\lambda^{k-4}\,(24) \prod_{i=5}^{k} (i-4)}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{(-1)^4\lambda^k\,(4!) \prod_{i=5}^{k} (i-4)}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{(-1)^4\lambda^k\,(4!)\,(k-4)!}$$

Generalizing the results above, the residue at the pole $s = -\lambda(n_0 + k)$ is given by;

$$\operatorname{Re} sidue\left[e^{st}\overline{f}(s)\right] = \frac{e^{-\lambda(n_0+k)t}}{(-1)^k \lambda^k k!\,(k-k)!}$$
$$= \frac{e^{-\lambda(n_0+k)t}}{(-1)^k \lambda^k k!}$$

Therefore summing the residues at each pole yields,

$$L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]}\right\} = \frac{1}{2\pi i}\int_{r-c\infty}^{r+c\infty}\left(\frac{e^{st}}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]}\right)ds$$
$$= \sum \operatorname{Re} sidue\left(\frac{e^{st}}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]}\right)$$

$$= \frac{e^{-\lambda n_0 t}}{\lambda^k 0!k!} + \frac{e^{-\lambda(n_0+1)t}}{(-1)\lambda^k 1!\,(k-1)!} + \frac{e^{-\lambda(n_0+2)t}}{(-1)^2\lambda^k 2!\,(k-2)!}$$
$$+ \frac{e^{-\lambda(n_0+3)t}}{(-1)^3\lambda^k 3!\,(k-3)!} + \frac{e^{-\lambda(n_0+4)t}}{(-1)^4\lambda^4 4!\,(k-4)!}... + \frac{e^{-\lambda(n_0+k)t}}{(-1)^k\lambda^k k!\,(k-k)!}$$

$$= \sum_{j=0}^{k}\left\{\frac{e^{-\lambda(n_0+j)t}}{(-1)^j\lambda^k j!\,(k-j)!}\right\}$$
$$= \frac{1}{\lambda^k}\sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}}{(-1)^j j!\,(k-j)!}\right\} \tag{5.15}$$

But from equation (5.14) we had

$$P_n(t) = \lambda^k \prod_{i=0}^{k-1}(n_0+i)L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}[s+(n_0+i)]}\right\}$$

Thus

$$P_n\left(t\right) = \lambda^k \prod_{i=0}^{k-1}\left(n_0+i\right) \times \frac{1}{\lambda^k}\sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}}{\left(-1\right)^j j!\left(k-j\right)!}\right\}$$

$$= \prod_{i=0}^{k-1}\left(n_0+i\right) \times \sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}}{\left(-1\right)^j j!\left(k-j\right)!}\right\} \tag{5.16}$$

But

$$\prod_{i=0}^{k-1}\left(n_0+i\right) = n_0(n_0+1)(n_0+2)(n_0+3)...(n_0+k-2)(n_0+k-1)$$

And by the definition of the factorial function

$$n! = n(n-1)(n-2)(n-3)...(3)(2)(1)$$

$$\Rightarrow (n_0+k-1)! = (n_0+k-1)(n_0+k-3)...(n_0+2)(n_0+1)(n_0)(n_0-1)!$$

$$\Rightarrow \frac{(n_0+k-1)!}{(n_0-1)!} = (n_0+k-1)(n_0+k-3)...(n_0+2)(n_0+1)n_0$$

$$\therefore \prod_{i=0}^{k-1}\left(n_0+i\right) = \frac{(n_0+k-1)!}{(n_0-1)!} \tag{5.17}$$

Therefore

$$P_n\left(t\right) = \prod_{i=0}^{k-1}\left(n_0+i\right) \times \sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}}{\left(-1\right)^j j!\left(k-j\right)!}\right\}$$

$$= \frac{(n_0+k-1)!}{(n_0-1)!}\sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}}{\left(-1\right)^j j!\left(k-j\right)!}\right\}$$

Multiplying the above equation by $\frac{k!}{k!}$ we have

$$P_n(t) = \frac{(n_0+k-1)!}{(n_0-1)!}\sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}}{(-1)^j\, j!\,(k-j)!}\right\} \times \frac{k!}{k!}$$

$$= \frac{(n_0+k-1)!}{k!(n_0-1)!}\sum_{j=0}^{n}\left\{\frac{e^{-\lambda(n_0+j)t}k!}{(-1)^j\, j!\,(k-j)!}\right\}$$

$$= \frac{(n_0+k-1)!e^{-\lambda t n_0}}{k!(n_0-1)!}\sum_{j=0}^{n}\left\{\frac{(-1)^j\, e^{-\lambda j t}k!}{j!\,(k-j)!}\right\}$$

$$= \frac{(n_0+k-1)!e^{-\lambda n_0 t}}{k!(n_0-1)!}\sum_{j=0}^{n}\left\{\frac{\left(-e^{-\lambda t}\right)^j k!}{j!\,(k-j)!}\right\}$$

But

$$\frac{(n_0+k-1)!}{k!(n_0-1)!} = \binom{n_0+k-1}{n_0-1}$$

And

$$\frac{k!}{j!\,(k-j)!} = \binom{k}{j}$$

Thus

$$P_n(t) = \binom{n_0+k-1}{n_0-1} e^{-\lambda t n_0}\sum_{j=0}^{k}\binom{k}{j}\left(-e^{-\lambda t}\right)^j$$

$$= \binom{n_0+k-1}{n_0-1}\left(e^{-\lambda t}\right)^{n_0}\left(1-e^{-\lambda t}\right)^{n-n_0} \qquad n = n_0,\ n_0+1,\ n_0+2...$$

$$= \binom{n_0+k-1}{k}\left(e^{-\lambda t}\right)^{n_0}\left(1-e^{-\lambda t}\right)^{k} \qquad k = 0, 1, 2, ...$$

Which is the pmf of a negative binomial distribution with parameters $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

Special case

If the initial population at time $t = 0$ is 1, then $n_0 = 1$, thus

$$\begin{aligned}
P_n(t) &= \binom{1+k-1}{1-1} e^{-\lambda t}\left(1-e^{-\lambda t}\right)^k \\
&= e^{-\lambda t}\left(1-e^{-\lambda t}\right)^k \; n = 1+k,\;\; k = 0,1,2,... \\
&= e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1} \; n = 1,2,... \\
&= pq^{n-1}, n = 1,2,...
\end{aligned}$$

Which is the pmf of a shifted geometric distribution with parameters $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

Partial Fractions Method

By equation (5.14) we had

$$P_n(t) = \lambda^k \prod_{i=0}^{k-1}(n_0+i) L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+(n_0+i)\right]}\right\}$$

But from equation (5.17)

$$\prod_{i=0}^{k-1}(n_0+i) = \frac{(n_0+k-1)!}{(n_0-1)!}$$

Therefore

$$\begin{aligned}
P_n(t) &= \lambda^k \prod_{i=0}^{k-1}(n_0+i) L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+\lambda(n_0+i)\right]}\right\} \\
&= \frac{\lambda^k(n_0+k-1)!}{(n_0-1)!} L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+\lambda(n_0+i)\right]}\right\}
\end{aligned}$$

We now determine

$$L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+(n_0+i)\right]}\right\}$$

But

$$\frac{1}{\prod\limits_{i=0}^{k}\left[s+\lambda\left(n_0+i\right)\right]}=\left(\frac{1}{s+\lambda n_0}\right)\left(\frac{1}{s+\lambda\left(n_0+1\right)}\right)\cdots\left(\frac{1}{s+\lambda\left(j-1\right)}\right)\left(\frac{1}{s+\lambda j}\right)\times$$

$$\left(\frac{1}{s+\lambda\left(j+1\right)}\right)\cdots\left(\frac{1}{s+\lambda\left(n_0+k\right)}\right)$$

Using partial fractions

$$\left\{\begin{array}{c}\left(\frac{1}{s+\lambda n_0}\right)\left(\frac{1}{s+\lambda(n_0+1)}\right)\cdots \\ \\ \left(\frac{1}{s+\lambda(j-1)}\right)\left(\frac{1}{s+\lambda j}\right) \\ \\ \left(\frac{1}{s+\lambda(j+1)}\right)\cdots\left(\frac{1}{s+\lambda(n_0+k)}\right)\end{array}\right\}=\left\{\begin{array}{c}\frac{a_{n_0}}{s+\lambda n_0}+\frac{a_{n_0+1}}{s+\lambda(n_0+1)}+\ldots \\ \\ +\frac{a_{j-1}}{s+\lambda(j-1)}+\frac{a_j}{s+\lambda j}+ \\ \\ \frac{a_{j+1}}{s+\lambda(j+1)}+\ldots+\frac{a_{n_0+k}}{s+\lambda(n_0+k)}\end{array}\right\}$$

Multiplying both sides by $s+\lambda j$ yields

$$\left\{\begin{array}{c}\left(\frac{1}{s+\lambda n_0}\right)\left(\frac{1}{s+\lambda(n_0+1)}\right)\cdots\\ \\ \left(\frac{1}{s+\lambda(j-1)}\right)\left(\frac{1}{s+\lambda j}\right)\\ \\ \left(\frac{1}{s+\lambda(j+1)}\right)\cdots\left(\frac{1}{s+\lambda(n_0+k)}\right)\end{array}\right\}=\left\{\begin{array}{c}\frac{a_{n_0}(s+\lambda j)}{s+\lambda n_0}+\frac{a_{n_0+1}(s+\lambda j)}{s+\lambda(n_0+1)}+\ldots\\ \\ +\frac{a_{j-1}(s+\lambda j)}{s+\lambda(j-1)}+\frac{a_j(s+\lambda j)}{s+\lambda j}+\\ \\ \frac{a_{j+1}(s+\lambda j)}{s+\lambda(j+1)}+\ldots+\frac{a_{n_0+k}(s+\lambda j)}{s+\lambda(n_0+k)}\end{array}\right\}$$

$$\left\{\begin{array}{c}\left(\frac{1}{s+\lambda n_0}\right)\left(\frac{1}{s+\lambda(n_0+1)}\right)\\ \\ \ldots\left(\frac{1}{s+\lambda(j-1)}\right)\left(\frac{1}{s+\lambda(j+1)}\right)\\ \\ \ldots\left(\frac{1}{s+\lambda(n_0+k)}\right)\end{array}\right\}=\left\{\begin{array}{c}\frac{a_{n_0}(s+\lambda j)}{s+\lambda n_0}+\frac{a_{n_0+1}(s+\lambda j)}{s+\lambda(n_0+1)}+\ldots\\ \\ +\frac{a_{j-1}(s+\lambda j)}{s+\lambda(j-1)}+a_j+\\ \\ \frac{a_{j+1}(s+\lambda j)}{s+\lambda(j+1)}+\ldots+\frac{a_{n_0+k}(s+\lambda j)}{s+\lambda(n_0+k)}\end{array}\right\}$$

Setting $s=-\lambda j$, implies

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda j+\lambda n_0}\right)\left(\frac{1}{-\lambda j+\lambda(n_0+1)}\right)\cdots\\ \\ \left(\frac{1}{-\lambda j+\lambda(j-1)}\right)\left(\frac{1}{-\lambda j+\lambda(j+1)}\right)\cdots\\ \\ \left(\frac{1}{-\lambda j+\lambda(n_0+k)}\right)\end{array}\right\}=\left\{\begin{array}{c}\frac{a_{n_0}(-\lambda j+\lambda j)}{-\lambda j+\lambda n_0}+\frac{a_{n_0+1}(-\lambda j+\lambda j)}{-\lambda j+\lambda(n_0+1)}+\ldots+\\ \\ \frac{a_{j-1}(-\lambda j+\lambda j)}{-\lambda j+\lambda(j-1)}+a_j+\frac{a_{j+1}(-\lambda j+\lambda j)}{-\lambda j+\lambda(j+1)}\\ \\ +\ldots+\frac{a_{n_0+k}(-\lambda j+\lambda j)}{-\lambda j+\lambda(n_0+k)}\end{array}\right\}$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda j+\lambda n_0}\right)\left(\frac{1}{-\lambda j+\lambda(n_0+1)}\right)\cdots\left(\frac{1}{-\lambda j+\lambda(j-2)}\right)\left(\frac{1}{-\lambda j+\lambda(j-1)}\right)\\ \\ \times\left(\frac{1}{-\lambda j+\lambda(j+1)}\right)\left(\frac{1}{-\lambda j+\lambda(j+2)}\right)\cdots\left(\frac{1}{-\lambda j+\lambda(n_0+k)}\right)\end{array}\right\}=a_j$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda(j-n_0)}\right)\left(\frac{1}{-\lambda(j-(n_0+1))}\right)\cdots\left(\frac{1}{-\lambda(2)}\right)\left(\frac{1}{-\lambda(1)}\right)\\ \\ \times\left(\frac{1}{\lambda(1)}\right)\left(\frac{1}{\lambda(2)}\right)\cdots\left(\frac{1}{\lambda(n_0+k-j)}\right)\end{array}\right\}=a_j$$

$$\left\{\begin{array}{c}\frac{1}{(-\lambda)^{j-n_0}}\left(\frac{1}{j-n_0}\right)\left(\frac{1}{j-(n_0+1)}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{1}{1}\right)\\ \\ \times\frac{1}{\lambda^{n_0+k-j}}\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{n_0+k-j}\right)\end{array}\right\}=a_j$$

$$\frac{1}{(-\lambda)^{j-n_0}(j-n_0)!}\times\frac{1}{\lambda^{n_0+k-j}(n_0+k-j)!}=a_j$$

$$\Rightarrow a_j=\frac{1}{(-1)^{j-n_0}\lambda^{j-n_0}(j-n_0)!}\times\frac{1}{\lambda^{n_0+k-j}(n_0+k-j)!}$$

$$=\frac{1}{(-1)^{j-n_0}\lambda^{j-n_0}(j-n_0)!\lambda^{n_0+k-j}(n_0+k-j)!}$$

$$= \frac{(-1)^{j-n_0}}{\lambda^{j-n_0+n_0+k-j}\,(j-n_0)!\,(n_0+k-j)!}$$

$$= \frac{(-1)^{j-n_0}}{\lambda^{k}\,(j-n_0)!\,(n_0+k-j)!}$$

$$= \left(\frac{1}{\lambda}\right)^{k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} \qquad j = n_0, \quad n_0+1, \quad n_0+2,$$

With this it follows that

$$\frac{1}{\prod_{i=0}^{k}\left[s+\lambda\,(n_0+i)\right]} = \sum_{j=n_0}^{n_0+k} \frac{a_j}{s+j\lambda}$$

$$= \sum_{j=n_0}^{n_0+k} \left(\frac{1}{\lambda}\right)^{k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} \left(\frac{1}{s+j\lambda}\right)$$

Thus

$$P_n\,(t) = \frac{\lambda^k (n_0+k-1)!}{(n_0-1)!} L^{-1} \left\{ \frac{1}{\prod_{i=0}^{k}\left[s+\lambda\,(n_0+i)\right]} \right\}$$

$$= \frac{\lambda^k (n_0+k-1)!}{(n_0-1)!} L^{-1} \left\{ \sum_{j=n_0}^{n_0+k} \left(\frac{1}{\lambda}\right)^{k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} \left(\frac{1}{s+j\lambda}\right) \right\}$$

$$= \frac{\lambda^k (n_0+k-1)!}{(n_0-1)!} \left(\frac{1}{\lambda}\right)^{k} \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} L^{-1} \left\{ \frac{1}{s+j\lambda} \right\}$$

$$= \frac{(n_0+k-1)!}{(n_0-1)!} \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} L^{-1} \left\{ \frac{1}{s+j\lambda} \right\}$$

But

$$L\left\{e^{-\lambda jt}\right\} = \int_{0}^{\infty} e^{-st} e^{-\lambda jt} dt$$

$$= \int_{0}^{\infty} e^{-(\lambda j+s)t} dt$$

$$= \left.\frac{e^{-(\lambda j+s)t}}{-(\lambda j+s)}\right|_{0}^{\infty}$$

$$= \frac{1}{-(\lambda j+s)}\left[0-1\right]$$

$$= \frac{1}{s+j\lambda}$$

$$\Rightarrow e^{-\lambda jt} = L^{-1}\left\{\frac{1}{s+j\lambda}\right\}$$

Thus

$$P_n\left(t\right) = \frac{(n_0+k-1)!}{(n_0-1)!}\sum_{j=n_0}^{n_0+k}\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}L^{-1}\left\{\frac{1}{s+j\lambda}\right\}$$

$$= \frac{(n_0+k-1)!}{(n_0-1)!}\sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}\right] e^{-j\lambda t}$$

Multiplying the above equation by $\frac{k!}{k!}$ yields;

$$P_n\left(t\right) = \frac{(n_0+k-1)!}{(n_0-1)!}\sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}\right] e^{-j\lambda t}\frac{k!}{k!}$$

$$= \frac{(n_0+k-1)!}{(n_0-1)!k!}\sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}\,k!}{(j-n_0)!\,(n_0+k-j)!}\right] e^{-j\lambda t}$$

$$= \binom{n_0+k-1}{n_0-1} \sum_{j=n_0}^{n_0+k} (-1)^{j-n_0} \binom{k}{j-n_0} e^{-j\lambda t}$$

$$= \binom{n_0+k-1}{k} \sum_{j=n_0}^{n_0+k} (-1)^{j-n_0} \binom{k}{j-n_0} e^{-j\lambda t}$$

Letting $m = j - n_0 \Rightarrow j = m + n_0$ implies

$$P_n(t) = \binom{n_0+k-1}{n_0-1} \sum_{m=0}^{k} (-1)^m \binom{k}{m} e^{-(m+n_0)\lambda t}$$

$$= \binom{n_0+k-1}{n_0-1} e^{-\lambda t n_0} \sum_{m=0}^{k} (-1)^m \binom{k}{m} e^{-\lambda t m}$$

$$= \binom{n_0+k-1}{n_0-1} e^{-\lambda t n_0} \sum_{m=0}^{k} \binom{k}{m} \left(-e^{-\lambda t}\right)^m$$

$$= \binom{n_0+k-1}{n_0-1} e^{-\lambda t n_0} \left(1 - e^{-\lambda t}\right)^k$$

But $n = n_0 + k$, hence

$$P_n(t) = \binom{n-1}{n_0-1} e^{-\lambda t n_0} \left(1 - e^{-\lambda t}\right)^{n-n_0} \quad n = n_0, n_0+1, n_0+2, ...$$

$$= \binom{n_0+k-1}{k} e^{-\lambda t n_0} \left(1 - e^{-\lambda t}\right)^k \qquad k = 0,\ 1,\ 2,\ 3,\ ...$$

Which is the negative binomial distribution with $r = n_0, \quad p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

Method 2: Using the pgf approach

The basic difference-differential equations for the simple process birth process as defined by equations (5.11a) and (5.11b) are;

$$P_0'(t) = 0 \tag{5.11a}$$

$$P_n'(t) = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t) \quad n \geqslant 1 \tag{5.11b}$$

Multiplying both sides of equation (5.11b) by z^n and summing the result over n, we have

$$\sum_{n=1}^{\infty} P_n'(t)z^n = -\lambda \sum_{n=1}^{\infty} nP_n(t)z^n + \lambda \sum_{n=1}^{\infty} (n-1)P_{n-1}(t)z^n \tag{5.18}$$

Let $G(z,t)$ be the probability generating function of $X(t)$ defined as follows;

$$\begin{aligned} G(z,t) &= \sum_{n=0}^{\infty} P_n(t)z^n \\ &= P_0(t) + \sum_{n=1}^{\infty} P_n(t)z^n \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial t}G(z,t) &= \sum_{n=0}^{\infty} P_n'(t)z^n \\ &= P_0'(t) + \sum_{n=1}^{\infty} P_n'(t)z^n \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial z}G(z,t) &= \sum_{n=0}^{\infty} nP_n(t)z^{n-1} \\ &= \frac{1}{z}\sum_{n=1}^{\infty} nP_n(t)z^n \\ &= \sum_{n=1}^{\infty} (n-1)P_n(t)z^{n-2} \end{aligned}$$

We now simplify the three parts of equation (5.18) separately

Part I

$$\sum_{n=1}^{\infty} P_n'(t) z^n = \frac{\partial}{\partial t} G(z,t) - P_0'(t)$$

Part II

$$\begin{aligned}\lambda \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) z^n &= \lambda z^2 \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) z^{n-2} \\ &= \lambda z^2 \frac{\partial}{\partial z} G(z,t)\end{aligned}$$

Part III

$$-\lambda \sum_{n=1}^{\infty} n P_n(t) z^n = -\lambda z \frac{\partial}{\partial z} G(z,t)$$

Substituting the above results in equation (5.18) yields

$$\begin{aligned}\frac{\partial}{\partial t} G(z,t) - P_0'(t) &= \lambda z^2 \frac{\partial}{\partial z} G(z,t) - \lambda z \frac{\partial}{\partial z} G(z,t) \\ &= \lambda z \left(z-1\right) \frac{\partial}{\partial z} G(z,t)\end{aligned}$$

But

$$\begin{aligned}&P_0'(t) = 0 \\ &\Rightarrow \frac{\partial}{\partial t} G(z,t) = \lambda z \left(z-1\right) \frac{\partial}{\partial z} G(z,t) \\ &\Rightarrow \frac{\partial}{\partial t} G(z,t) - \lambda z \left(z-1\right) \frac{\partial}{\partial z} G(z,t) = 0\end{aligned}$$

Which is a linear partial differential equation. Appying Laplace transform to both sides yields.

$$
\begin{aligned}
L\left\{\frac{\partial}{\partial t}G(z,t)-\lambda z\,(z-1)\,\frac{\partial}{\partial z}G(z,t)\right\} &= L\left\{0\right\}\\
L\left\{\frac{\partial}{\partial t}G(z,t)\right\}-\lambda z\,(z-1)\,L\left\{\frac{\partial}{\partial z}G(z,t)\right\} &= 0
\end{aligned}
\tag{5.19}
$$

But by the definition of Laplace transform

$$
\begin{aligned}
L\left\{\frac{\partial}{\partial t}G(z,t)\right\} &= \overline{G}\,(z,s)\\
&= \int_0^\infty e^{-st}\frac{\partial}{\partial t}G(z,t)dt
\end{aligned}
$$

Using integration by parts
Let $u=e^{-st}\Rightarrow du=-se^{-st}$ and

$$
dv=\frac{\partial}{\partial t}G(z,t)dt\Rightarrow v=G(z,t)
$$

Thus

$$
\begin{aligned}
L\left\{\frac{\partial}{\partial t}G(z,t)\right\} &= e^{-st}G(z,t)\Big|_0^\infty-\left\{\int_0^\infty -se^{-st}G(z,t)\right\}\\
&= [0-G(z,t)]+s\underbrace{\int_0^\infty e^{-st}G(z,t)}_{\overline{G}(z,s)}\\
&= s\overline{G}(z,s)-G(z,0)
\end{aligned}
$$

Similarly

$$
L\left\{\frac{\partial}{\partial z}G(z,t)\right\}=\int_0^\infty e^{-st}\frac{\partial}{\partial z}G(z,t)dt
$$

By Leibnitz rule for differentiating under an integral, we have

$$L\left\{\frac{\partial}{\partial z}G(z,t)\right\} = \frac{d}{dz}\underbrace{\int_0^\infty e^{-st}G(z,t)dt}_{\overline{G}(z,s)}$$
$$= \frac{d}{dz}\overline{G}(z,s)$$

With this now equation (5.19) becomes

$$s\overline{G}(z,s) - G(z,0) - \lambda z\,(z-1)\,\frac{d}{dz}\overline{G}(z,s) = 0$$
$$s\overline{G}(z,s) - \lambda z\,(z-1)\,\frac{d}{dz}\overline{G}(z,s) = G(z,0)$$

$$-\lambda z\,(z-1)\,\frac{d}{dz}\overline{G}\,(z,s) + s\overline{G}\,(z,s) = G\,(z,0)$$

$$\frac{d}{dz}\overline{G}\,(z,s) - \frac{s}{\lambda z\,(z-1)}\overline{G}\,(z,s) = -\frac{G\,(z,0)}{\lambda z\,(z-1)} \tag{5.20a}$$

Which is an ordinary differential equation.

The remaining task is to find a solution to this differential equation. Two methods are considered.

- Dirac delta function approach
- Hyper geometric function approach

Dirac delta function approach

Equation (5.20a) is an ODE of 1^{st} order. It is of the form $y' + Py = Q$ where

$$y = \overline{G}\left(z,s\right), P = -\frac{s}{\lambda z\left(z-1\right)} \text{ and } Q = -\frac{G(z,0)}{\lambda z\left(z-1\right)}$$

Now the solution formula for such an ODE is given by

$$ye^{\int Pdz} = f + \int Qe^{\int Pdz}$$

Where f does not depend on the variable z but can depend on the s parameter which implies that $f = f(s)$, in our case y is a function of z and s ,that is $y = y(z,s)$.
$e^{\int Pdz}$ is called the integrating factor. We first calculate the P integral.

$$\begin{aligned}\int Pdz &= \int -\frac{s}{\lambda z(z-1)}dz \\ &= -\int \frac{s}{\lambda z(z-1)}dz\end{aligned}$$

But using partial fractions

$$\frac{1}{z\left(z-1\right)} = \frac{A}{z} + \frac{B}{z-1}$$

Multiplying both sides by $z\left(z-1\right)$ yields $1 = A\left(z-1\right) + Bz$ which holds for all values of z . Setting $z = 0$, we have

$$\begin{aligned}&1 = A\left(0-1\right) + B\left(0\right) \\ &\Rightarrow A = -1\end{aligned}$$

Setting $z = 1$, we have

$$\begin{aligned}&1 = A\left(1-1\right) + B\left(1\right) \\ &\Rightarrow B = 1\end{aligned}$$

Thus

$$\frac{1}{z\left(z-1\right)} = -\frac{1}{z} + \frac{1}{z-1}$$

With this P integral becomes

$$\begin{aligned}\int Pdz &= -\frac{s}{\lambda}\int\frac{1}{z\left(z-1\right)}\\ &= -\frac{s}{\lambda}\left[\int -\frac{1}{z}dz + \int\frac{1}{z-1}dz\right]\\ &= -\frac{s}{\lambda}\left[-\int\frac{1}{z}dz + \int\frac{1}{z-1}dz\right]\\ &= -\frac{s}{\lambda}\left[-\ln\left|z\right| + \ln\left|z-1\right|\right]\\ &= -\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|\end{aligned}$$

Thus integrating factor becomes

$$\begin{aligned}I.F &= e^{\int Pdz}\\ &= e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}\end{aligned}$$

From above the general solution of the ODE is given by

$$ye^{\int Pdz} = f + \int Qe^{\int Pdz}$$

So substituting y, P and Q we obtain

$$\overline{G}\left(z,s\right)e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) + \int -\frac{G(z,0)}{\lambda z\left(z-1\right)}e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}dz$$

$$\Rightarrow \overline{G}\left(z,s\right)e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{G(z,0)}{\lambda z\left(z-1\right)}e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}dz \qquad (5.20b)$$

We consider two cases

1. when $X(0) = 1$
2. when $X(0) = n_0$

Case 1: When initial population $X(0) = 1$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$
$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$
$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = 1$, we have

$$P_1(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq 1$$

$$\therefore G(z,0) = z$$

With this equation (5.20b) becomes

$$\overline{G}\,(z,s)\, e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{z}{\lambda z\,(z-1)} e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} dz$$

$$\overline{G}\,(z,s)\, e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{1}{\lambda\,(z-1)} e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} dz \tag{5.21}$$

Now this looks like a complicated equation to solve, but at this juncture we can licitly apply the inverse Laplace transform to both sides.

We observe that $f(s)$ can be regarded as a Laplace transform of some unknown function$f(t)$

Applying inverse Laplace transform to both sides of equation (5.21) we have from the table of Laplace transform in chapter 2 or from examples 7 and 8 (see pages 38-39)

1. e^{-cs} is the Laplace transform of the Dirac delta function $\delta(t-c)$

2. $\overline{G}(z,s)e^{cs}$ is the Laplace transform of $G(t-c)\eta(t-c)$ where η is the Heaviside step function.

In our case $c = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$, With this equation (5.21) can be rewritten as

$$\Rightarrow \overline{G}\left(z,s\right)e^{-cs} = f(s) - \int \frac{1}{\lambda\left(z-1\right)}e^{-cs}dz$$

So all together inversely transforming both sides the above equation, we come to

$$G(z,t-c)\eta(t-c) = f(t) - \int \frac{1}{\lambda\left(z-1\right)}\delta(t-c)dz$$

Substituting the value of c yields

$$G\left(z,t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t) - \int \frac{1}{\lambda\left(z-1\right)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz \tag{5.22}$$

Now this is a pretty large equation but it can be simplified, but to do that we need to make one big detour. We can free ourselves of this terrible integral by using the Dirac delta function, but what we have is delta of function of variable z , so we need to first simplify it to a common delta of variable. To do so, we shall use the following property of delta function

$$\delta\left(g\left(z\right)\right) = \sum_{z_i} \frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right) \tag{5.23}$$

Where $g'\left(z\right)$ is the first derivative of $g\left(z\right)$, z_i is a simple root of $g\left(z\right)$ such that $g'\left(z_i\right) \neq 0$.

In our case $g(z) = t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$. To obtain the roots of $g(z)$ we solve $g(z) = 0$

$$\begin{aligned}
&\Rightarrow t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right| = 0 \\
&\Rightarrow t = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right| \\
&\Rightarrow t\lambda = \ln\left|\frac{z-1}{z}\right| \\
&\Rightarrow e^{t\lambda} = \left|\frac{z-1}{z}\right| \\
&\Rightarrow e^{t\lambda} = \pm\frac{z-1}{z}
\end{aligned}$$

First root

$$\begin{aligned}
&e^{t\lambda} = \frac{z-1}{z} \\
&ze^{t\lambda} = z - 1 \\
&1 = z - ze^{t\lambda} \\
&1 = z\left(1 - e^{t\lambda}\right) \\
&z_1 = \frac{1}{1 - e^{t\lambda}}
\end{aligned}$$

Second root

$$\begin{aligned}
&e^{t\lambda} = -\frac{z-1}{z} \\
&ze^{t\lambda} = -(z - 1) \\
&1 = z + ze^{t\lambda} \\
&1 = z\left(1 + e^{t\lambda}\right) \\
&z_2 = \frac{1}{1 + e^{t\lambda}}
\end{aligned}$$

Therefore

$$z_i = \frac{1}{1 \pm e^{t\lambda}}$$

The next step is to determine

$$g'(z) = \frac{d}{dz}\left[t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right]$$
$$= -\frac{1}{\lambda}\frac{d}{dz}\ln\left|\frac{z-1}{z}\right|$$

Using the property $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$ we get

$$g'(z) = -\frac{1}{\lambda}\left(\frac{\frac{d}{dz}\left|\frac{z-1}{z}\right|}{\left|\frac{z-1}{z}\right|}\right)$$

$$\therefore g'(z) = -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \tag{5.24}$$

Now remembering that $|x| = x\,\mathrm{sgn}\,(x)$ and $\mathrm{sgn}\,(x) = 2\eta(x) - 1$, therefore

$$\left|\frac{z}{z-1}\right| = \frac{z}{z-1}\,\mathrm{sgn}\left(\frac{z}{z-1}\right) \tag{5.25}$$

Also

$$\left|\frac{z-1}{z}\right| = \frac{z-1}{z}\,\mathrm{sgn}\left(\frac{z-1}{z}\right)$$
$$= \frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]$$

Where sgn is the sign distribution and eta is the Heaviside distribution .Hence

$$\left|\frac{z-1}{z}\right|' = \frac{d}{dz}\left|\frac{z-1}{z}\right|$$
$$= \frac{d}{dz}\left\{\frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\right\} \tag{5.26}$$

At this step we need to recall that

$$\eta'_{h(z)} = h'(z)\frac{\partial \eta(z)}{\partial h}$$

Use product rule of differentiation, we simplify equation (5.26) as follows Let

$$\begin{aligned}
U &= \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow U' = \frac{1}{z^2} \\
V &= 2\eta\left(\frac{z-1}{z}\right) - 1 \\
&= 2\eta\left(1-\frac{1}{z}\right) - 1 \Rightarrow V' = \frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)
\end{aligned}$$

We now have

$$\begin{aligned}
\left|\frac{z-1}{z}\right|' &= U'V + V'U \\
&= \frac{1}{z^2}\underbrace{\left[2\eta\left(\frac{z-1}{z}\right)-1\right]}_{\operatorname{sgn}\left(\frac{z-1}{z}\right)} + \frac{z-1}{z}\left[\frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)\right] \\
&= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\eta'\left(1-\frac{1}{z}\right)
\end{aligned}$$

$$\therefore \left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(1-\frac{1}{z}\right) \tag{5.27}$$

Since $\eta'(x) = \delta(x)$
But

$$\delta(g(z)) = \sum_{z_i}\frac{1}{|g'(z_i)|}\delta(z-z_i)$$

Let

$$\begin{aligned}
g(z) &= 1 - \frac{1}{z} \\
g'(z) &= \frac{1}{z^2}
\end{aligned}$$

Solving for roots of $g(z)$, we have

$$\begin{aligned}
&g(z) = 0 \\
&\Rightarrow 1 - \frac{1}{z} = 0 \\
&1 = \frac{1}{z} \Rightarrow z = 1 \\
&g'(z) = \frac{1}{1^2} \\
&\therefore \delta\left(1 - \frac{1}{z}\right) = \delta(z-1)
\end{aligned}$$

Hence equation (5.27) becomes

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta(z-1) \tag{5.28}$$

We now substitute equations (5.25) and (5.28) in equation (5.24) we have

$$\begin{aligned}
g'(z) &= -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \\
&= -\frac{1}{\lambda}\left\{\frac{z}{z-1}\operatorname{sgn}\left(\frac{z}{z-1}\right)\left[\frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta(z-1)\right]\right\} \\
&= -\frac{1}{\lambda}\left[\frac{1}{z^2}\left(\frac{z}{z-1}\right)\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + 2\left(\frac{z}{z-1}\right)\left(\frac{z-1}{z^3}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta(z-1)\right] \\
&= -\frac{1}{\lambda}\left[\frac{1}{z(z-1)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta(z-1)\right]
\end{aligned}$$

Recalling that we obtained the roots of $g(z)$ as

$$z_1 = \frac{1}{1+e^{t\lambda}}, z_2 = \frac{1}{1-e^{t\lambda}}$$

And the Dirac delta function is zero everywhere except when its argument is zero.

$$\arg\delta(z-1) = z-1$$

For both roots, the delta term in $g'(z)$ vanishes because for it to exist it requires that argument of delta function must be zero. That is

$$0 = z_i - 1 = \frac{1}{1 \pm e^{t\lambda}} - 1$$

which is equivalent to $e^{t\lambda} = 0$ and that is never true in Mathematics since both λ and t are non negative, so only one term remains after neglecting delta in $g'(z_i)$. Therefore

$$g'(z_1) = -\frac{1}{\lambda}\left[\frac{1}{z(z-1)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta(z-1)\right]$$

But

$$\operatorname{sgn}(x)\operatorname{sgn}(y) = \operatorname{sgn}(xy)$$

$$\begin{aligned}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) &= \operatorname{sgn}\left[\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right)\right] \\ &= \operatorname{sgn}(1) \\ &= 1\end{aligned}$$

This implies that

$$\begin{aligned}g'(z_1) &= -\frac{1}{\lambda}\left[\frac{1}{z(z-1)}\right] \\ g'(z_1) &= -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1+e^{\lambda t}}\left(\frac{1}{1+e^{\lambda t}} - 1\right)}\right] \\ &= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1}{1+e^{\lambda t}} - 1}\right] \\ &= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1-\left(1+e^{\lambda t}\right)}{1+e^{\lambda t}}}\right] \\ &= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{1-\left(1+e^{\lambda t}\right)}\right] \\ &= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{-e^{\lambda t}}\right] \\ &= \frac{\left(1+e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}\end{aligned}$$

$$
\begin{aligned}
g'(z_2) &= -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1-e^{\lambda t}}\left(\frac{1}{1-e^{\lambda t}}-1\right)}\right] \\
&= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1}{1-e^{\lambda t}}-1}\right] \\
&= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1-\left(1-e^{\lambda t}\right)}{1-e^{\lambda t}}}\right] \\
&= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{1-\left(1-e^{\lambda t}\right)}\right] \\
&= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{e^{\lambda t}}\right] \\
&= -\frac{\left(1-e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}
\end{aligned}
$$

Therefore

$$
g'(z_i) = \pm\left(1 \pm e^{\lambda t}\right)^2 \frac{1}{\lambda e^{\lambda t}}
$$

But $g'(z_i)$ must not be zero, and for the initial condition that is at $t = 0$ the negative root makes $g'(z_i)$ to be singular $[g'(z_i) = 0]$ so that root needs to be discarded in the summation only one root of z_i is left and will be from now on $z_i = z_1$

It is now time to go back to equation (5.23)

$$
\begin{aligned}
\delta(g(z)) &= \sum_{z_i} \frac{1}{|g'(z_i)|}\delta(z - z_i) \\
&= \frac{1}{|g'(z_1)|}\delta(z - z_1) \\
&= \frac{1}{\left|(1+e^{\lambda t})^2 \frac{1}{\lambda e^{\lambda t}}\right|}\delta\left(z - \frac{1}{1+e^{\lambda t}}\right) \\
&= \frac{\lambda e^{\lambda t}}{(1+e^{\lambda t})^2}\delta\left[z - \left(1+e^{\lambda t}\right)^{-1}\right]
\end{aligned}
$$

And this is the formula we searched all this time. We can now use the delta function to eliminate the integral in equation (5.22) In doing so we need to recall that

$$\int f(x)\delta(x-a)dx = f(a)$$

That is the integral of a function multiplied by a Dirac delta shifted by a units is the value of that function at a . With this we now have

$$-\int \frac{1}{\lambda(z-1)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz = \int \frac{1}{\lambda(z-1)}\frac{\lambda e^{\lambda t}}{(1+e^{\lambda t})^2}\delta\left[z-\left(1+e^{\lambda t}\right)^{-1}\right]dz$$

Integral disappears and instead we get value of integrand for $z = \frac{1}{1+e^{t\lambda}}$

$$= \frac{1}{\lambda(z-1)}\frac{\lambda e^{\lambda t}}{(1+e^{\lambda t})^2}\Big|_{z=\frac{1}{1+e^{t\lambda}}}$$

$$= \frac{1}{\lambda\left(\frac{1}{1+e^{t\lambda}}-1\right)}\frac{\lambda e^{\lambda t}}{(1+e^{\lambda t})^2}$$

$$= \frac{e^{\lambda t}}{(1+e^{\lambda t})^2\left(\frac{1}{1+e^{t\lambda}}-1\right)}$$

$$= \frac{e^{\lambda t}}{(1+e^{\lambda t})^2\left[\frac{1-\left(1+e^{t\lambda}\right)}{1+e^{t\lambda}}\right]}$$

$$= \frac{\left(1+e^{\lambda t}\right)e^{\lambda t}}{(1+e^{\lambda t})^2\left[1-(1+e^{\lambda t})\right]}$$

$$= \frac{\left(1+e^{\lambda t}\right)e^{\lambda t}}{(1+e^{\lambda t})^2\left[1-1-e^{\lambda t}\right]}$$

$$= \frac{e^{\lambda t}}{(1+e^{\lambda t})\left[-e^{\lambda t}\right]}$$
$$= -\frac{1}{(1+e^{\lambda t})}$$
$$= -\left(1+e^{\lambda t}\right)^{-1}$$

$$\therefore \int \frac{1}{\lambda(z-1)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz = -\left(1+e^{t\lambda}\right)^{-1}$$

But this is not the end of our trouble, with the integral solved, our problem as in equation (5.22) is reduced to

$$G\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t)+\left(1+e^{t\lambda}\right)^{-1}$$

There is no harm if we make another substitution. Thus the final step towards the solution is to change the variable from t to

$$T = t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$$

Multiplying both sides by λ yields

$$\lambda T = \lambda t - \ln\left|\frac{z-1}{z}\right|$$
$$\lambda t = \lambda T + \ln\left|\frac{z-1}{z}\right|$$

Exponentiating this we get

$$e^{\lambda t} = e^{\lambda T+\ln\left|\frac{z-1}{z}\right|}$$
$$= e^{\lambda T}e^{\ln\left|\frac{z-1}{z}\right|}$$
$$= e^{\lambda T}\left|\frac{z-1}{z}\right|$$

Remembering that for our root

$$z = \frac{1}{1+e^{t\lambda}}$$

$$\begin{aligned}\Rightarrow \frac{z-1}{z} &= 1 - \frac{1}{z} \\ &= 1 - \frac{1}{\left[\frac{1}{1+e^{t\lambda}}\right]} \\ &= 1 - \left[1+e^{t\lambda}\right] \\ &= -e^{t\lambda}\end{aligned}$$

$$\therefore \left|-e^{t\lambda}\right| = -\left(-e^{t\lambda}\right)$$

$$\Rightarrow \left|\frac{z-1}{z}\right| = -\frac{z-1}{z}$$

$$\begin{aligned}\therefore e^{t\lambda} &= e^{T\lambda}\left|\frac{z-1}{z}\right| \\ &= e^{T\lambda}\left[-\frac{z-1}{z}\right]\end{aligned}$$

$$\Rightarrow 1+e^{t\lambda} = 1 - e^{T\lambda}\left[\frac{z-1}{z}\right]$$

Multiplying the last time in the right hand side by $1 = \frac{e^{-\lambda T}}{e^{-\lambda T}}$ yields

$$1+e^{t\lambda} = 1 - e^{T\lambda}\left[\frac{z-1}{z}\right]\left(\frac{e^{-\lambda T}}{e^{-\lambda T}}\right)$$

$$= 1 - \left[\frac{z-1}{ze^{-\lambda T}}\right]$$

$$= \frac{ze^{-\lambda T} - (z-1)}{ze^{-\lambda T}}$$

$$= \frac{ze^{-\lambda T} - z + 1}{ze^{-\lambda T}}$$

$$= \frac{1 - z\left[1 - e^{-\lambda T}\right]}{ze^{-\lambda T}}$$

$$\therefore 1 + e^{t\lambda} = \frac{1 - z\left[1 - e^{-\lambda T}\right]}{ze^{-\lambda T}}$$

Described by T variable equation for G becomes

$$\begin{aligned} G(z,T)\,\eta(T) &= f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left(1 + e^{t\lambda}\right)^{-1} \\ &= f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1} \end{aligned}$$

But we know that f does not depend on z

$$G(z,T)\,\eta(T) = f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1}$$

When $T = 0$ it follows that

$$G(z,0)\,\eta(0) = f\left(0 + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z\left(1 - e^{-\lambda(0)}\right)}{ze^{-\lambda(0)}}\right]^{-1}$$

But $\eta(0) = 1$ hence

$$G(z,0) = f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1-z(1-1)}{z(1)}\right]^{-1}$$
$$= f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1}{z}\right]^{-1}$$
$$= f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + z$$

But knowing that

$$z = G(z,0) = f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + z$$

It follows that

$$f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = 0$$

So we have finally eliminated f. Thus the solution is

$$G(z,T)\,\eta(T) = \left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1}$$

Since $T \geqslant 0$ it follows by definition of tau that $\eta(T) = 1$

Thus we have

$$G(z,T) = \left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1}$$
$$= \frac{ze^{-\lambda T}}{1-z\left(1-e^{-\lambda T}\right)}$$

We can rewrite it in terms of t as

$$G(z,t) = \frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}$$

Which by identification is the pgf of a shifted geometric distribution with parameters $p = e^{-\lambda t}, q = 1 - e^{-\lambda t}$

The next step is to invert the pgf so that we may obtain $P_n(t)$. $P_n(t)$ is the coefficient of z^n in the expansion of $G(z,t)$ .But

$$G(z,t) = \frac{ze^{-\lambda t}}{1 - z(1 - e^{-\lambda t})}$$

$$= ze^{-\lambda t}\left[1 - z\left(1 - e^{-\lambda t}\right)\right]^{-1}$$

$$= ze^{-\lambda t}\sum_{n=0}^{\infty}\left[z\left(1 - e^{-\lambda t}\right)\right]^{n}$$

$$= e^{-\lambda t}\sum_{n=0}^{\infty}\left(1 - e^{-\lambda t}\right)^{n} z^{n+1}$$

$$= e^{-\lambda t}\sum_{n=1}^{\infty}\left(1 - e^{-\lambda t}\right)^{n-1} z^{n}$$

$$= \sum_{n=1}^{\infty} e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1} z^{n}$$

Therefore the coefficient of z^n is $e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$

Hence

$$P_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1} \qquad n = 1, 2, 3, ...$$

Which is the pmf of a shifted geometric distribution

Case 2: When initial population $X(0) = n_0$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$
$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$
$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = n_0$, we have

$$P_{n_0}(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq n_0$$

$$\therefore G(z,0) = z^{n_0}$$

With this equation (5.20b) becomes

$$\overline{G}\,(z,s)\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{z^{n_0}}{\lambda z\,(z-1)} e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} dz \tag{5.29}$$

The next step is to appy inverse Laplace transform to both sides of equation (5.29).

We observe that $f(s)$ can be regarded as a Laplace transform of some unknown function$f(t)$

Applying inverse Laplace transform to both sides of equation (5.29) we get from the table of Laplace transform in chapter 2 or from examples 7 and 8 (see pages 38-39)

1. e^{-cs} is the Laplace transform of the Dirac delta function $\delta(t-c)$
2. $\overline{G}(z,s)e^{cs}$ is the Laplace transform of $G(t-c)\eta(t-c)$ where η is the Heaviside step function. In our case $c = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$

With this equation (5.29) can be rewritten as

$$\Rightarrow \overline{G}\,(z,s)\,e^{-cs} = f(s) - \int \frac{z^{n_0}}{\lambda z\,(z-1)} e^{-cs} dz$$

So all together inversely transforming both sides the above equation, we come to

$$G(z,t-c)\eta(t-c) = f(t) - \int \frac{z^{n_0}}{\lambda z\,(z-1)}\delta(t-c)dz$$

Substituting the value of we obtain

$$G\left(z, t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t) - \int \frac{z^{n_0}}{\lambda z\,(z-1)}\delta\left(t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz \tag{5.30}$$

Now this equation looks complicated but it can be simplified with the help of the Dirac delta function. See that what we have is delta of function of variable z , so there is need to first simplify it to a common delta of variable. To do so, we make use of the following property of delta function

$$\delta\left(g\left(z\right)\right) = \sum_{z_i} \frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z - z_i\right) \tag{5.31}$$

Where $g'\left(z\right)$ is the first derivative of $g\left(z\right)$, z_i is a simple root of $g\left(z\right)$ such that $g'\left(z_i\right)$. In our case $g\left(z\right) = t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$. To obtain the roots of $g(z)$ we solve $g\left(z\right) = 0$

$$\begin{aligned}
&\Rightarrow t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right| = 0 \\
&\Rightarrow t = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right| \\
&\Rightarrow t\lambda = \ln\left|\frac{z-1}{z}\right| \\
&\Rightarrow e^{t\lambda} = \left|\frac{z-1}{z}\right| \\
&\Rightarrow e^{t\lambda} = \pm\frac{z-1}{z}
\end{aligned}$$

First root

$$\begin{aligned}
e^{t\lambda} &= \frac{z-1}{z} \\
ze^{t\lambda} &= z-1 \\
1 &= z - ze^{t\lambda} \\
1 &= z\left(1-e^{t\lambda}\right) \\
z_1 &= \frac{1}{1-e^{t\lambda}}
\end{aligned}$$

Second root

$$\begin{aligned}
e^{t\lambda} &= -\frac{z-1}{z} \\
ze^{t\lambda} &= -(z-1) \\
1 &= z + ze^{t\lambda} \\
1 &= z\left(1+e^{t\lambda}\right) \\
z_2 &= \frac{1}{1+e^{t\lambda}}
\end{aligned}$$

Therefore

$$z_i = \frac{1}{1\pm e^{t\lambda}}$$

The next step is to determine

$$\begin{aligned}
g'(z) &= \frac{d}{dz}\left[t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right] \\
&= -\frac{1}{\lambda}\frac{d}{dz}\ln\left|\frac{z-1}{z}\right|
\end{aligned}$$

Using the property $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

$$\begin{aligned}
g'(z) &= -\frac{1}{\lambda}\left(\frac{\frac{d}{dz}\left|\frac{z-1}{z}\right|}{\left|\frac{z-1}{z}\right|}\right) \\
&= -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right)
\end{aligned} \tag{5.32}$$

Now remembering that $|x| = x\,\mathrm{sgn}\,(x)$ and $\mathrm{sgn}\,(x) = 2\eta(x) - 1$, therefore

$$\left|\frac{z}{z-1}\right| = \frac{z}{z-1}\,\mathrm{sgn}\left(\frac{z}{z-1}\right) \tag{5.33}$$

Also

$$\begin{aligned}\left|\frac{z-1}{z}\right| &= \frac{z-1}{z}\,\mathrm{sgn}\left(\frac{z-1}{z}\right)\\ &= \frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\end{aligned}$$

Where sgn is the sign distribution and eta is the Heaviside distribution .Hence

$$\begin{aligned}\left|\frac{z-1}{z}\right|' &= \frac{d}{dz}\left|\frac{z-1}{z}\right|\\ &= \frac{d}{dz}\left\{\frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\right\}\end{aligned} \tag{5.34}$$

At this step we need to recall that

$$\eta'_{h(z)} = h'(z)\frac{\partial\eta(z)}{\partial h}$$

Use product rule of differentiation, we simplify equation (5.34) as follows Let

$$\begin{aligned}U &= \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow U' = \frac{1}{z^2}\\ V &= 2\eta\left(\frac{z-1}{z}\right) - 1\\ &= 2\eta\left(1 - \frac{1}{z}\right) - 1 \Rightarrow V' = \frac{2}{z^2}\eta'\left(1 - \frac{1}{z}\right)\end{aligned}$$

We now have

$$\begin{aligned}\left|\frac{z-1}{z}\right|' &= U'V + V'U \\ &= \frac{1}{z^2}\underbrace{\left[2\eta\left(\frac{z-1}{z}\right)-1\right]}_{\operatorname{sgn}\left(\frac{z-1}{z}\right)} + \frac{z-1}{z}\left[\frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)\right] \\ &= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\eta'\left(1-\frac{1}{z}\right)\end{aligned}$$

$$\therefore \left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(1-\frac{1}{z}\right) \tag{5.35}$$

Since $\eta'(x) = \delta(x)$

But

$$\delta(g(z)) = \sum_{z_i}\frac{1}{|g'(z_i)|}\delta(z - z_i)$$

Let

$$\begin{aligned} g(z) &= 1 - \frac{1}{z} \\ g'(z) &= \frac{1}{z^2}\end{aligned}$$

Solving for roots of $g(z)$, we have

$$\begin{aligned} & g(z) = 0 \\ & \Rightarrow 1 - \frac{1}{z} = 0 \\ & 1 = \frac{1}{z} \Rightarrow z = 1 \\ & g'(z) = \frac{1}{1^2} \\ & \therefore \delta\left(1-\frac{1}{z}\right) = \delta(z-1)\end{aligned}$$

Hence equation (5.35) becomes

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(z-1\right) \tag{5.36}$$

We now substitute equations (5.33) and (5.36) in equation (5.32) we have

$$\begin{aligned} g'(z) &= -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \\ &= -\frac{1}{\lambda}\left\{\frac{z}{z-1}\operatorname{sgn}\left(\frac{z}{z-1}\right)\left[\frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(z-1\right)\right]\right\} \\ &= -\frac{1}{\lambda}\left[\frac{1}{z^2}\left(\frac{z}{z-1}\right)\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + 2\left(\frac{z}{z-1}\right)\left(\frac{z-1}{z^3}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right] \\ &= -\frac{1}{\lambda}\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right] \end{aligned}$$

Recalling that we obtained the roots of $g(z)$ as

$$z_1 = \frac{1}{1+e^{t\lambda}}, z_2 = \frac{1}{1-e^{t\lambda}}$$

And the Dirac delta function is zero everywhere except when its argument is zero.

$$\arg\delta\left(z-1\right) = z-1$$

For both roots, the delta term in

$$g'(z)$$

vanishes because for it to exist it requires that argument of delta function must be zero. That is

$$0 = z_i - 1 = \frac{1}{1 \pm e^{t\lambda}} - 1$$

which is equivalent to $e^{t\lambda} = 0$

which can't be true since both λ and t are greater than zero. Therefore only one term remains after neglecting delta in $g'(z_i)$. Therefore

$$g'(z_1) = -\frac{1}{\lambda}\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

But

$$\operatorname{sgn}(x)\operatorname{sgn}(y) = \operatorname{sgn}(xy)$$

$$\begin{aligned}
\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) &= \operatorname{sgn}\left[\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right)\right] \\
&= \operatorname{sgn}(1) \\
&= 1
\end{aligned}$$

This implies that

$$\begin{aligned}
g'(z_1) &= -\frac{1}{\lambda}\left[\frac{1}{z_1(z_1-1)}\right] \\
\\
&= -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1+e^{\lambda t}}\left(\frac{1}{1+e^{\lambda t}}-1\right)}\right] \\
\\
&= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1}{1+e^{\lambda t}}-1}\right] \\
\\
&= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1-\left(1+e^{\lambda t}\right)}{1+e^{\lambda t}}}\right] \\
\\
&= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{1-\left(1+e^{\lambda t}\right)}\right] \\
\\
&= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{-e^{\lambda t}}\right] \\
\\
&= \frac{\left(1+e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}
\end{aligned}$$

$$g'(z_2) = -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1-e^{\lambda t}}\left(\frac{1}{1-e^{\lambda t}}-1\right)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1}{1-e^{\lambda t}}-1}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1-\left(1-e^{\lambda t}\right)}{1-e^{\lambda t}}}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{1-(1-e^{\lambda t})}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{e^{\lambda t}}\right]$$

$$= -\frac{\left(1-e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}$$

Therefore

$$g'(z_i) = \pm\left(1 \pm e^{\lambda t}\right)^2 \frac{1}{\lambda e^{\lambda t}}$$

But $g'(z_i)$ must not be zero, and for the initial condition that is at $t = 0$ the negative root makes $g'(z_i)$ to be singular $[g'(z_i) = 0]$ so that root needs to be discarded in the summation only one root of z_i is left and will be from now on $z_i = z_1$

It is now time to go back to equation (5.31)

$$\delta\left(g\left(z\right)\right)=\sum_{z_i}\frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right)$$

$$=\frac{1}{\left|g'\left(z_1\right)\right|}\delta\left(z-z_1\right)$$

$$=\frac{1}{\left|\left(1+e^{\lambda t}\right)^2\frac{1}{\lambda e^{\lambda t}}\right|}\delta\left(z-\frac{1}{1+e^{\lambda t}}\right)$$

$$=\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\delta\left[z-\left(1+e^{\lambda t}\right)^{-1}\right]$$

We now use the delta function to eliminate the integral in equation (5.30) In doing so we need to recall that

$$\int f(x)\delta\left(x-a\right)dx=f(a)$$

That is the integral of a function multiplied by a Dirac delta shifted by a units is the value of that function at a . With this we now have

$$\int\frac{z^{n_0}}{\lambda z\left(z-1\right)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz=\int\frac{z^{n_0}}{\lambda z\left(z-1\right)}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\delta\left[z-\left(1+e^{\lambda t}\right)^{-1}\right]dz$$

Integral disappears and instead we get value of integrand for $z=\frac{1}{1+e^{t\lambda}}$

Thus

$$\int\frac{z^{n_0}}{\lambda z\left(z-1\right)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz=\frac{z^{n_0}}{\lambda z\left(z-1\right)}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\Big|_{z=\frac{1}{1+e^{t\lambda}}}$$

$$=\frac{\left(\frac{1}{1+e^{t\lambda}}\right)^{n_0}}{\lambda\left(\frac{1}{1+e^{t\lambda}}\right)\left(\frac{1}{1+e^{t\lambda}}-1\right)}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}$$

$$= \frac{\left(1+e^{t\lambda}\right)^{-n_0}\left(1+e^{\lambda t}\right)e^{\lambda t}}{(1+e^{\lambda t})^2\left(\frac{1}{1+e^{t\lambda}}-1\right)}$$

$$= \frac{\left(1+e^{t\lambda}\right)^{-n_0}e^{\lambda t}}{(1+e^{\lambda t})\left(\frac{1}{1+e^{t\lambda}}-1\right)}$$

$$= \frac{\left(1+e^{t\lambda}\right)^{-n_0}e^{\lambda t}}{1-(1+e^{\lambda t})}$$

$$= \frac{\left(1+e^{t\lambda}\right)^{-n_0}e^{\lambda t}}{-e^{\lambda t}}$$

$$= -\left(1+e^{t\lambda}\right)^{-n_0}$$

$$\therefore -\int \frac{z^{n_0}}{\lambda z\,(z-1)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz = \left(1+e^{t\lambda}\right)^{-n_0}$$

But this is not the end of our trouble, with the integral solved, our problem as in equation (5.30) is reduced to

$$G\left(z, t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t)+\left(1+e^{t\lambda}\right)^{-n_0}$$

Making another substitution. We let $T = t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$. Multiplying both sides by λ yields

$$\lambda T = \lambda t - \ln\left|\frac{z-1}{z}\right|$$
$$\lambda t = \lambda T + \ln\left|\frac{z-1}{z}\right|$$

Exponentiating this we get

$$\begin{aligned} e^{\lambda t} &= e^{\lambda T + \ln\left|\frac{z-1}{z}\right|} \\ &= e^{\lambda T} e^{\ln\left|\frac{z-1}{z}\right|} \\ &= e^{\lambda T} \left|\frac{z-1}{z}\right| \end{aligned}$$

Remembering that for our root

$$\begin{aligned} z &= \frac{1}{1+e^{t\lambda}} \\ \Rightarrow \frac{z-1}{z} &= 1 - \frac{1}{z} \\ &= 1 - \frac{1}{\left[\frac{1}{1+e^{t\lambda}}\right]} \\ &= 1 - \left[1 + e^{t\lambda}\right] \\ &= -e^{t\lambda} \\ \therefore \left|-e^{t\lambda}\right| &= -\left(-e^{t\lambda}\right) \\ \Rightarrow \left|\frac{z-1}{z}\right| &= -\frac{z-1}{z} \end{aligned}$$

$$\begin{aligned} \therefore e^{t\lambda} &= e^{T\lambda} \left|\frac{z-1}{z}\right| \\ &= e^{T\lambda} \left[-\frac{z-1}{z}\right] \\ \Rightarrow 1 + e^{t\lambda} &= 1 - e^{T\lambda} \left[\frac{z-1}{z}\right] \end{aligned}$$

Multiplying the right hand side by $1 = \frac{e^{-\lambda T}}{e^{-\lambda T}}$ we have

$$\begin{aligned} 1 + e^{t\lambda} &= 1 - e^{T\lambda} \left[\frac{z-1}{z}\right] \left(\frac{e^{-\lambda T}}{e^{-\lambda T}}\right) \\ &= 1 - \left[\frac{z-1}{ze^{-\lambda T}}\right] \\ &= \frac{ze^{-\lambda T} - (z-1)}{ze^{-\lambda T}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{ze^{-\lambda T} - z + 1}{ze^{-\lambda T}} \\
&= \frac{1 - z\left[1 - e^{-\lambda T}\right]}{ze^{-\lambda T}} \\
\therefore 1 + e^{t\lambda} &= \frac{1 - z\left[1 - e^{-\lambda T}\right]}{ze^{-\lambda T}}
\end{aligned}
$$

Described by$T$ variable equation for G becomes

$$
\begin{aligned}
G(z,T)\,\eta(T) &= f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left(1 + e^{t\lambda}\right)^{-n_0} \\
&= f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0}
\end{aligned}
$$

But we know that f does not depend on z

$$
G(z,T)\,\eta(T) = f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0}
$$

When $T = 0$ it follows that

$$
G(z,0)\,\eta(0) = f\left(0 + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z\left(1 - e^{-\lambda(0)}\right)}{ze^{-\lambda(0)}}\right]^{-n_0}
$$

But $\eta(0) = 1$ hence

$$
\begin{aligned}
G(z,0) &= f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1 - z(1-1)}{z(1)}\right]^{-n_0} \\
&= f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + \left[\frac{1}{z}\right]^{-n_0} \\
&= f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + z^{n_0}
\end{aligned}
$$

But knowing that

$$
z^{n_0} = G(z,0) = f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + z^{n_0}
$$

It follows that

$$f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)=0$$

So we have finally eliminated f. Thus the solution is

$$G\left(z,T\right)\eta\left(T\right)=\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0}$$

Since $T\geqslant 0$ it follows by definition of tau that $\eta\left(T\right)=1$

Hence

$$\begin{aligned}G\left(z,T\right)&=\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0}\\&=\left[\frac{ze^{-\lambda T}}{1-z\left(1-e^{-\lambda T}\right)}\right]^{n_0}\end{aligned}$$

We can rewrite it as

$$G\left(z,t\right)=\left[\frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0}$$

Which is the pgf of a negative binomial distribution with parameters $r=n_0$, and $p=e^{-\lambda t}$
We now obtain $P_n(t)$ as follows;

$P_n\left(t\right)$ is the coefficient of z^n in $G\left(z,t\right)$

But $G\left(z,t\right)$ is of the form

$$G\left(z,t\right)=\left(\frac{zp}{1-zq}\right)^{n_0}$$

$$=\left(zp\right)^{n_0}\left(1-zq\right)^{-n_0}$$

$$= (zp)^{n_0} \sum_{k=0}^{\infty} \binom{-n_0}{k} (-zq)^k$$

$$= (zp)^{n_0} \sum_{k=0}^{\infty} \binom{-n_0}{k} (-1)^k (zq)^k$$

$$= (zp)^{n_0} \sum_{k=0}^{\infty} \binom{n_0 + k - 1}{k} (zq)^k$$

$$= p^{n_0} \sum_{k=0}^{\infty} \binom{n_0 + k - 1}{k} q^k z^{n_0+k}$$

Letting $n = n_0 + k$ implies

$$G(z,t) = p^{n_0} \sum_{n=n_0}^{\infty} \binom{n-1}{n-n_0} q^{n-n_0} z^n$$

Thus the coefficient of z^n is

$$p^{n_0} \binom{n-1}{n-n_0} q^{n-n_0}$$

But $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$ implying that

$$P_n(t) = \binom{n-1}{n-n_0} \left(e^{-\lambda t}\right)^{n_0} \left(1 - e^{-\lambda t}\right)^{n-n_0} \qquad n = n_0, n_0+1, n_0+2, ...$$

or

$$P_n(t) = \binom{n_0 + k - 1}{k} \left(e^{-\lambda t}\right)^{n_0} \left(1 - e^{-\lambda t}\right)^{k} \qquad k = 0 \ \ ,1 \ \ 2, ...$$

Which is the pmf of a negative binomial distribution

Hyper geometric function approach

We had from equation (5.20a)

$$\frac{d}{dz}\overline{G}(z,s) - \frac{s}{\lambda z(z-1)}\overline{G}(z,s) = -\frac{G(z,0)}{\lambda z(z-1)}$$

This can be rewritten as

$$\frac{d}{dz}\overline{G}(z,s) + \frac{s}{\lambda z(1-z)}\overline{G}(z,s) = \frac{G(z,0)}{\lambda z(1-z)} \tag{5.37a}$$

which is an ODE of first order.

Using the Integrating Factor technique it follows that

$$I.F = e^{\int \frac{s}{\lambda z(1-z)}dz} = e^{\frac{s}{\lambda}\int \frac{1}{z(1-z)}}$$

By Partial Fractions, We have

$$\frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$

Multiplying both sides by $z(1-z)$ yields

$$1 = A(1-z) + Bz$$

Which holds true for all values of s Setting $s=0$ we have

$$1 = A(1-0) + B(0) \Rightarrow A = 1$$

Setting $s=1$ We have

$$1 = A(1-1) + B(1) \Rightarrow B = 1$$

Thus

$$\int \frac{1}{z\,(1-z)}dz = \int \frac{1}{z}dz + \int \frac{1}{1-z}dz$$
$$= \ln z - \ln\,(1-z)$$
$$= \ln\left(\frac{z}{1-z}\right)$$

$$\therefore I.F = e^{\frac{s}{\lambda}\int \frac{z}{1-z}dz}$$
$$= e^{\frac{s}{\lambda}\ln\left(\frac{z}{1-z}\right)}$$
$$= e^{\ln\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}}$$
$$= \left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}$$

Multiplying both sides of equation (5.37) by the integrating factor yields

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \frac{d}{dz}\overline{G}\,(z,s) + \left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \frac{s}{\lambda z\,(1-z)}\overline{G}\,(z,s) = \left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \frac{G(z,0)}{\lambda z\,(1-z)}$$

$$\frac{d}{dz}\left[\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\,(z,s)\right] = \frac{z^{\frac{s}{\lambda}}G(z,0)}{\lambda z\,(1-z)^{\frac{s}{\lambda}+1}}$$

Integrating both sides with respect to z we have

$$\int \frac{d}{dz}\left[\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\,(z,s)\right]dz = \int \frac{z^{\frac{s}{\lambda}}G(z,0)}{\lambda z\,(1-z)^{\frac{s}{\lambda}+1}}dz$$

$$\int d\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\,(z,s) = \frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}}G(z,0)}{z(1-z)^{\frac{s}{\lambda}+1}}dz$$

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\,(z,s) = \frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}}G(z,0)}{z(1-z)^{\frac{s}{\lambda}+1}}dz \qquad (5.37b)$$

We consider two cases

1. when $X(0) = 1$
2. when $X(0) = n_0$

Case 1: When initial population $X(0) = 1$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$

$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$

$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = 1$, we have

$$P_1(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq 1$$

$$\therefore G(z,0) = z$$

With this equation (5.37b) becomes

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\,(z,s) = \frac{1}{\lambda} \int \frac{z^{\frac{s}{\lambda}} z}{z(1-z)^{\frac{s}{\lambda}+1}} dz$$

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\,(z,s) = \frac{1}{\lambda} \int \frac{z^{\frac{s}{\lambda}}}{(1-z)^{\frac{s}{\lambda}+1}} dz \tag{5.38}$$

We first simplify the right hand side as follows Recall that

$$\int \frac{x^a}{(1-x)^{a+1}} dx = \frac{x^{a+1}\ {}_2F_1\left(a+1, a+1; a+2; x\right)}{a+1} + \text{constant}$$

Where ${}_2F_1\left(a+1,a+1;a+2;x\right)$ is the gauss hyper geometric function

Thus

$$\frac{1}{\lambda}\int\frac{z^{\frac{s}{\lambda}}}{(1-z)^{\frac{s}{\lambda}+1}}dz=\frac{1}{\lambda}\left[\frac{z^{\frac{s}{\lambda}+1}\ {}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+1;\frac{s}{\lambda}+2;z\right)}{\frac{s}{\lambda}+1}\right]+\frac{c_1}{\lambda}$$

Where c_1 is a constant of integration. Using this in equation (5.38) We get

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}\overline{G}\left(z,s\right)=\frac{1}{\lambda}\left[\frac{z^{\frac{s}{\lambda}+1}\ {}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+1;\frac{s}{\lambda}+2;z\right)}{\frac{s}{\lambda}+1}\right]+\frac{c_1}{\lambda}$$

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}\overline{G}\left(z,s\right)=\left[\frac{z^{\frac{s}{\lambda}+1}\ {}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+1;\frac{s}{\lambda}+2;z\right)}{\lambda\left(\frac{s}{\lambda}+1\right)}\right]+\frac{c_1}{\lambda}\tag{5.39}$$

But

$${}_2F_1\left(a,b;c;z\right)=(1-z)^{c-a-b}\ {}_2F_1\left(c-a,c-b;c;z\right)$$

We now use this property to simplify the term

$${}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+1;\frac{s}{\lambda}+2;z\right)$$

Here

$$a=\frac{s}{\lambda}+1,\qquad b=\frac{s}{\lambda}+1,\qquad c=\frac{s}{\lambda}+2$$

$$
\begin{aligned}
c-a &= \frac{s}{\lambda}+2-\left(\frac{s}{\lambda}+1\right)\\
&= \frac{s}{\lambda}+2-\frac{s}{\lambda}-1=1
\end{aligned}
$$

$$
\begin{aligned}
c-b &= \frac{s}{\lambda}+2-\left(\frac{s}{\lambda}+1\right)\\
&= \frac{s}{\lambda}+2-\frac{s}{\lambda}-1=1
\end{aligned}
$$

$$
\begin{aligned}
c-a-b &= \frac{s}{\lambda}+2-\left(\frac{s}{\lambda}+1\right)-\left(\frac{s}{\lambda}+1\right)\\
&= \frac{s}{\lambda}+2-\frac{s}{\lambda}-1-\frac{s}{\lambda}-1\\
&= -\frac{s}{\lambda}
\end{aligned}
$$

Thus

$$
{}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+1;\frac{s}{\lambda}+2;z\right)=(1-z)^{-\frac{s}{\lambda}}\ {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right)
$$

With this equation (5.39) becomes

$$
\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}\overline{G}\left(z,s\right)=\left[\frac{z^{\frac{s}{\lambda}+1}\left(1-z\right)^{-\frac{s}{\lambda}}\ {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right)}{\lambda\left(\frac{s}{\lambda}+1\right)}\right]+\frac{c_1}{\lambda}
$$

$$
=\frac{z^{\frac{s}{\lambda}+1}\left(1-z\right)^{-\frac{s}{\lambda}}}{\lambda\left(\frac{s}{\lambda}+1\right)}\ {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right)+\frac{c_1}{\lambda}
$$

$$\therefore \overline{G}(z,s) = \frac{1}{\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}} \frac{z^{\frac{s}{\lambda}+1}(1-z)^{-\frac{s}{\lambda}}}{\lambda\left(\frac{s}{\lambda}+1\right)} \;\; {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right) + \frac{1}{\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}}\left(\frac{c_1}{\lambda}\right)$$

$$= \left(\frac{1-z}{z}\right)^{\frac{s}{\lambda}} \frac{z^{\frac{s}{\lambda}+1}(1-z)^{-\frac{s}{\lambda}}}{\lambda\left(\frac{s}{\lambda}+1\right)} \;\; {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right) + \frac{c_1}{\lambda}\left(\frac{1-z}{z}\right)^{\frac{s}{\lambda}}$$

$$= \frac{z}{\lambda\left(\frac{s}{\lambda}+1\right)} \;\; {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right) + \frac{c_1}{\lambda}\left(\frac{1-z}{z}\right)^{\frac{s}{\lambda}}$$

But since for all t, $z \leqslant 1$; $G(z,t) \leqslant 1$, It follows that $c_1 = 0$

Thus

$$\overline{G}(z,s) = \frac{z}{\lambda\left(\frac{s}{\lambda}+1\right)} \;\; {}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right) \tag{5.40}$$

But according to Euler the Gauss hyper geometric series can be expressed as

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

$$= 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + ...$$

Where a, b and c are complex numbers and

$$(a)_k = \prod_{i=0}^{k-i} (a+i) = a(a+1)(a+2)(a+3)...(a+k-1)$$

Thus Letting $a = 1, b = 1$ and $c = \frac{s}{\lambda} + 2$ implies

$$F\left(1,1;\frac{s}{\lambda}+2;z\right)=F\left(a,b;c;z\right)=\left\{\begin{array}{c} 1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+ \\ \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+... \end{array}\right\}$$

$$=1+\frac{z}{\left(\frac{s}{\lambda}+2\right)}+\frac{2!2!}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)}\frac{z^2}{2!}+\frac{3!3!}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)\left(\frac{s}{\lambda}+4\right)}\frac{z^2}{3!}+...$$

$$=1+\frac{z}{\left(\frac{s}{\lambda}+2\right)}+\frac{2!z^2}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)}+\frac{3!z^3}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)\left(\frac{s}{\lambda}+4\right)}+...$$

With this equation (5.40) becomes

$$\overline{G}\left(z,s\right)=\frac{z}{\lambda\left(\frac{s}{\lambda}+1\right)}\,{}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right)$$

$$=\frac{z}{\lambda\left(\frac{s}{\lambda}+1\right)}\left\{1+\frac{z}{\left(\frac{s}{\lambda}+2\right)}+\frac{2!z^2}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)}+\frac{3!z^3}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)\left(\frac{s}{\lambda}+4\right)}+...\right\}$$

$$=\frac{z}{s+\lambda}\left\{1+\frac{z}{\left(\frac{s}{\lambda}+2\right)}+\frac{2!z^2}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)}+\frac{3!z^3}{\left(\frac{s}{\lambda}+2\right)\left(\frac{s}{\lambda}+3\right)\left(\frac{s}{\lambda}+4\right)}+...\right\}$$

$$=\frac{z}{s+\lambda}\left\{1+\frac{z}{\frac{1}{\lambda}\left(s+2\lambda\right)}+\frac{2!z^2}{\frac{1}{\lambda^2}\left(s+2\lambda\right)\left(s+3\lambda\right)}+\frac{3!z^3}{\frac{1}{\lambda^3}\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}+...\right\}$$

$$=\frac{z}{s+\lambda}\left\{1+\frac{z\lambda}{\left(s+2\lambda\right)}+\frac{2!z^2\lambda^2}{\left(s+2\lambda\right)\left(s+3\lambda\right)}+\frac{3!z^3\lambda^3}{\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}+...\right\}$$

$$\therefore\overline{G}\left(z,s\right)=\left\{\begin{matrix}\frac{z}{s+\lambda}+\frac{z^2\lambda}{(s+\lambda)(s+2\lambda)}+\frac{2!z^3\lambda^2}{(s+\lambda)(s+2\lambda)(s+3\lambda)}+\\ \\ \frac{3!z^4\lambda^3}{(s+\lambda)(s+2\lambda)(s+3\lambda)(s+4\lambda)}+...\end{matrix}\right\}$$

Applying inverse Laplace transform to both sides We get

$$L^{-1}\left\{\overline{G}\left(z,s\right)\right\}=L^{-1}\left\{\begin{array}{c}\frac{z}{s+\lambda}+\frac{z^2\lambda}{(s+\lambda)(s+2\lambda)}+\frac{2!z^3\lambda^2}{(s+\lambda)(s+2\lambda)(s+3\lambda)}+\\ \\ \frac{3!z^4\lambda^3}{(s+\lambda)(s+2\lambda)(s+3\lambda)(s+4\lambda)}+\ldots\end{array}\right\}$$

$$G\left(z,t\right)=\underbrace{L^{-1}\left\{\frac{z}{s+\lambda}\right\}}_{PartI}+\underbrace{L^{-1}\left\{\frac{z^2\lambda}{\left(s+\lambda\right)\left(s+2\lambda\right)}\right\}}_{PartII}+\underbrace{L^{-1}\left\{\frac{2!z^3\lambda^2}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)}\right\}}_{PartIII}$$

$$+\underbrace{L^{-1}\left\{\frac{3!z^4\lambda^3}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}\right\}}_{PartIV}+\ldots$$

The next is to simplify the four Parts of the above equation

Part 1

$$L^{-1}\left\{\frac{z}{s+\lambda}\right\}=zL^{-1}\left\{\frac{1}{s+\lambda}\right\}$$

Miscellaneous Method

By the first shifting property

$$L\left\{e^{-at}f(t)\right\}=\overline{f}(s+a)$$
$$\text{and } L\left\{1\right\}=\frac{1}{s}$$
$$\Rightarrow L\left\{e^{-at}\right\}=\frac{1}{s+a}$$

Letting $a=\lambda$, we have

$$
\begin{aligned}
L\left\{e^{-\lambda t}\right\} &= \frac{1}{s+\lambda} \\
L^{-1}\left\{\frac{1}{s+\lambda}\right\} &= e^{-\lambda t} \\
\therefore L^{-1}\left\{\frac{z}{s+\lambda}\right\} &= ze^{-\lambda t}
\end{aligned}
$$

Use of tables

$$
L^{-1}\left\{\frac{z}{s+\lambda}\right\} = zL^{-1}\left\{\frac{1}{s+\lambda}\right\}
$$

The Laplace transform of $f(t)$ is of the form

$$
L\{f(t)\} = \frac{k}{s-a}
$$

Which from the table is the table of transform pairs yields

$$
f(t) = ke^{at}
$$

In our case $k = z,\ a = -\lambda$

$$
L^{-1}\left\{\frac{z}{s+\lambda}\right\} = ze^{-\lambda t}
$$

Part II

$$
L^{-1}\left\{\frac{z^2\lambda}{(s+\lambda)(s+2\lambda)}\right\} = z^2\lambda L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)}\right\}
$$

Complex Inversion Formula

The function

$$
\frac{1}{(s+\lambda)(s+2\lambda)}
$$

has simple poles at $s=-\lambda$ and $s=-2\lambda$

Its residues are given by

At the pole $s=-\lambda$

$$\begin{aligned} a_{-i} &= \lim_{s\to-\lambda}(s+\lambda)\frac{e^{st}}{(s+\lambda)(s+2\lambda)} \\ &= \lim_{s\to-\lambda}\frac{e^{st}}{s+2\lambda} \\ &= \frac{e^{-\lambda t}}{-\lambda+2\lambda} \\ &= \frac{e^{-\lambda t}}{\lambda} \end{aligned}$$

Similarly at $s=-2\lambda$

$$\begin{aligned} a_{-i} &= \lim_{s\to-2\lambda}(s+2\lambda)\frac{e^{st}}{(s+\lambda)(s+2\lambda)} \\ &= \lim_{s\to-2\lambda}\frac{e^{st}}{s+\lambda} \\ &= \frac{e^{-2\lambda t}}{-2\lambda+\lambda} \\ &= \frac{e^{-\lambda t}}{-\lambda} \end{aligned}$$

Thus

$$L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)}\right\} = \sum a_{-i} = \frac{e^{-\lambda t}}{\lambda} - \frac{e^{-2\lambda t}}{\lambda} = \frac{e^{-\lambda t}}{\lambda}\left(1-e^{-\lambda t}\right)$$

$$\begin{aligned} \therefore L^{-1}\left\{\frac{z^2\lambda}{(s+\lambda)(s+2\lambda)}\right\} &= z^2\lambda L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)}\right\} \\ &= z^2\lambda\frac{e^{-\lambda t}}{\lambda}\left(1-e^{-\lambda t}\right) \\ &= z^2 e^{-\lambda t}\left(1-e^{-\lambda t}\right) \end{aligned}$$

Part III

$$L^{-1}\left\{\frac{2!z^3\lambda^2}{(s+\lambda)(s+2\lambda)(s+3\lambda)}\right\} = 2!z^3\lambda^2 L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)(s+3\lambda)}\right\}$$

The function

$$\frac{1}{(s+\lambda)(s+2\lambda)(s+3\lambda)}$$

has simple poles at $s = -\lambda$ $s = -2\lambda$ and $s = -3\lambda$

Its residues are given by

At the pole $s = -\lambda$

$$\begin{aligned}
a_{-i} &= \lim_{s\to-\lambda}(s+\lambda)\frac{e^{st}}{(s+\lambda)(s+2\lambda)(s+3\lambda)} \\
&= \lim_{s\to-\lambda}\frac{e^{st}}{(s+2\lambda)(s+3\lambda)} \\
&= \frac{e^{-\lambda t}}{(-\lambda+2\lambda)(-\lambda+3\lambda)} \\
&= \frac{e^{-\lambda t}}{\lambda(2\lambda)} \\
&= \frac{e^{-\lambda t}}{\lambda^2 2!}
\end{aligned}$$

For $s = -2\lambda$

$$\begin{aligned}
a_{-i} &= \lim_{s\to-2\lambda}(s+2\lambda)\frac{e^{st}}{(s+\lambda)(s+2\lambda)(s+3\lambda)} \\
&= \lim_{s\to-2\lambda}\frac{e^{st}}{(s+\lambda)(s+3\lambda)} \\
&= \frac{e^{-2\lambda t}}{(-2\lambda+\lambda)(-2\lambda+3\lambda)} \\
&= \frac{e^{-2\lambda t}}{-\lambda(\lambda)} \\
&= \frac{e^{-2\lambda t}}{-\lambda^2}
\end{aligned}$$

Similarly at $s = -3\lambda$

$$\begin{aligned}
a_{-i} &= \lim_{s \to -3\lambda} (s+3\lambda) \frac{e^{st}}{(s+\lambda)(s+2\lambda)(s+3\lambda)} \\
&= \lim_{s \to -3\lambda} \frac{e^{st}}{(s+\lambda)(s+2\lambda)} \\
&= \frac{e^{-3\lambda t}}{(-3\lambda+\lambda)(-3\lambda+2\lambda)} \\
&= \frac{e^{-2\lambda t}}{-2\lambda(-\lambda)} \\
&= \frac{e^{-2\lambda t}}{2!\lambda^2}
\end{aligned}$$

Thus

$$\begin{aligned}
L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)(s+3\lambda)}\right\} &= \sum a_{-i} = \frac{e^{-\lambda t}}{2!\lambda^2} - \frac{e^{-2\lambda t}}{\lambda^2} + \frac{e^{-3\lambda t}}{2!\lambda^2} \\
&= \frac{e^{-\lambda t}}{2!\lambda^2}\left(1 - 2e^{-\lambda t} + e^{-2\lambda t}\right) \\
&= \frac{e^{-\lambda t}}{2!\lambda^2}\left(1 - e^{-\lambda t}\right)^2
\end{aligned}$$

$$\begin{aligned}
\therefore L^{-1}\left\{\frac{z^2\lambda}{(s+\lambda)(s+2\lambda)}\right\} &= z^2\lambda L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)}\right\} \\
&= z^2\lambda\frac{e^{-\lambda t}}{\lambda}\left(1 - e^{-\lambda t}\right) \\
&= z^2 e^{-\lambda t}\left(1 - e^{-\lambda t}\right)
\end{aligned}$$

Part IV

$$L^{-1}\left\{\frac{3!z^4\lambda^3}{(s+\lambda)(s+2\lambda)(s+3\lambda)(s+4\lambda)}\right\} = 3!z^4\lambda^3 L^{-1}\left\{\frac{1}{(s+\lambda)(s+2\lambda)(s+3\lambda)(s+4\lambda)}\right\}$$

The function

$$\frac{1}{(s+\lambda)(s+2\lambda)(s+3\lambda)(s+4\lambda)}$$

has simple poles at $s = -\lambda$ $s = -2\lambda$ $s = -3\lambda$ and $s = -4\lambda$

Its residues are given by At the pole $s = -\lambda$

$$
\begin{aligned}
a_{-i} &= \lim_{s \to -\lambda} (s + \lambda) \frac{e^{st}}{(s + \lambda)(s + 2\lambda)(s + 3\lambda)(s + 4\lambda)} \\
&= \lim_{s \to -\lambda} \frac{e^{st}}{(s + 2\lambda)(s + 3\lambda)(s + 4\lambda)} \\
&= \frac{e^{-\lambda t}}{(-\lambda + 2\lambda)(-\lambda + 3\lambda)(-\lambda + 4\lambda)} \\
&= \frac{e^{-\lambda t}}{\lambda (2\lambda)(3\lambda)} = \frac{e^{-\lambda t}}{\lambda^3 3!}
\end{aligned}
$$

Similarly at $s = -2\lambda$

$$
\begin{aligned}
a_{-i} &= \lim_{s \to -2\lambda} (s + 2\lambda) \frac{e^{st}}{(s + \lambda)(s + 2\lambda)(s + 3\lambda)(s + 4\lambda)} \\
&= \lim_{s \to -2\lambda} \frac{e^{st}}{(s + \lambda)(s + 3\lambda)(s + 4\lambda)} \\
&= \frac{e^{-2\lambda t}}{(-2\lambda + \lambda)(-2\lambda + 3\lambda)(-2\lambda + 4\lambda)} \\
&= \frac{e^{-2\lambda t}}{-\lambda (\lambda)(2\lambda)} = \frac{e^{-2\lambda t}}{-\lambda^3 2!}
\end{aligned}
$$

Similarly at $s = -3\lambda$

$$
\begin{aligned}
a_{-i} &= \lim_{s \to -3\lambda} (s + 3\lambda) \frac{e^{st}}{(s + \lambda)(s + 2\lambda)(s + 3\lambda)(s + 4\lambda)} \\
&= \lim_{s \to -3\lambda} \frac{e^{st}}{(s + \lambda)(s + 2\lambda)(s + 4\lambda)} \\
&= \frac{e^{-3\lambda t}}{(-3\lambda + \lambda)(-3\lambda + 2\lambda)(-3\lambda + 4\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{-3\lambda t}}{-2\lambda\left(-\lambda\right)\left(\lambda\right)}\\
&= \frac{e^{-3\lambda t}}{\lambda^3 2!}
\end{aligned}
$$

Similarly at $s = -4\lambda$

$$
\begin{aligned}
a_{-i} &= \lim_{s\to -4\lambda}\left(s+4\lambda\right)\frac{e^{st}}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}\\
&= \lim_{s\to -4\lambda}\frac{e^{st}}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)}\\
&= \frac{e^{-4\lambda t}}{\left(-4\lambda+\lambda\right)\left(-4\lambda+2\lambda\right)\left(-4\lambda+3\lambda\right)}\\
&= \frac{e^{-4\lambda t}}{-3\lambda\left(-2\lambda\right)\left(-\lambda\right)}\\
&= \frac{e^{-4\lambda t}}{-\lambda^3 3!}
\end{aligned}
$$

Thus

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}\right\} &= \sum a_{-i} = \frac{e^{-\lambda t}}{3!\lambda^3} - \frac{e^{-2\lambda t}}{2!\lambda^3} + \frac{e^{-3\lambda t}}{2!\lambda^3} - \frac{e^{-4\lambda t}}{3!\lambda^3}\\
&= \frac{e^{-\lambda t}}{3!\lambda^3}\left(1-3e^{-\lambda t}+3e^{-2\lambda t}-e^{-3\lambda t}\right)\\
&= \frac{e^{-\lambda t}}{3!\lambda^3}\left(1-e^{-\lambda t}\right)^3
\end{aligned}
$$

$$
\begin{aligned}
\therefore L^{-1}\left\{\frac{3!z^4\lambda^3}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}\right\} &= 3!z^4\lambda^3 L^{-1}\left\{\frac{1}{\left(s+\lambda\right)\left(s+2\lambda\right)\left(s+3\lambda\right)\left(s+4\lambda\right)}\right\}\\
&= 3!z^4\lambda^3\frac{e^{-\lambda t}}{3!\lambda^3}\left(1-e^{-\lambda t}\right)^3\\
&= z^4 e^{-\lambda t}\left(1-e^{-\lambda t}\right)^3
\end{aligned}
$$

Consolidating the above results we get

$$
\begin{aligned}
G(z,t) &= ze^{-\lambda t} + z^2 e^{-\lambda t}\left(1-e^{-\lambda t}\right) + z^3 e^{-\lambda t}\left(1-e^{-\lambda t}\right) + z^4 e^{-\lambda t}\left(1-e^{-\lambda t}\right)^3 + ... \\
&= ze^{-\lambda t}\underbrace{\left[1 + z\left(1-e^{-\lambda t}\right) + z^2\left(1-e^{-\lambda t}\right)^2 + z^3\left(1-e^{-\lambda t}\right)^3 + ...\right]}_{geometricseries} \\
&= ze^{-\lambda t}\frac{1}{1-z\left(1-e^{-\lambda t}\right)}
\end{aligned}
$$

$$
\therefore G(z,t) = \frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}
$$

Which by identification is the pgf of a shifted geometric distribution with $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$. Thus

$$
\Pr ob\left[X(t) = n\right] = pq^{n-1}, n = 1, 2, ...
$$

Alternatively, $P_n(t)$ is the coefficient of z^n in $G(z,t)$ .But

$$
\begin{aligned}
G(z,t) &= \frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)} \\
&= ze^{-\lambda t}\left[1 - z\left(1-e^{-\lambda t}\right)\right]^{-1} \\
&= ze^{-\lambda t}\sum_{n=0}^{\infty}\left[z\left(1-e^{-\lambda t}\right)\right]^n \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\left(1-e^{-\lambda t}\right)^n z^{n+1} \\
&= e^{-\lambda t}\sum_{n=1}^{\infty}\left(1-e^{-\lambda t}\right)^{n-1} z^n \\
&= \sum_{n=1}^{\infty} e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1} z^n
\end{aligned}
$$

Therefore the coefficient of z^n is $e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}$
Hence

$$
P_n(t) = e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1} \qquad n = 1, 2, 3, ...
$$

Which is the pmf of a shifted geometric distribution

Case 2: When initial population $X(0) = n_0$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$

$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$

$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = n_0$, we have

$$P_{n_0}(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq n_0$$

$$\therefore G(z,0) = z^{n_0}$$

With this equation (5.37b) becomes

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\left(z,s\right) = \frac{1}{\lambda} \int \frac{z^{\frac{s}{\lambda}} z^{n_0}}{z(1-z)^{\frac{s}{\lambda}+1}} dz$$

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} \overline{G}\left(z,s\right) = \frac{1}{\lambda} \int \frac{z^{\frac{s}{\lambda}+n_0-1}}{(1-z)^{\frac{s}{\lambda}+1}} dz \tag{5.42}$$

We first simplify the right hand side as follows. Recall that

$$\int \frac{x^{a+r-1}}{(1-x)^{a+1}} dx = x^{a+r} \left[\frac{{}_2F_1\left(a, a+r; a+r+1; x\right)}{a+r} + x \frac{{}_2F_1\left(a+1, a+r+1; a+r+2; x\right)}{a+r+1}\right] + \text{constant}$$

Where ${}_2F_1\left(a, a+r; a+r+1; x\right)$ and ${}_2F_1\left(a+1, a+r+1; a+r+2; x\right)$ are gauss hyper geometric functions

With this, we have

$$\frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}+n_0-1}}{(1-z)^{\frac{s}{\lambda}+1}}dz = \frac{z^{\frac{s}{\lambda}+n_0}}{\lambda}\left\{\begin{array}{c} \frac{{}_2F_1\left(\frac{s}{\lambda},\frac{s}{\lambda}+n_0;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0}+ \\ \\ z\frac{{}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+n_0+1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1} \end{array}\right\} + \frac{c_2}{\lambda}$$

where c_2 is a constant of integration

But

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{c-a-b}\ {}_2F_1\left(c-a,c-b;c;z\right)$$

We now use this property to simplify the terms in the RHS, We start with

$${}_2F_1\left(\frac{s}{\lambda},\frac{s}{\lambda}+n_0;\frac{s}{\lambda}+n_0+1;z\right)$$

Here

$$a = \frac{s}{\lambda}, \quad b = \frac{s}{\lambda}+n_0, \quad c = \frac{s}{\lambda}+n_0+1$$

$$\begin{aligned} c-a &= \frac{s}{\lambda}+n_0+1-\frac{s}{\lambda} \\ &= n_0+1 \end{aligned}$$

$$\begin{aligned} c-b &= \frac{s}{\lambda}+n_0+1-\left(\frac{s}{\lambda}+n_0\right) \\ &= \frac{s}{\lambda}+n_0+1-\frac{s}{\lambda}-n_0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} c-a-b &= \frac{s}{\lambda}+n_0+1-\frac{s}{\lambda}-\left(\frac{s}{\lambda}+n_0\right) \\ &= \frac{s}{\lambda}+n_0+1-\frac{s}{\lambda}-\frac{s}{\lambda}-n_0 \\ &= 1-\frac{s}{\lambda} \end{aligned}$$

Thus

$$ {}_2F_1\left(\frac{s}{\lambda}, \frac{s}{\lambda}+n_0; \frac{s}{\lambda}+n_0+1; z\right) = (1-z)^{1-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+1, 1; \frac{s}{\lambda}+n_0+1; z\right) $$

Similarly for

$$ {}_2F_1\left(\frac{s}{\lambda}+1, \frac{s}{\lambda}+n_0+1; \frac{s}{\lambda}+n_0+2; z\right) $$

We have

$$ a = \frac{s}{\lambda}+1, \quad b = \frac{s}{\lambda}+n_0+1, \quad c = \frac{s}{\lambda}+n_0+2 $$

$$ \begin{aligned} c-a &= \frac{s}{\lambda}+n_0+2-\left(\frac{s}{\lambda}+1\right) \\ &= \frac{s}{\lambda}+n_0+2-\frac{s}{\lambda}-1 \\ &= n_0+1 \end{aligned} $$

$$ \begin{aligned} c-b &= \frac{s}{\lambda}+n_0+2-\left(\frac{s}{\lambda}+n_0+1\right) \\ &= \frac{s}{\lambda}+n_0+2-\frac{s}{\lambda}-n_0-1 \\ &= 1 \end{aligned} $$

$$ \begin{aligned} c-a-b &= \frac{s}{\lambda}+n_0+2-\left(\frac{s}{\lambda}+1\right)-\left(\frac{s}{\lambda}+n_0+1\right) \\ &= \frac{s}{\lambda}+n_0+2-\frac{s}{\lambda}-1-\frac{s}{\lambda}-n_0-1 \\ &= -\frac{s}{\lambda} \end{aligned} $$

Thus

$$ {}_2F_1\left(\frac{s}{\lambda}+1, \frac{s}{\lambda}+n_0+1; \frac{s}{\lambda}+n_0+2; z\right) = (1-z)^{-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+1, 1; \frac{s}{\lambda}+n_0+2; z\right) $$

Using the above results, We get

$$\frac{1}{\lambda}\int\frac{z^{\frac{s}{\lambda}+n_0-1}}{(1-z)^{\frac{s}{\lambda}+1}}dz=\frac{z^{\frac{s}{\lambda}+n_0}}{\lambda}\left\{\begin{matrix}\frac{(1-z)^{1-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0}+\\ \\ z\frac{(1-z)^{-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1}\end{matrix}\right\}+\frac{c_2}{\lambda}$$

Therefore

$$\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}}\overline{G}\left(z,s\right)=\frac{1}{\lambda}\int\frac{z^{\frac{s}{\lambda}+n_0-1}}{(1-z)^{\frac{s}{\lambda}+1}}dz$$

$$=\frac{z^{\frac{s}{\lambda}+n_0}}{\lambda}\left\{\begin{matrix}\frac{(1-z)^{1-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0}+\\ \\ z\frac{(1-z)^{-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1}\end{matrix}\right\}+\frac{c_2}{\lambda}$$

$$=\frac{z^{\frac{s}{\lambda}+n_0}\left(1-z\right)^{-\frac{s}{\lambda}}}{\lambda}\left\{\begin{matrix}\frac{(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0}+\\ \\ z\frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1}\end{matrix}\right\}+\frac{c_2}{\lambda}$$

$$\Rightarrow\overline{G}\left(z,s\right)=\left(\frac{z}{1-z}\right)^{-\frac{s}{\lambda}}\frac{z^{\frac{s}{\lambda}+n_0}\left(1-z\right)^{-\frac{s}{\lambda}}}{\lambda}\left\{\begin{matrix}\frac{(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0}+\\ \\ z\frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1}\end{matrix}\right\}+\frac{c_2}{\lambda}\left(\frac{z}{1-z}\right)^{-\frac{s}{\lambda}}$$

$$\therefore\overline{G}\left(z,s\right)=\frac{z^{n_0}}{\lambda}\left\{\begin{matrix}\frac{(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0}+\\ \\ z\frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1}\end{matrix}\right\}+\frac{c_2}{\lambda}\left(\frac{z}{1-z}\right)^{-\frac{s}{\lambda}}$$

But since for all $t, z \leqslant 1$ we have $G(z,t) \leqslant 1$, It follows that $c_2 = 0$
We thus have

$$\overline{G}(z,s) = \frac{z^{n_0}}{\lambda}\left[\frac{(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{s}{\lambda}+n_0} + z\ \frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{s}{\lambda}+n_0+1}\right]$$

$$= \frac{z^{n_0}}{\lambda}\left[\frac{(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{\frac{1}{\lambda}(s+\lambda n_0)} + z\ \frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{\frac{1}{\lambda}[s+\lambda(n_0+1)]}\right]$$

$$= \frac{z^{n_0}}{\lambda}\left[\frac{\lambda(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{(s+\lambda n_0)} + \lambda z\ \frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{[s+\lambda(n_0+1)]}\right]$$

$$= \frac{\lambda z^{n_0}}{\lambda}\left[\frac{(1-z)\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{(s+\lambda n_0)} + z\ \frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{[s+\lambda(n_0+1)]}\right]$$

$$= z^{n_0}\left[(1-z)\frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)}{(s+\lambda n_0)} + z\ \frac{{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)}{[s+\lambda(n_0+1)]}\right]$$

But according to Euler the Gauss hyper geometric series can be expressed as

$$\begin{aligned}{}_2F_1(a,b;c;x) &= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}\\ &= 1+\frac{ab}{c}\frac{x}{1!}+\frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)\,b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!}+...\end{aligned}$$

Where a, b and c are complex numbers and

$$(a)_k = \prod_{i=0}^{k-i}(a+i) = a(a+1)(a+2)(a+3)\,...\,(a+k-1)$$

We shall use this property to simplify the terms in the RHS For

$${}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)$$

Thus Letting $a = n_0 + 1$ $b = 1$ and $c = \frac{s}{\lambda} + n_0 + 1$ we have

$$F\left(n_0 + 1, 1; \frac{s}{\lambda} + n_0 + 1; z\right) = F\left(a, b; c; z\right)$$

$$= \left\{ \begin{array}{c} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \\ \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 1 + \frac{(n_0+1)z}{\left(\frac{s}{\lambda}+n_0+1\right)} + \frac{(n_0+1)(n_0+2)2!}{\left(\frac{s}{\lambda}+n_0+1\right)\left(\frac{s}{\lambda}+n_0+2\right)}\frac{z^2}{2!} + \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)3!}{\left(\frac{s}{\lambda}+n_0+1\right)\left(\frac{s}{\lambda}+n_0+2\right)\left(\frac{s}{\lambda}+n_0+3\right)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 1 + \frac{(n_0+1)z}{\frac{1}{\lambda}[s+\lambda(n_0+1)]} + \frac{(n_0+1)(n_0+2)z^2}{\frac{1}{\lambda^2}[s+\lambda(n_0+1)][s+\lambda(n_0+2)]} + \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)z^3}{\frac{1}{\lambda^3}[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]} + \ldots \end{array} \right\}$$

$$\therefore F\left(n_0 + 1, 1; \frac{s}{\lambda} + n_0 + 1; z\right) = \left\{ \begin{array}{c} 1 + \frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)]} + \frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]} + \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]} + \ldots \end{array} \right\}$$

Similarly for

$${}_2F_1\left(n_0 + 1, 1; \frac{s}{\lambda} + n_0 + 2; z\right)$$

We let $a = n_0 + 1$ $b = 1$ and $c = \frac{s}{\lambda} + n_0 + 2$ which implies that

$$F\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)=F\left(a,b;c;z\right)$$

$$=\left\{\begin{matrix} 1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+ \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\ldots \end{matrix}\right\}$$

$$=\left\{\begin{matrix} 1+\frac{(n_0+1)z}{\left(\frac{s}{\lambda}+n_0+2\right)}+\frac{(n_0+1)(n_0+2)2!}{\left(\frac{s}{\lambda}+n_0+2\right)\left(\frac{s}{\lambda}+n_0+3\right)}\frac{z^2}{2!}+ \\ \frac{(n_0+1)(n_0+2)(n_0+3)3!}{\left(\frac{s}{\lambda}+n_0+2\right)\left(\frac{s}{\lambda}+n_0+3\right)\left(\frac{s}{\lambda}+n_0+4\right)}\frac{z^3}{3!}+\ldots \end{matrix}\right\}$$

$$=\left\{\begin{matrix} 1+\frac{(n_0+1)z}{\frac{1}{\lambda}[s+\lambda(n_0+2)]}+\frac{(n_0+1)(n_0+2)z^2}{\frac{1}{\lambda^2}[s+\lambda(n_0+3)][s+\lambda(n_0+4)]}+ \\ \frac{(n_0+1)(n_0+2)(n_0+3)z^3}{\frac{1}{\lambda^3}[s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}+\ldots \end{matrix}\right\}$$

$$\therefore F\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)=\left\{\begin{matrix} 1+\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+2)]}+\frac{(n_0+1)(n_0+2)\lambda^2z^2}{[s+\lambda(n_0+2)][s+\lambda(n_0+3)]}+ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3z^3}{[s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}+\ldots \end{matrix}\right\}$$

Using the above results, We get

$$\overline{G}(z,s) = z^{n_0}\left\{\begin{array}{c}\frac{(1-z)}{(s+\lambda n_0)}\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)+ \\ \\ \frac{z}{[s+\lambda(n_0+1)]}\ {}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)\end{array}\right\}$$

$$= z^{n_0}\left\{\begin{array}{c}\frac{(1-z)}{(s+\lambda n_0)}\left[\begin{array}{c}1+\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)]}+\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}+ \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}+\ldots\end{array}\right]+ \\ \\ \frac{z}{[s+\lambda(n_0+1)]}\left[\begin{array}{c}1+\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+2)]}+\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+2)][s+\lambda(n_0+3)]}+ \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}+\ldots\end{array}\right]\end{array}\right\}$$

$$= z^{n_0}\left\{\begin{array}{c}(1-z)\left[\begin{array}{c}\frac{1}{(s+\lambda n_0)}+\frac{(n_0+1)\lambda z}{(s+\lambda n_0)[s+\lambda(n_0+1)]}+\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)]} \\ \\ +\frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}+\ldots\end{array}\right] \\ \\ +z\left[\begin{array}{c}\frac{1}{[s+\lambda(n_0+1)]}+\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}+ \\ \\ \frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}+ \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}+\ldots\end{array}\right]\end{array}\right\}$$

From this it follows that $\overline{G}(z,s)$ is of the form

$$\overline{G}(z,s) = z^{n_0}\left[(1-z)A + zB\right] \tag{5.43}$$

Where

$$A = \left\{ \begin{array}{c} \frac{1}{(s+\lambda n_0)} + \frac{(n_0+1)\lambda z}{(s+\lambda n_0)[s+\lambda(n_0+1)]} + \frac{(n_0+1)(n_0+2)\lambda^2 z^2}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)]} + \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]} + \ldots \end{array} \right\}$$

and

$$B = \left\{ \begin{array}{c} \frac{1}{[s+\lambda(n_0+1)]} + \frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]} + \\ \\ \frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]} + \\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]} + \ldots \end{array} \right\}$$

The next step is to apply inverse Laplace transform to both sides of equation (5.43).

$$\begin{aligned} L^{-1}\left\{\overline{G}(z,s)\right\} &= L^{-1}\left\{z^{n_0}\left[(1-z)A + zB\right]\right\} \\ &= z^{n_0}\left[(1-z)L^{-1}\{A\} + zL^{-1}\{B\}\right] \end{aligned}$$

$$\therefore G(z,t) = z^{n_0}\left[(1-z)L^{-1}\{A\} + zL^{-1}\{B\}\right]$$

Various methods can be used, We shall however use the complex inversion formula . For convenience purposes We deal with A and B separately.

Dealing with A

$$L^{-1}\{A\} = L^{-1}\left\{\begin{array}{c}\frac{1}{(s+\lambda n_0)} + \frac{(n_0+1)\lambda z}{(s+\lambda n_0)[s+\lambda(n_0+1)]} + \frac{(n_0+1)(n_0+2)\lambda^2 z^2}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)]} \\ + \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]} + \ldots \end{array}\right\}$$

$$= \left[\begin{array}{c} L^{-1}\left\{\frac{1}{(s+\lambda n_0)}\right\} + L^{-1}\left\{\frac{(n_0+1)\lambda z}{(s+\lambda n_0)[s+\lambda(n_0+1)]}\right\} + \\ L^{-1}\left\{\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}\right\} + \\ L^{-1}\left\{\frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}\right\} + \ldots \end{array}\right]$$

We again simplify the above first four terms separately

First term

From the table of transform pairs

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$
$$\Rightarrow L^{-1}\left\{\frac{1}{s+\lambda n_0}\right\} = e^{-\lambda n_0 t}$$

Second term

$$L^{-1}\left\{\frac{(n_0+1)\lambda z}{(s+\lambda n_0)[s+\lambda(n_0+1)]}\right\} = (n_0+1)\lambda z L^{-1}\left\{\frac{1}{(s+\lambda n_0)[s+\lambda(n_0+1)]}\right\}$$

The function

$$\frac{1}{(s+\lambda n_0)\left[s+\lambda\left(n_0+1\right)\right]}$$

has simple poles at $s=-\lambda n_0$ and $s=-\lambda\left(n_0+1\right)$

Thus its residues at each pole are obtained as follows

At $s=-\lambda n_0$

$$\begin{aligned}
a_{-i} &= \lim_{s\to-\lambda n_0}\frac{(s+\lambda n_0)\,e^{st}}{(s+\lambda n_0)\left[s+\lambda\left(n_0+1\right)\right]}\\
&= \lim_{s\to-\lambda n_0}\frac{e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]}\\
&= \frac{e^{-\lambda n_0 t}}{\left[-\lambda n_0+\lambda\left(n_0+1\right)\right]}\\
&= \frac{e^{-\lambda n_0 t}}{\left[-\lambda n_0+\lambda n_0+\lambda\right]}\\
&= \frac{e^{-\lambda n_0 t}}{\lambda}
\end{aligned}$$

At $s=-\lambda\left(n_0+1\right)$

$$\begin{aligned}
a_{-i} &= \lim_{s\to-\lambda(n_0+1)}\frac{\left[s+\lambda\left(n_0+1\right)\right]e^{st}}{(s+\lambda n_0)\left[s+\lambda\left(n_0+1\right)\right]}\\
&= \lim_{s\to-\lambda(n_0+1)}\frac{e^{st}}{(s+\lambda n_0)}\\
&= \frac{e^{-\lambda(n_0+1)t}}{\left[-\lambda\left(n_0+1\right)+\lambda n_0\right]}\\
&= \frac{e^{-\lambda(n_0+1)t}}{\left[-\lambda n_0-\lambda+\lambda n_0\right]}\\
&= \frac{e^{-\lambda(n_0+1)t}}{-\lambda}
\end{aligned}$$

Thus

$$L^{-1}\left\{\frac{1}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]}\right\}=\sum a_{-i}$$
$$=\frac{e^{-\lambda n_0 t}}{\lambda}-\frac{e^{-\lambda(n_0+1)t}}{\lambda}$$
$$=\frac{e^{-\lambda n_0 t}}{\lambda}\left(1-e^{-\lambda t}\right)$$

Therefore

$$L^{-1}\left\{\frac{(n_0+1)\lambda z}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]}\right\}=(n_0+1)\lambda z\frac{e^{-\lambda n_0 t}}{\lambda}\left(1-e^{-\lambda t}\right)$$
$$=(n_0+1)ze^{-\lambda n_0 t}\left(1-e^{-\lambda t}\right)$$

Third term

$$L^{-1}\left\{\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]}\right\}=(n_0+1)(n_0+2)\lambda^2 z^2\times$$

$$L^{-1}\left\{\frac{1}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]}\right\}$$

The function

$$\frac{1}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]}$$

has simple poles at $s=-\lambda n_0$,$s=-\lambda(n_0+1)$ and $s=-\lambda(n_0+2)$

Thus its residues at each pole are obtained as follows At $s=-\lambda n_0$

$$a_{-i}=\lim_{s\to-\lambda n_0}\frac{(s+\lambda n_0)e^{st}}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]}$$

$$=\lim_{s\to-\lambda n_0}\frac{e^{st}}{\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]}$$

$$=\frac{e^{-\lambda n_0 t}}{\left[-\lambda n_0+\lambda(n_0+1)\right]\left[-\lambda n_0+\lambda(n_0+2)\right]}$$

$$= \frac{e^{-\lambda n_0 t}}{[-\lambda n_0 + \lambda n_0 + \lambda]\,[-\lambda n_0 + \lambda n_0 + 2\lambda]}$$

$$= \frac{e^{-\lambda n_0 t}}{\lambda\,(2\lambda)}$$

$$= \frac{e^{-\lambda n_0 t}}{\lambda^2 2!}$$

At $s = -\lambda\,(n_0 + 1)$

$$a_{-i} = \lim_{s \to -\lambda(n_0+1)} \frac{[s + \lambda\,(n_0 + 1)]\,e^{st}}{(s + \lambda n_0)\,[s + \lambda\,(n_0 + 1)]\,[s + \lambda\,(n_0 + 2)]}$$

$$= \lim_{s \to -\lambda(n_0+1)} \frac{e^{st}}{(s + \lambda n_0)\,[s + \lambda\,(n_0 + 2)]}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{[-\lambda\,(n_0 + 1) + \lambda n_0]\,[-\lambda\,(n_0 + 1) + \lambda\,(n_0 + 2)]}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{[-\lambda n_0 - \lambda + \lambda n_0]\,[-\lambda n_0 - \lambda + \lambda n_0 + 2\lambda]}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{-\lambda\,(\lambda)}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{-\lambda^2}$$

At $s = -\lambda(n_0+2)$

$$
\begin{aligned}
a_{-i} &= \lim_{s \to -\lambda(n_0+2)} \frac{[s+\lambda(n_0+2)]\, e^{st}}{(s+\lambda n_0)\,[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]} \\
&= \lim_{s \to -\lambda(n_0+2)} \frac{e^{st}}{(s+\lambda n_0)\,[s+\lambda(n_0+1)]} \\
&= \frac{e^{-\lambda(n_0+2)t}}{[-\lambda(n_0+2)+\lambda n_0]\,[-\lambda(n_0+2)+\lambda(n_0+1)]} \\
&= \frac{e^{-\lambda(n_0+2)t}}{[-\lambda n_0 - 2\lambda + \lambda n_0]\,[-\lambda n_0 - 2\lambda + \lambda n_0 + \lambda]} \\
&= \frac{e^{-\lambda(n_0+2)t}}{-2\lambda(-\lambda)} \\
&= \frac{e^{-\lambda(n_0+2)t}}{\lambda^2 2!}
\end{aligned}
$$

Thus

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{(s+\lambda n_0)\,[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]}\right\} &= \sum a_{-i} \\
&= \frac{e^{-\lambda n_0 t}}{\lambda^2 2!} - \frac{e^{-\lambda(n_0+1)t}}{\lambda^2} + \frac{e^{-\lambda(n_0+2)t}}{\lambda^2 2!} \\
&= \frac{e^{-\lambda n_0 t}}{\lambda 2!}\left(1 - 2e^{-\lambda t} + e^{-2\lambda t}\right) \\
&= \frac{e^{-\lambda n_0 t}}{\lambda 2!}\left(1 - e^{-\lambda t}\right)^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
L^{-1}\left\{\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{(s+\lambda n_0)\,[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]}\right\} &= (n_0+1)(n_0+2)\lambda^2 z^2 \frac{e^{-\lambda n_0 t}}{\lambda^2 2!}\left(1-e^{-\lambda t}\right)^2 \\
&= (n_0+1)(n_0+2)\frac{z^2}{2!} e^{-\lambda n_0 t}\left(1-e^{-\lambda t}\right)^2
\end{aligned}
$$

Fourth term

$$
L^{-1}\left\{\frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{(s+\lambda n_0)\,[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]\,[s+\lambda(n_0+3)]}\right\} = (n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3
$$

$$
\times L^{-1}\left\{\frac{1}{(s+\lambda n_0)\,[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]\,[s+\lambda(n_0+3)]}\right\}
$$

The function

$$\frac{1}{(s+\lambda n_0)\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]}$$

has simple poles at $s=-\lambda n_0$,$s=-\lambda\left(n_0+1\right)$, $s=-\lambda\left(n_0+2\right)$ and $s=-\lambda\left(n_0+3\right)$

Thus its residues at each pole are obtained as follows

At $s=-\lambda n_0$

$$a_{-i}=\lim_{s\to-\lambda n_0}\frac{(s+\lambda n_0)\,e^{st}}{(s+\lambda n_0)\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]}$$

$$=\lim_{s\to-\lambda n_0}\frac{e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]}$$

$$=\frac{e^{-\lambda n_0 t}}{\left[-\lambda n_0+\lambda\left(n_0+1\right)\right]\left[-\lambda n_0+\lambda\left(n_0+2\right)\right]\left[-\lambda n_0+\lambda\left(n_0+3\right)\right]}$$

$$=\frac{e^{-\lambda n_0 t}}{\left[-\lambda n_0+\lambda n_0+\lambda\right]\left[-\lambda n_0+\lambda n_0+2\lambda\right]\left[-\lambda n_0+\lambda n_0+3\lambda\right]}$$

$$=\frac{e^{-\lambda n_0 t}}{\lambda\left(2\lambda\right)\left(3\lambda\right)}$$

$$=\frac{e^{-\lambda n_0 t}}{\lambda^3 3!}$$

At $s=-\lambda\left(n_0+1\right)$

$$a_{-i}=\lim_{s\to-\lambda(n_0+1)}\frac{\left[s+\lambda\left(n_0+1\right)\right]e^{st}}{(s+\lambda n_0)\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]}$$

$$= \lim_{s\to -\lambda(n_0+1)} \frac{e^{st}}{(s+\lambda n_0)\left[s+\lambda(n_0+2)\right]\left[s+\lambda(n_0+3)\right]}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{\left[-\lambda(n_0+1)+\lambda n_0\right]\left[-\lambda(n_0+1)+\lambda(n_0+2)\right]\left[-\lambda(n_0+1)+\lambda(n_0+3)\right]}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{\left[-\lambda n_0-\lambda+\lambda n_0\right]\left[-\lambda n_0-\lambda+\lambda n_0+2\lambda\right]\left[-\lambda n_0-\lambda+\lambda n_0+3\lambda\right]}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{-\lambda(\lambda)(2\lambda)}$$

$$= \frac{e^{-\lambda(n_0+1)t}}{-\lambda^3 2!}$$

At $s = -\lambda(n_0+2)$

$$a_{-i} = \lim_{s\to -\lambda(n_0+2)} \frac{\left[s+\lambda(n_0+2)\right]e^{st}}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]\left[s+\lambda(n_0+3)\right]}$$

$$= \lim_{s\to -\lambda(n_0+2)} \frac{e^{st}}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+3)\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\left[-\lambda(n_0+2)+\lambda n_0\right]\left[-\lambda(n_0+2)+\lambda(n_0+1)\right]\left[-\lambda(n_0+2)+\lambda(n_0+3)\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\left[-\lambda n_0-2\lambda+\lambda n_0\right]\left[-\lambda n_0-2\lambda+\lambda n_0+\lambda\right]\left[-\lambda n_0-2\lambda+\lambda n_0+3\lambda\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-2\lambda(-\lambda)\lambda}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\lambda^3 2!}$$

At $s = -\lambda(n_0+3)$

$$a_{-i} = \lim_{s\to-\lambda(n_0+3)} \frac{[s+\lambda(n_0+3)]\,e^{st}}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}$$

$$= \lim_{s\to-\lambda(n_0+3)} \frac{e^{st}}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{[-\lambda(n_0+3)+\lambda n_0][-\lambda(n_0+3)+\lambda(n_0+1)][-\lambda(n_0+3)+\lambda(n_0+2)]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{[-\lambda n_0-3\lambda+\lambda n_0][-\lambda n_0-3\lambda+\lambda n_0+\lambda][-\lambda n_0-3\lambda+\lambda n_0+2\lambda]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{-3\lambda(-2\lambda)(-\lambda)}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\lambda^3 3!}$$

Thus

$$L^{-1}\left\{\frac{1}{(s+\lambda n_0)[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}\right\} = \sum a_{-i}$$

but

$$\begin{aligned}\sum a_{-i} &= \frac{e^{-\lambda n_0 t}}{\lambda^3 3!} - \frac{e^{-\lambda(n_0+1)t}}{\lambda^3 2!} + \frac{e^{-\lambda(n_0+2)t}}{\lambda^3 2!} - \frac{e^{-\lambda(n_0+3)t}}{\lambda^3 3!}\\ &= \frac{e^{-\lambda n_0 t}}{\lambda^3 3!}\left(1-\frac{3!}{2!}e^{-\lambda t}+\frac{3!}{2!}e^{-2\lambda t}-e^{-3\lambda t}\right)\\ &= \frac{e^{-\lambda n_0 t}}{\lambda^3 3!}\left(1-3e^{-\lambda t}+3e^{-2\lambda t}-e^{-3\lambda t}\right)\\ &= \frac{e^{-\lambda n_0 t}}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{(s+\lambda n_0)\left[s+\lambda(n_0+1)\right]\left[s+\lambda(n_0+2)\right]\left[s+\lambda(n_0+3)\right]}\right\}$$

$$=(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3\frac{e^{-\lambda n_0 t}}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$

$$=(n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!}e^{-\lambda n_0 t}\left(1-e^{-\lambda t}\right)^3$$

Consolidating the above results We get

$$L^{-1}\{A\}=e^{-\lambda n_0 t}+(n_0+1)ze^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)+(n_0+1)(n_0+2)\frac{z^2}{2!}e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^2$$
$$+(n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!}e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^3+...$$

Similarly

Dealing with B

$$L^{-1}\{B\}=L^{-1}\left\{\begin{array}{c}\frac{1}{[s+\lambda(n_0+1)]}+\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}+\\ \\ \frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}+\\ \\ \frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}+...\end{array}\right\}$$

$$=\left[\begin{array}{c}L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)]}\right\}+L^{-1}\left\{\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}\right\}+\\ \\ L^{-1}\left\{\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}\right\}+\\ \\ L^{-1}\left\{\frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}\right\}+...\end{array}\right]$$

We again simplify the above first four terms of separately

First term

From the table of transform pairs

$$L^{-1}\left\{\frac{1}{s+a}\right\}=e^{-at}$$
$$\Rightarrow L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)]}\right\}=e^{-\lambda(n_0+1)t}$$

Second term

$$L^{-1}\left\{\frac{(n_0+1)\lambda z}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}\right\}=(n_0+1)\lambda z L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}\right\}$$

The function

$$\frac{1}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}$$

has simple poles at $s=-\lambda(n_0+1)$ and $s=-\lambda(n_0+2)$

Thus its residues at each pole are obtained as follows

At $s=-\lambda(n_0+1)$

$$\begin{aligned}a_{-i}&=\lim_{s\to-\lambda(n_0+1)}\frac{[s+\lambda(n_0+1)]e^{st}}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)]}\\&=\lim_{s\to-\lambda(n_0+1)}\frac{e^{st}}{[s+\lambda(n_0+2)]}\\&=\frac{e^{-\lambda(n_0+1)t}}{[-\lambda(n_0+1)+\lambda(n_0+2)]}\\&=\frac{e^{-\lambda(n_0+1)t}}{[-\lambda n_0-\lambda+\lambda n_0+2\lambda]}\\&=\frac{e^{-\lambda(n_0+1)t}}{\lambda}\end{aligned}$$

At $s = -\lambda(n_0+2)$

$$a_{-i} = \lim_{s \to -\lambda(n_0+2)} \frac{[s+\lambda(n_0+2)]\, e^{st}}{[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]}$$

$$= \lim_{s \to -\lambda(n_0+2)} \frac{e^{st}}{[s+\lambda(n_0+1)]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{[-\lambda(n_0+2)+\lambda(n_0+1)]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{[-\lambda n_0 - 2\lambda + \lambda n_0 + \lambda]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-\lambda}$$

Thus

$$\begin{aligned} L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]}\right\} &= \sum a_{-i} \\ &= \frac{e^{-\lambda(n_0+1)t}}{\lambda} - \frac{e^{-\lambda(n_0+2)t}}{\lambda} \\ &= \frac{e^{-\lambda(n_0+1)t}}{\lambda}\left(1-e^{-\lambda t}\right) \end{aligned}$$

Therefore

$$\begin{aligned} L^{-1}\left\{\frac{(n_0+1)\,\lambda z}{[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]}\right\} &= (n_0+1)\,\lambda z \frac{e^{-\lambda(n_0+1)t}}{\lambda}\left(1-e^{-\lambda t}\right) \\ &= (n_0+1)\, z e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda t}\right) \end{aligned}$$

Third term

$$L^{-1}\left\{\frac{(n_0+1)(n_0+2)\,\lambda^2 z^2}{[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]\,[s+\lambda(n_0+3)]}\right\} = (n_0+1)(n_0+2)\,\lambda^2 z^2 \; \times$$
$$L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)]\,[s+\lambda(n_0+2)]\,[s+\lambda(n_0+3)]}\right\}$$

The function

$$\frac{1}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}$$

has simple poles at $s=-\lambda(n_0+1)$,$s=-\lambda(n_0+2)$ and $s=-\lambda(n_0+3)$

Thus its residues at each pole are obtained as follows

At $s=-\lambda(n_0+1)$

$$a_{-i}=\lim_{s\to-\lambda(n_0+1)}\frac{[s+\lambda(n_0+1)]e^{st}}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}$$

$$=\lim_{s\to-\lambda(n_0+1)}\frac{e^{st}}{[s+\lambda(n_0+2)][s+\lambda(n_0+3)]}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{[-\lambda(n_0+1)+\lambda(n_0+2)][-\lambda(n_0+1)+\lambda(n_0+3)]}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{[-\lambda n_0-\lambda+\lambda n_0+2\lambda][-\lambda n_0-\lambda+\lambda n_0+3\lambda]}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{\lambda(2\lambda)}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{\lambda^2 2!}$$

At $s=-\lambda(n_0+2)$

$$a_{-i}=\lim_{s\to-\lambda(n_0+2)}\frac{[s+\lambda(n_0+2)]e^{st}}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}$$

$$=\lim_{s\to-\lambda(n_0+2)}\frac{e^{st}}{[s+\lambda(n_0+1)][s+\lambda(n_0+3)]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\left[-\lambda\left(n_0+2\right)+\lambda\left(n_0+1\right)\right]\left[-\lambda\left(n_0+2\right)+\lambda\left(n_0+3\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\left[-\lambda n_0-2\lambda+\lambda n_0+\lambda\right]\left[-\lambda n_0-2\lambda+\lambda n_0+3\lambda\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-\lambda\left(\lambda\right)}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-\lambda^2}$$

At $s = -\lambda\left(n_0+3\right)$

$$a_{-i} = \lim_{s\to-\lambda(n_0+3)} \frac{\left[s+\lambda\left(n_0+3\right)\right]e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]}$$

$$= \lim_{s\to-\lambda(n_0+3)} \frac{e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\left[-\lambda\left(n_0+3\right)+\lambda\left(n_0+1\right)\right]\left[-\lambda\left(n_0+3\right)+\lambda\left(n_0+2\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\left[-\lambda n_0-3\lambda+\lambda n_0+\lambda\right]\left[-\lambda n_0-3\lambda+\lambda n_0+2\lambda\right]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{-2\lambda\left(-\lambda\right)}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\lambda^2 2!}$$

Thus

$$L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}\right\}=\sum a_{-i}$$

$$\begin{aligned}&=\frac{e^{-\lambda(n_0+1)t}}{\lambda^2 2!}-\frac{e^{-\lambda(n_0+2)t}}{\lambda^2}+\frac{e^{-\lambda(n_0+3)t}}{\lambda^2 2!}\\&=\frac{e^{-\lambda(n_0+1)t}}{\lambda 2!}\left(1-2e^{-\lambda t}+e^{-2\lambda t}\right)\\&=\frac{e^{-\lambda(n_0+1)t}}{\lambda 2!}\left(1-e^{-\lambda t}\right)^2\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{(n_0+1)(n_0+2)\lambda^2 z^2}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)]}\right\}=\left\{\begin{array}{c}(n_0+1)(n_0+2)\lambda^2 z^2\\ \\ \frac{e^{-\lambda(n_0+1)t}}{\lambda^2 2!}\left(1-e^{-\lambda t}\right)^2\end{array}\right\}$$

$$=\left\{\begin{array}{c}(n_0+1)(n_0+2)\frac{z^2}{2!}\\ \\ e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda t}\right)^2\end{array}\right\}$$

Fourth term

$$L^{-1}\left\{\frac{(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3}{\begin{array}{c}[s+\lambda(n_0+1)][s+\lambda(n_0+2)]\\ [s+\lambda(n_0+3)][s+\lambda(n_0+4)]\end{array}}\right\}=(n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3$$

$$\times L^{-1}\left\{\frac{1}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}\right\}$$

The function

$$\frac{1}{[s+\lambda(n_0+1)][s+\lambda(n_0+2)][s+\lambda(n_0+3)][s+\lambda(n_0+4)]}$$

has simple poles at $s=-\lambda(n_0+1)$,$s=-\lambda(n_0+2)$,$s=-\lambda(n_0+3)$ and $s=-\lambda(n_0+4)$

Thus the residue of

$$\frac{1}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

at each pole is obtained as follows

At $s=-\lambda\left(n_0+1\right)$

$$a_{-i}=\lim_{s\to-\lambda(n_0+1)}\frac{\left[s+\lambda\left(n_0+1\right)\right]e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

$$=\lim_{s\to-\lambda(n_0+1)}\frac{e^{st}}{\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{\left[-\lambda\left(n_0+1\right)+\lambda\left(n_0+2\right)\right]\left[-\lambda\left(n_0+1\right)+\lambda\left(n_0+3\right)\right]\left[-\lambda\left(n_0+1\right)+\lambda\left(n_0+4\right)\right]}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{\left[-\lambda n_0-\lambda+\lambda n_0+2\lambda\right]\left[-\lambda n_0-\lambda+\lambda n_0+3\lambda\right]\left[-\lambda n_0-\lambda+\lambda n_0+4\lambda\right]}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{\lambda\left(2\lambda\right)\left(3\lambda\right)}$$

$$=\frac{e^{-\lambda(n_0+1)t}}{\lambda^3 3!}$$

At $s=-\lambda\left(n_0+2\right)$

$$a_{-i}=\lim_{s\to-\lambda(n_0+2)}\frac{\left[s+\lambda\left(n_0+2\right)\right]e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

$$=\lim_{s\to-\lambda(n_0+2)}\frac{e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+3\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\left[-\lambda\left(n_0+2\right)+\lambda\left(n_0+1\right)\right]\left[-\lambda\left(n_0+2\right)+\lambda\left(n_0+3\right)\right]\left[-\lambda\left(n_0+2\right)+\lambda\left(n_0+4\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{\left[-\lambda n_0-2\lambda+\lambda n_0+\lambda\right]\left[-\lambda n_0-2\lambda+\lambda n_0+3\lambda\right]\left[-\lambda n_0-2\lambda+\lambda n_0+4\lambda\right]}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-\lambda\left(\lambda\right)\left(2\lambda\right)}$$

$$= \frac{e^{-\lambda(n_0+2)t}}{-\lambda^3 2!}$$

At $s = -\lambda\left(n_0+3\right)$

$$a_{-i} = \lim_{s\to-\lambda(n_0+3)} \frac{\left[s+\lambda\left(n_0+3\right)\right]e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+3\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

$$= \lim_{s\to-\lambda(n_0+3)} \frac{e^{st}}{\left[s+\lambda\left(n_0+1\right)\right]\left[s+\lambda\left(n_0+2\right)\right]\left[s+\lambda\left(n_0+4\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\left[-\lambda\left(n_0+3\right)+\lambda\left(n_0+1\right)\right]\left[-\lambda\left(n_0+3\right)+\lambda\left(n_0+2\right)\right]\left[-\lambda\left(n_0+3\right)+\lambda\left(n_0+4\right)\right]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\left[-\lambda n_0-3\lambda+\lambda n_0+\lambda\right]\left[-\lambda n_0-3\lambda+\lambda n_0+2\lambda\right]\left[-\lambda n_0-3\lambda+\lambda n_0+4\lambda\right]}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{-2\lambda\left(-\lambda\right)\lambda}$$

$$= \frac{e^{-\lambda(n_0+3)t}}{\lambda^3 2!}$$

At $s = -\lambda (n_0 + 4)$

$$a_{-i} = \lim_{s \to -\lambda(n_0+4)} \frac{[s + \lambda (n_0 + 4)] e^{st}}{[s + \lambda (n_0 + 1)] [s + \lambda (n_0 + 2)] [s + \lambda (n_0 + 3)] [s + \lambda (n_0 + 4)]}$$

$$= \lim_{s \to -\lambda(n_0+4)} \frac{e^{st}}{[s + \lambda (n_0 + 1)] [s + \lambda (n_0 + 2)] [s + \lambda (n_0 + 3)]}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{[-\lambda (n_0 + 4) + \lambda (n_0 + 1)] [-\lambda (n_0 + 4) + \lambda (n_0 + 2)] [-\lambda (n_0 + 4) + \lambda (n_0 + 3)]}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{[-\lambda n_0 - 4\lambda + \lambda n_0 + \lambda] [-\lambda n_0 - 4\lambda + \lambda n_0 + 2\lambda] [-\lambda n_0 - 4\lambda + \lambda n_0 + 3\lambda]}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{-3\lambda (-2\lambda) (-\lambda)}$$

$$= \frac{e^{-\lambda(n_0+4)t}}{-\lambda^3 3!}$$

Thus

$$L^{-1} \left\{ \frac{1}{[s + \lambda (n_0 + 1)] [s + \lambda (n_0 + 2)] [s + \lambda (n_0 + 3)] [s + \lambda (n_0 + 4)]} \right\} = \sum a_{-i}$$

But

$$\begin{aligned} \sum a_{-i} &= \frac{e^{-\lambda(n_0+1)t}}{\lambda^3 3!} - \frac{e^{-\lambda(n_0+2)t}}{\lambda^3 2!} + \frac{e^{-\lambda(n_0+3)t}}{\lambda^3 2!} - \frac{e^{-\lambda(n_0+4)t}}{\lambda^3 3!} \\ &= \frac{e^{-\lambda(n_0+1)t}}{\lambda^3 3!} \left(1 - \frac{3!}{2!} e^{-\lambda t} + \frac{3!}{2!} e^{-2\lambda t} - e^{-3\lambda t} \right) \\ &= \frac{e^{-\lambda(n_0+1)t}}{\lambda^3 3!} \left(1 - 3e^{-\lambda t} + 3e^{-2\lambda t} - e^{-3\lambda t} \right) \\ &= \frac{e^{-\lambda(n_0+1)t}}{\lambda^3 3!} \left(1 - e^{-\lambda t} \right)^3 \end{aligned}$$

Therefore

$$L^{-1} \left\{ \frac{(n_0 + 1) (n_0 + 2) (n_0 + 3) \lambda^3 z^3}{[s + \lambda (n_0 + 1)] [s + \lambda (n_0 + 2)] [s + \lambda (n_0 + 3)] [s + \lambda (n_0 + 4)]} \right\}$$

will be

$$= (n_0+1)(n_0+2)(n_0+3)\lambda^3 z^3 \frac{e^{-\lambda(n_0+1)t}}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$
$$= (n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!}e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda t}\right)^3$$

Consolidating the above results We get

$$L^{-1}\{B\} = \left\{\begin{matrix} e^{-\lambda(n_0+1)t} + (n_0+1)\, z e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda n_0 t}\right) \\ + (n_0+1)(n_0+2)\frac{z^2}{2!}e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda n_0 t}\right)^2 \\ + (n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!}e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda n_0 t}\right)^3 + ... \end{matrix}\right\}$$

By equation (5.43) have

$G(z,t) = z^{n_0}\left[(1-z)L^{-1}\{A\} + zL^{-1}\{B\}\right]$

$$= z^{n_0}\left\{\begin{matrix} (1-z)\left[\begin{matrix} e^{-\lambda n_0 t} + (n_0+1)\, z e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right) \\ + (n_0+1)(n_0+2)\frac{z^2}{2!}e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^2 \\ + (n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!}e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^3 + ... \end{matrix}\right] \\ +z\left[\begin{matrix} e^{-\lambda(n_0+1)t} + (n_0+1)\, z e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda n_0 t}\right) \\ + (n_0+1)(n_0+2)\frac{z^2}{2!}e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda n_0 t}\right)^2 \\ + (n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!}e^{-\lambda(n_0+1)t}\left(1-e^{-\lambda n_0 t}\right)^3 + ... \end{matrix}\right] \end{matrix}\right\}$$

$$= z^{n_0}\left\{\begin{array}{c} e^{-\lambda n_0 t} + (n_0+1)\, z e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right) + \\ (n_0+1)(n_0+2)\,\frac{z^2}{2!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^2 + \\ (n_0+1)(n_0+2)(n_0+3)\,\frac{z^3}{3!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^3 - \\ z e^{-\lambda n_0 t} - (n_0+1)\, z^2 e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right) - \\ (n_0+1)(n_0+2)\,\frac{z^3}{2!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^2 - \\ (n_0+1)(n_0+2)(n_0+3)\,\frac{z^4}{3!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^3 + \\ z e^{-\lambda (n_0+1) t} + (n_0+1)\, z^2 e^{-\lambda (n_0+1) t}\left(1-e^{-\lambda n_0 t}\right) + \\ (n_0+1)(n_0+2)\,\frac{z^3}{2!} e^{-\lambda (n_0+1) t}\left(1-e^{-\lambda n_0 t}\right)^2 + \\ (n_0+1)(n_0+2)(n_0+3)\,\frac{z^4}{3!} e^{-\lambda (n_0+1) t}\left(1-e^{-\lambda n_0 t}\right)^3 + \ldots \end{array}\right\}$$

We thus have

$$G(z,t) = z^{n_0} C \tag{5.44}$$

Where

$$C = \left\{ \begin{array}{c} e^{-\lambda n_0 t} + (n_0+1)\, z e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right) + \\ \\ (n_0+1)(n_0+2)\,\frac{z^2}{2!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^2 + \\ \\ (n_0+1)(n_0+2)(n_0+3)\,\frac{z^3}{3!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^3 - \\ \\ z e^{-\lambda n_0 t} - (n_0+1)\, z^2 e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right) - \\ \\ (n_0+1)(n_0+2)\,\frac{z^3}{2!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^2 - \\ \\ (n_0+1)(n_0+2)(n_0+3)\,\frac{z^4}{3!} e^{-\lambda n_0 t}\left(1-e^{-\lambda n_0 t}\right)^3 + \\ \\ z e^{-\lambda (n_0+1) t} + (n_0+1)\, z^2 e^{-\lambda (n_0+1) t}\left(1-e^{-\lambda n_0 t}\right) + \\ \\ (n_0+1)(n_0+2)\,\frac{z^3}{2!} e^{-\lambda (n_0+1) t}\left(1-e^{-\lambda n_0 t}\right)^2 + \\ \\ (n_0+1)(n_0+2)(n_0+3)\,\frac{z^4}{3!} e^{-\lambda (n_0+1) t}\left(1-e^{-\lambda n_0 t}\right)^3 + \ldots \end{array} \right\}$$

$$C = e^{-\lambda n_0 t} \left\{ \begin{array}{c} 1 + (n_0+1)\, z \left(1 - e^{-\lambda n_0 t}\right) + \\ (n_0+1)(n_0+2)\, \frac{z^2}{2!} \left(1 - e^{-\lambda n_0 t}\right)^2 + \\ (n_0+1)(n_0+2)(n_0+3)\, \frac{z^3}{3!} \left(1 - e^{-\lambda n_0 t}\right)^3 - \\ z - (n_0+1)\, z^2 \left(1 - e^{-\lambda n_0 t}\right) - \\ (n_0+1)(n_0+2)\, \frac{z^3}{2!} \left(1 - e^{-\lambda n_0 t}\right)^2 - \\ (n_0+1)(n_0+2)(n_0+3)\, \frac{z^4}{3!} \left(1 - e^{-\lambda n_0 t}\right)^3 + \\ z e^{-\lambda t} + (n_0+1)\, z^2 e^{-\lambda t} \left(1 - e^{-\lambda n_0 t}\right) + \\ (n_0+1)(n_0+2)\, \frac{z^3}{2!} e^{-\lambda t} \left(1 - e^{-\lambda n_0 t}\right)^2 + \\ (n_0+1)(n_0+2)(n_0+3)\, \frac{z^4}{3!} e^{-\lambda t} \left(1 - e^{-\lambda n_0 t}\right)^3 + \ldots \end{array} \right\}$$

The next step is to simplify C, this involves equating corresponding powers of $\left(1-e^{-\lambda t}\right)$. To make our life easier we express this in a table as shown below where the terms of each power of $\left(1-e^{-\lambda t}\right)$ are specified in the corresponding columns.

Power of $\left(1-e^{-\lambda t}\right)$	Coefficient
0	$1-z+ze^{-\lambda t}$
1	$(n_0+1)\,z-(n_0+1)\,z^2+(n_0+1)\,z^2e^{-\lambda t}$
2	$(n_0+1)\,(n_0+2)\,\frac{z^2}{2!}-(n_0+1)\,(n_0+2)\,\frac{z^3}{2!}+(n_0+1)\,(n_0+2)\,\frac{z^3}{2!}e^{-\lambda t}$
3	$(n_0+1)\,(n_0+2)\,(n_0+3)\,\frac{z^3}{3!}-(n_0+1)\,(n_0+2)\,(n_0+3)\,\frac{z^4}{3!}+(n_0+1)\,(n_0+2)\,(n_0+3)\,\frac{z^4}{3!}e^{-\lambda t}$

From the table above, it follows that

$$C = e^{-\lambda n_0 t} \left\{ \begin{array}{c} 1 - z + ze^{-\lambda t} + \\ \left[(n_0+1)\, z - (n_0+1)\, z^2 + (n_0+1)\, z^2 e^{-\lambda t} \right] \left(1 - e^{-\lambda t}\right) + \\ \left[\begin{array}{c} (n_0+1)(n_0+2)\frac{z^2}{2!} - \\ (n_0+1)(n_0+2)\frac{z^3}{2!} + \\ (n_0+1)(n_0+2)\frac{z^3}{2!} e^{-\lambda t} \end{array} \right] \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} (n_0+1)(n_0+2)(n_0+3)\frac{z^3}{3!} - \\ (n_0+1)(n_0+2)(n_0+3)\frac{z^4}{3!} + \\ (n_0+1)(n_0+2)(n_0+3)\frac{z^4}{3!} e^{-\lambda t} \end{array} \right] \left(1 - e^{-\lambda t}\right)^3 + ... \end{array} \right\}$$

But by equation (5.44) we had

$$G(z,t) = z^{n_0} C$$

$$= z^{n_0} e^{-\lambda n_0 t} \left\{ \begin{array}{c} 1 - z + z e^{-\lambda t} + (n_0+1) z \left[1 - z + z e^{-\lambda t}\right] \left(1 - e^{-\lambda t}\right) + \\ (n_0+1)(n_0+2) \frac{z^2}{2!} \left[1 - z + z e^{-\lambda t}\right] \left(1 - e^{-\lambda t}\right)^2 + \\ (n_0+1)(n_0+2)(n_0+3) \frac{z^3}{3!} \left[1 - z + z e^{-\lambda t}\right] \left(1 - e^{-\lambda t}\right)^3 + ... \end{array} \right\}$$

Factoring out $1 - z + z e^{-\lambda t}$ in the RHS implies that

$$G(z,t) = z^{n_0} e^{-\lambda n_0 t} \left[1 - z\left(1 - e^{-\lambda t}\right)\right] \left\{ \begin{array}{c} 1 + (n_0+1) z \left(1 - e^{-\lambda t}\right) + \\ (n_0+1)(n_0+2) \frac{z^2}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ (n_0+1)(n_0+2)(n_0+3) \frac{z^3}{3!} \left(1 - e^{-\lambda t}\right)^3 \\ +... \end{array} \right\}$$

$$= z^{n_0} e^{-\lambda n_0 t} \left[1 - z\left(1 - e^{-\lambda t}\right)\right] \left\{ \begin{array}{c} 1 + \frac{(n_0+1)}{1!} \left[z\left(1 - e^{-\lambda t}\right)\right] + \\ \frac{(n_0+1)(n_0+2)}{2!} \left[z\left(1 - e^{-\lambda t}\right)\right]^2 + \\ \frac{(n_0+1)(n_0+2)(n_0+3)}{3!} \left[z\left(1 - e^{-\lambda t}\right)\right]^3 + ... \end{array} \right\}$$

To simplify this, We use the binomial expansion properties, Let us multiply the RHS by 1 in a nice way that is $\frac{n_0!}{n_0!}$. This yields

$$G(z,t) = z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1 - e^{-\lambda t}\right)\right]\frac{n_0!}{n_0!}\left\{\begin{array}{c} 1 + \frac{(n_0+1)}{1!}\left[z\left(1 - e^{-\lambda t}\right)\right] + \\ \frac{(n_0+1)(n_0+2)}{2!}\left[z\left(1 - e^{-\lambda t}\right)\right]^2 + \\ \frac{(n_0+1)(n_0+2)(n_0+3)}{3!}\left[z\left(1 - e^{-\lambda t}\right)\right]^3 + ... \end{array}\right\}$$

$$= z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1 - e^{-\lambda t}\right)\right]\left\{\begin{array}{c} \frac{n_0!}{n_0!} + \frac{(n_0+1)n_0!}{n_0!1!}\left[z\left(1 - e^{-\lambda t}\right)\right] + \\ \frac{(n_0+1)(n_0+2)n_0!}{n_0!2!}\left[z\left(1 - e^{-\lambda t}\right)\right]^2 + \\ \frac{(n_0+1)(n_0+2)(n_0+3)n_0!}{n_0!3!}\left[z\left(1 - e^{-\lambda t}\right)\right]^3 + ... \\ \end{array}\right\}$$

$$\therefore G(z,t) = z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1 - e^{-\lambda t}\right)\right]\left\{\begin{array}{c} 1 + \frac{(n_0+1)n_0!}{n_0!1!}\left[z\left(1 - e^{-\lambda t}\right)\right] + \\ \frac{(n_0+1)(n_0+2)n_0!}{n_0!2!}\left[z\left(1 - e^{-\lambda t}\right)\right]^2 + \\ \frac{(n_0+1)(n_0+2)(n_0+3)n_0!}{n_0!3!}\left[z\left(1 - e^{-\lambda t}\right)\right]^3 + ... \end{array}\right\} \tag{5.45}$$

Since

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We have

$$\binom{n_0+0}{0} = 1$$

$$\binom{n_0+1}{1} = \frac{(n_0+1)!}{n_0!1!} = \frac{(n_0+1)\, n_0!}{n_0!1!}$$

$$\binom{n_0+2}{2} = \frac{(n_0+2)!}{n_0!2!} = \frac{(n_0+2)!\,(n_0+1)\, n_0!}{n_0!2!}$$

$$\binom{n_0+3}{3} = \frac{(n_0+3)!}{n_0!3!} = \frac{(n_0+3)\,(n_0+2)!\,(n_0+1)\, n_0!}{n_0!3!}$$

And so on With this equation (5.45) becomes

$$G(z,t) = z^{n_0} e^{-\lambda n_0 t} \left[1 - z\left(1-e^{-\lambda t}\right)\right] \left\{ \begin{array}{c} 1 + \binom{n_0+1}{1} \left[z\left(1-e^{-\lambda t}\right)\right] + \\ \binom{n_0+2}{2} \left[z\left(1-e^{-\lambda t}\right)\right]^2 + \\ \binom{n_0+3}{3} \left[z\left(1-e^{-\lambda t}\right)\right]^3 + ... \end{array} \right\}$$

$$= z^{n_0} e^{-\lambda n_0 t} \left[1 - z\left(1-e^{-\lambda t}\right)\right] \sum_{j=0}^{\infty} \binom{n_0+j}{j} \left[z\left(1-e^{-\lambda t}\right)\right]^j$$

But

$$\binom{n_0+j}{j} = \binom{[n_0+1]+j-1}{j}$$

Thus

$$G(z,t) = z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1-e^{-\lambda t}\right)\right] \sum_{j=0}^{\infty} \binom{[n_0+1]+j-1}{j} \left[z\left(1-e^{-\lambda t}\right)\right]^j$$

Also

$$\binom{-r}{j}(-1)^j = \binom{r+j-1}{j}$$

$$\Rightarrow \binom{-[n_0+1]}{j}(-1)^j = \binom{[n_0+1]+j-1}{j}$$

Therefore

$$\begin{aligned}
G(z,t) &= z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1-e^{-\lambda t}\right)\right] \sum_{j=0}^{\infty} \binom{-[n_0+1]}{j} (-1)^j \left[z\left(1-e^{-\lambda t}\right)\right]^j \\
&= z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1-e^{-\lambda t}\right)\right] \sum_{j=0}^{\infty} \binom{-[n_0+1]}{j} \left[-z\left(1-e^{-\lambda t}\right)\right]^j \\
&= z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1-e^{-\lambda t}\right)\right] \sum_{j=0}^{\infty} \binom{-[n_0+1]}{j} \left[-z\left(1-e^{-\lambda t}\right)\right]^j \\
&= z^{n_0} e^{-\lambda n_0 t}\left[1 - z\left(1-e^{-\lambda t}\right)\right] \left[1 - z\left(1-e^{-\lambda t}\right)\right]^{-(n_0+1)} \\
&= \left(ze^{-\lambda t}\right)^{n_0} \left[1 - z\left(1-e^{-\lambda t}\right)\right]^{-n_0}
\end{aligned}$$

$$\therefore G(z,t) = \left[\frac{ze^{-\lambda t}}{1 - z\left(1-e^{-\lambda t}\right)}\right]^{n_0}$$

By identification, this is the pgf of a negative binomial distribution with $r = n_0$ and $p = e^{-\lambda t}$

$P_n(t)$ is the coefficient of z^n in the expansion of $G(z,t)$

But $G(z,t)$ is of the form

$$G(z,t) = \left(\frac{zp}{1-zq}\right)^{n_0}$$

$$= (zp)^{n_0}(1-zq)^{-n_0}$$

$$= (zp)^{n_0}\sum_{k=0}^{\infty}\binom{-n_0}{k}(-zq)^k$$

$$= (zp)^{n_0}\sum_{k=0}^{\infty}\binom{-n_0}{k}(-1)^k(zq)^k$$

$$= (zp)^{n_0}\sum_{k=0}^{\infty}\binom{n_0+k-1}{k}(zq)^k$$

$$= p^{n_0}\sum_{k=0}^{\infty}\binom{n_0+k-1}{k}q^k z^{n_0+k}$$

Letting $n = n_0 + k$ implies

$$G(z,t) = p^{n_0}\sum_{n=n_0}^{\infty}\binom{n-1}{n-n_0}q^{n-n_0}z^n$$

Thus the coefficient of z^n is

$$p^{n_0}\binom{n-1}{n-n_0}q^{n-n_0}$$

But $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$ implying that

$$P_n(t) = \binom{n-1}{n-n_0}\left(e^{-\lambda t}\right)^{n_0}\left(1-e^{-\lambda t}\right)^{n-n_0} \qquad n = n_0, n_0+1, n_0+2, ...$$

Which is the pmf of a negative binomial distribution

5.5 Simple Birth Process With Immigration

The basic difference-differential equations for this process are obtained from the basic difference-differential equations for the general birth process by $\lambda n = n\lambda + v$ for all n where v is the immigration rate.

The equations are

$$P_0'(t) = -vP_0(t) \tag{5.46a}$$

$$P_n'(t) = [(n-1)\lambda + v]\, P_{n-1}(t) - (n\lambda + v)P_n(t) \qquad n \geqslant 1 \tag{5.46b}$$

We now solve these equations;

Method 1: By Iteration

Taking the Laplace transform of both sides of equation (5.46b) yields;

$$L\{P_n'(t)\} = L\{[(n-1)\lambda + v]\, P_{n-1}(t) - (n\lambda + v)P_n(t)\}$$

But by the linearity property of Laplace transform (Property 1)

$$L\{[(n-1)\lambda + v]\, P_{n-1}(t) - (n\lambda + v)P_n(t)\} = [(n-1)\lambda + v]\, L\{P_{n-1}(t)\} - (n\lambda + v)L\{P_n(t)\}$$

And by the Laplace transform of derivatives (Property 6)

If

$$L\{f(t)\} = \overline{f}(s)$$

Then

$$L\{f'(t)\} = s\overline{f}(s) - f(0)$$

This implies that

$$L\{P_n'(t)\} = sL\{P_n(t)\} - P_n(0)$$

Hence we have;

$$L\{P_n'(t)\} = [(n-1)\lambda + v]\,L\{P_{n-1}(t)\} - (n\lambda + v)\,L\{P_n(t)\}$$
$$L\{P_n'(t)\} - P_n(0) = [(n-1)\lambda + v]\,L\{P_{n-1}(t)\} - (n\lambda + v)\,L\{P_n(t)\}$$

$$[s + (n\lambda + v)]\,L\{P_n(t)\} = P_n(0) + [(n-1)\lambda + v]\,L\{P_{n-1}(t)\} \tag{5.47}$$

Remark
Assume that the initial population at time $t = 0$ is n_0 i.e. $X(0) = n_0$ thus $P_{n_0}(0) = 1$ and $P_n(0) = 0 \quad \forall \quad n \neq 0$. With this initial condition we solve equation (5.47) iteratively as follows.

For $n = n_0$ equation (5.47) becomes;

$$[s + (n_0\lambda + v)]\,L\{P_{n_0}(t)\} = P_{n_0}(0) + [(n_0-1)\lambda + v]\,L\{P_{n_0-1}(t)\}$$

But
$P_{n_0}(0) = 1$ and $P_{n_0-1}(0) = 0$

$$\Rightarrow L\{P_{n_0-1}(0)\} = L\{0\} = 0$$

We thus have

$$[s + (n_0\lambda + v)]\,L\{P_{n_0}(t)\} = 1 + \underbrace{[(n_0-1)\lambda + v]\,L\{P_{n_0-1}(t)\}}_{0}$$
$$[s + (n_0\lambda + v)]\,L\{P_{n_0}(t)\} = 1$$
$$\Rightarrow L\{P_{n_0}(t)\} = \frac{1}{s + (n_0\lambda + v)}$$

For $n = n_0 + 1$ equation (5.47) becomes;

$$[s + (n_0+1)\lambda + v]\,L\{P_{n_0+1}(t)\} = P_{n_0+1}(0) + [(n_0+1-1)\lambda + v]\,L\{P_{n_0+1-1}(t)\}$$
$$[s + (n_0+1)\lambda + v]\,L\{P_{n_0+1}(t)\} = P_{n_0+1}(0) + [n_0\lambda + v]\,L\{P_{n_0}(t)\}$$

But
$P_{n_0+1}(0) = 0$
and

$$L\{P_{n_0}(t)\} = \frac{1}{s + (n_0\lambda + v)}$$

Thus we have

$$[s+(n_0+1)\lambda+v]\,L\{P_{n_0+1}(t)\}=0+[n_0\lambda+v]\left[\frac{1}{s+(n_0\lambda+v)}\right]$$

$$[s+(n_0+1)\lambda+v]\,L\{P_{n_0+1}(t)\}=[n_0\lambda+v]\left[\frac{1}{s+(n_0\lambda+v)}\right]$$

$$\Rightarrow L\{P_{n_0+1}(t)\}=\frac{n_0\lambda+v}{[s+(n_0\lambda+v)]\,[s+(n_0+1)\lambda+v]}$$

For $n=n_0+2$ equation (5.47) becomes;

$$[s+(n_0+2)\lambda+v]\,L\{P_{n_0+2}(t)\}=P_{n_0+2}(0)+[(n_0+2-1)\lambda+v]\,L\{P_{n_0+2-1}(t)\}$$

$$[s+(n_0+2)\lambda+v]\,L\{P_{n_0+2}(t)\}=P_{n_0+2}(0)+[(n_0+1)\lambda+v]\,L\{P_{n_0+1}(t)\}$$

But
$P_{n_0+2}(0)=0$

and

$$L\{P_{n_0+1}(t)\}=\frac{n_0\lambda+v}{[s+(n_0\lambda+v)]\,[s+(n_0+1)\lambda+v]}$$

Thus we have

$$[s+(n_0+2)\lambda+v]\,L\{P_{n_0+2}(t)\}=0+[(n_0+1)\lambda+v]\,\frac{n_0\lambda+v}{[s+(n_0\lambda+v)]\,[s+(n_0+1)\lambda+v]}$$

$$[s+(n_0+2)\lambda+v]\,L\{P_{n_0+2}(t)\}=\frac{[(n_0+1)\lambda+v]\,[n_0\lambda+v]}{[s+(n_0\lambda+v)]\,[s+(n_0+1)\lambda+v]}$$

$$\Rightarrow L\{P_{n_0+2}(t)\}=\frac{[n_0\lambda+v]\,[(n_0+1)\lambda+v]}{[s+(n_0\lambda+v)]\,[s+(n_0+1)\lambda+v]\,[s+(n_0+1)\lambda+v]}$$

For $n=n_0+3$ equation (5.47) becomes

$$[s+(n_0+3)\lambda+v]\,L\{P_{n_0+3}(t)\}=P_{n_0+3}(0)+[(n_0+3-1)\lambda+v]\,L\{P_{n_0+3-1}(t)\}$$

$$[s+(n_0+3)\lambda+v]\,L\{P_{n_0+3}(t)\}=P_{n_0+3}(0)+[(n_0+2)\lambda+v]\,L\{P_{n_0+2}(t)\}$$

But

$$P_{n_0+3}(0) = 0$$

and

$$L\left\{P_{n_0+2}(t)\right\} = \frac{[n_0\lambda + v]\left[(n_0+1)\lambda + v\right]}{[s + (n_0\lambda + v)]\left[s + (n_0+1)\lambda + v\right]\left[s + (n_0+2)\lambda + v\right]}$$

Thus we have

$$[s + (n_0+3)\lambda + v]\, L\left\{P_{n_0+3}(t)\right\} = \left\{\begin{array}{c} [(n_0+2)\lambda + v] * \\ \\ \frac{(n_0\lambda+v)[(n_0+1)\lambda+v]}{[s+n_0\lambda+v][s+(n_0+1)\lambda+v][s+(n_0+2)\lambda+v]} \end{array}\right\}$$

$$\Rightarrow L\left\{P_{n_0+3}(t)\right\} = \frac{(n_0\lambda + v)\left[(n_0+1)\lambda + v\right]\left[(n_0+2)\lambda + v\right]}{[s + n_0\lambda + v]\left[s + (n_0+1)\lambda + v\right]\left[s + (n_0+2)\lambda + v\right]\left[s + (n_0+3)\lambda + v\right]}$$

Similarly for $n = n_0 + 4$ we have;

$$[s + (n_0+4)\lambda + v]\, L\left\{P_{n_0+4}(t)\right\} = P_{n_0+4}(0) + [(n_0+4-1)\lambda + v]\, L\left\{P_{n_0+4-1}(t)\right\}$$
$$[s + (n_0+4)\lambda + v]\, L\left\{P_{n_0+4}(t)\right\} = P_{n_0+4}(0) + [(n_0+3)\lambda + v]\, L\left\{P_{n_0+3}(t)\right\}$$

But

$$P_{n_0+4}(0) = 0$$

and

$$L\left\{P_{n_0+3}(t)\right\} = \frac{(n_0\lambda + v)\left[(n_0+1)\lambda + v\right]\left[(n_0+2)\lambda + v\right]}{[s + n_0\lambda + v]\left[s + (n_0+1)\lambda + v\right]\left[s + (n_0+2)\lambda + v\right]\left[s + (n_0+3)\lambda + v\right]}$$

Thus

$$[s+(n_0+4)\lambda+v]L\{P_{n_0+4}(t)\}$$
$$=[(n_0+3)\lambda+v]\left[\frac{(n_0\lambda+v)[(n_0+1)\lambda+v][(n_0+2)\lambda+v]}{[s+n_0\lambda+v][s+(n_0+1)\lambda+v][s+(n_0+2)\lambda+v][s+(n_0+3)\lambda+v][s+(n_0+4)\lambda+v]}\right]$$

$$\Rightarrow L\{P_{n_0+4}(t)\}=\frac{(n_0\lambda+v)[(n_0+1)\lambda+v][(n_0+2)\lambda+v][(n_0+3)\lambda+v]}{[s+n_0\lambda+v][s+(n_0+1)\lambda+v][s+(n_0+2)\lambda+v][s+(n_0+3)\lambda+v][s+(n_0+4)\lambda+v]}$$

$$=\frac{\prod_{i=0}^{3}[(n_0+i)\lambda+v]}{\prod_{i=0}^{4}[s+(n_0+i)\lambda+v]}$$

By mathematical induction, We assume that;

$$[s+(n\lambda+v)]L\{P_n(t)\}=P_n(0)+[(n-1)\lambda+v]L\{P_{n-1}(t)\}$$

Setting $n=n_0+k$ we obtain

$$[s+(n_0+k)\lambda+v]L\{P_{n_0+k}(t)\}=P_{n_0+k}(0)+[(n_0+k-1)\lambda+v]L\{P_{n_0+k-1}(t)\}$$

But

$$P_{n_0+k}(0)=0$$

and

$$L\{P_{n_0+k-1}(t)\}=\frac{\prod_{i=0}^{k-2}[(n_0+i)\lambda+v]}{\prod_{i=0}^{k-1}[s+(n_0+i)\lambda+v]}$$

Thus

$$[s+(n_0+k)\lambda+v]\,L\{P_{n_0+k}(t)\} = 0+[(n_0+k-1)\lambda+v]\,\frac{\prod\limits_{i=0}^{k-2}[(n_0+i)\lambda+v]}{\prod\limits_{i=0}^{k-1}[s+(n_0+i)\lambda+v]}$$

$$[s+(n_0+k)\lambda+v]\,L\{P_{n_0+k}(t)\} = [(n_0+k-1)\lambda+v]\,\frac{\prod\limits_{i=0}^{k-2}[(n_0+i)\lambda+v]}{\prod\limits_{i=0}^{k-1}[s+(n_0+i)\lambda+v]}$$

$$[s+(n_0+k)\lambda+v]\,L\{P_{n_0+k}(t)\} = \frac{\prod\limits_{i=0}^{k-1}[(n_0+i)\lambda+v]}{\prod\limits_{i=0}^{k-1}[s+(n_0+i)\lambda+v]}$$

$$\Rightarrow L\{P_{n_0+k}(t)\} = \frac{\prod\limits_{i=0}^{k-1}[(n_0+i)\lambda+v]}{\prod\limits_{i=0}^{k}[s+(n_0+i)\lambda+v]}$$

Letting $n=n_0+k$, We have

$$L\{P_n(t)\} = \frac{\prod\limits_{i=0}^{k-1}[(n_0+i)\lambda+v]}{\prod\limits_{i=0}^{k}[s+(n_0+i)\lambda+v]} \tag{5.48}$$

The next is to determine the inverse Laplace transform of $L\{P_n(t)\}$. Applying the inverse Laplace transform to both sides of equation (5.48) , we get

$$L^{-1}\left\{L\left\{P_n(t)\right\}\right\} = L^{-1}\left\{\frac{\prod\limits_{i=0}^{k-1}\left[(n_0+i)\,\lambda+v\right]}{\prod\limits_{i=0}^{k}\left[s+(n_0+i)\,\lambda+v\right]}\right\}$$

$$\Rightarrow P_n(t) = L^{-1}\left\{\frac{\prod\limits_{i=0}^{k-1}\left[(n_0+i)\,\lambda+v\right]}{\prod\limits_{i=0}^{k}\left[s+(n_0+i)\,\lambda+v\right]}\right\}$$

$$\therefore P_n(t) = \prod_{i=0}^{k-1}\left[(n_0+i)\,\lambda+v\right]L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+(n_0+i)\,\lambda+v\right]}\right\} \tag{5.49}$$

We now solve equation (5.49). The following methods have been considered

- Complex Inversion Formula
- Partial Fractions Method

Complex Inversion Formula

By the complex inversion formula, If

$$L\left\{f(t)\right\} = \overline{f}(s)$$

$$\begin{aligned} f(t) &= L^{-1}\left(\overline{f}(s)\right) \\ &= \frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty} e^{st}\overline{f}(s)ds \end{aligned}$$

Which is simply the sum of the residues of $e^{st}\overline{f}(s)$ at the poles of $\overline{f}(s)$. In our case;

$$\overline{f}(s) = \frac{1}{\prod_{i=0}^{k}\left[s+v+\lambda\left(n_0+i\right)\right]}$$

$$\Rightarrow e^{st}\overline{f}(s) = \frac{e^{st}}{\prod_{i=0}^{k}\left[s+v+\lambda\left(n_0+i\right)\right]}$$

It can be seen that $\overline{f}(s)$ is analytical is analytical at every other point except when

$$s = -\left[v+\lambda\left(n_0+i\right)\right], \qquad i = 0,1,2,3,...k$$

Thus $\overline{f}(s)$ has $k+1$ simple poles as listed below;

$$\begin{aligned} s &= -\left[v+\lambda n_0\right] \\ s &= -\left[v+\lambda\left(n_0+1\right)\right] \\ s &= -\left[v+\lambda\left(n_0+2\right)\right] \\ s &= -\left[v+\lambda\left(n_0+3\right)\right] \\ &. \\ &. \\ &. \\ s &= -\left[v+\lambda\left(n_0+k\right)\right] \end{aligned}$$

We now determine the Residue of $e^{st}\overline{f}(s)$ at each pole. In general for simple poles say $s=a$. The residue of $e^{st}\overline{f}(s)$ is given by

$$Residue\left[e^{st}\overline{f}(s)\right] = \lim_{s\to a}(s-a)e^{st}\overline{f}(s)$$

Thus the Residues of $e^{st}\overline{f}(s)$ at poles of $\overline{f}(s)$ are;

At $s = -[v + \lambda n_0]$

$$Residue\left[e^{st}\overline{f}(s)\right] = \lim_{s \to -(v+\lambda n_0)} (s + v + \lambda n_0) \frac{e^{st}}{\prod_{i=0}^{k} [s + v + \lambda (n_0 + i)]}$$

$$= \lim_{s \to -(v+\lambda n_0)} (s + v + \lambda n_0) \frac{e^{st}}{(s + v + \lambda n_0) \prod_{i=1}^{k} [s + v + \lambda (n_0 + i)]}$$

$$= \lim_{s \to -(v+\lambda n_0)} \frac{e^{st}}{\prod_{i=1}^{k} [s + v + \lambda (n_0 + i)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\prod_{i=1}^{k} [-(v + \lambda n_0) + v + \lambda (n_0 + i)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\prod_{i=1}^{k} [-v - \lambda n_0 + v + \lambda n_0 + \lambda i]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\prod_{i=1}^{k} \lambda i}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda^k \prod_{i=1}^{k} i}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda^n k!}$$

At $s = -\left[v + \lambda\left(n_0 + 1\right)\right]$

$$Residue\left[e^{st}\overline{f}(s)\right] = \lim_{s \to -[v+\lambda(n_0+1)]} \left[s + v + \lambda\left(n_0 + 1\right)\right] \frac{e^{st}}{\prod_{i=0}^{k}\left[s + v + \lambda\left(n_0 + i\right)\right]}$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \left[s + v + \lambda\left(n_0 + 1\right)\right] \left[\frac{e^{st}}{\left(s + v + \lambda n_0\right)\left[s + v + \lambda\left(n_0 + 1\right)\right] \prod_{i=2}^{k}\left[s + v + \lambda\left(n_0 + i\right)\right]}\right]$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \frac{e^{st}}{\left(s + v + \lambda n_0\right) \prod_{i=2}^{k}\left[s + v + \lambda\left(n_0 + i\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\left[-\left(v + \lambda\left(n_0 + 1\right)\right) + v + \lambda n_0\right] \prod_{i=2}^{k}\left[-\left(v + \lambda\left(n_0 + 1\right)\right) + v + \lambda\left(n_0 + i\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\left[-v - \lambda n_0 - \lambda + v + \lambda n_0\right] \prod_{i=2}^{k}\left[-v - \lambda n_0 - \lambda + v + \lambda n_0 + \lambda i\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda \prod_{i=2}^{k}\left(\lambda i - \lambda\right)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda \prod_{i=2}^{k}\lambda\left(i - 1\right)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda\lambda^{k-1}\prod_{i=2}^{k}(\lambda i-\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{(-1)\,\lambda^{k}\prod_{i=2}^{k}(i-1)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{(-1)\,\lambda^{k}\,(k-1)!}$$

At $s = -\left[v + \lambda\,(n_0 + 2)\right]$

Residue $\left[e^{st}\overline{f}(s)\right]$ is given by

$$Res = \lim_{s\to-[v+\lambda(n_0+2)]}\left[s+v+\lambda\,(n_0+2)\right]\frac{e^{st}}{\prod_{i=0}^{k}\left[s+v+\lambda\,(n_0+i)\right]}$$

$$= \lim_{s\to-[v+\lambda(n_0+2)]}\left[\frac{\left[s+v+\lambda\,(n_0+2)\right]e^{st}}{(s+v+\lambda n_0)\left[s+v+\lambda\,(n_0+1)\right]\left[s+v+\lambda\,(n_0+2)\right]\prod_{i=3}^{k}\left[s+v+\lambda\,(n_0+i)\right]}\right]$$

$$= \lim_{s\to-[v+\lambda(n_0+2)]}\frac{e^{st}}{(s+v+\lambda n_0)\left[s+v+\lambda\,(n_0+1)\right]\prod_{i=3}^{k}\left[s+v+\lambda\,(n_0+i)\right]}$$

$$= \left[\frac{e^{-[v+\lambda(n_0+2)]t}}{\left[-\left(v+\lambda\,(n_0+2)\right)+v+\lambda n_0\right]\left[-\left(v+\lambda\,(n_0+2)\right)+v+\lambda\,(n_0+1)\right]\prod_{i=3}^{k}\left[-\left(v+\lambda\,(n_0+2)\right)+v+\lambda\,(n_0+i)\right]}\right]$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{[-v-\lambda n_0-2\lambda+v+\lambda n_0]\,[-v-\lambda n_0-2\lambda+v+\lambda n_0+\lambda] \prod_{i=3}^{k} [-v-\lambda n_0-2\lambda+v+\lambda n_0+\lambda i]}$$

$$= \left[\frac{e^{-[v+\lambda(n_0+2)]t}}{[-v-\lambda n_0-2\lambda+v+\lambda n_0]\,[-v-\lambda n_0-2\lambda+v+\lambda n_0+\lambda] \prod_{i=3}^{k} [-v-\lambda n_0-2\lambda+v+\lambda n_0+\lambda i]} \right]$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-2\lambda\,(-\lambda) \prod_{i=3}^{k} (\lambda i-2\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-2\lambda\,(-\lambda) \prod_{i=3}^{k} \lambda\,(i-2)}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{(-1)^2\,2\lambda^2\,(\lambda)^{n-2} \prod_{i=3}^{k} (i-2)}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{(-1)^2\,2\lambda^n \prod_{i=3}^{k} (i-2)}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{(-1)^2\,2!\lambda^k\,(k-2)!}$$

At $s=-\left[v+\lambda\left(n_0+3\right)\right]$

Residue $\left[e^{st}\overline{f}(s)\right]$ is given by

$$Res=\lim_{s\to-[v+\lambda(n_0+3)]}\left[s+v+\lambda\left(n_0+3\right)\right]\frac{e^{st}}{\prod_{i=0}^{k}\left[s+v+\lambda\left(n_0+i\right)\right]}$$

$$=\lim_{s\to-[v+\lambda(n_0+3)]}\left[s+v+\lambda\left(n_0+3\right)\right]\left[\frac{e^{st}}{\prod_{i=0}^{2}\left[s+v+\lambda\left(n_0+i\right)\right]\left[s+v+\lambda\left(n_0+3\right)\right]\prod_{i=3}^{k}\left[s+v+\lambda\left(n_0+i\right)\right]}\right]$$

$$=\lim_{s\to-[v+\lambda(n_0+3)]}\frac{e^{st}}{\prod_{i=0}^{2}\left[s+v+\lambda\left(n_0+i\right)\right]\prod_{i=3}^{k}\left[s+v+\lambda\left(n_0+i\right)\right]}$$

$$=\frac{e^{-[v+\lambda(n_0+3)]t}}{\prod_{i=0}^{2}\left[-\left(v+\lambda\left(n_0+3\right)\right)+v+\lambda\left(n_0+i\right)\right]\prod_{i=3}^{k}\left[-\left(v+\lambda\left(n_0+3\right)\right)+v+\lambda\left(n_0+i\right)\right]}$$

$$=\frac{e^{-[v+\lambda(n_0+3)]t}}{\prod_{i=0}^{2}\left[-v-\lambda n_0-3\lambda+v+\lambda n_0+\lambda i\right]\prod_{i=3}^{k}\left[-v-\lambda n_0-3\lambda+v+\lambda n_0+\lambda i\right]}$$

$$=\frac{e^{-[v+\lambda(n_0+3)]t}}{\prod_{i=0}^{2}\left(\lambda i-3\lambda\right)\prod_{i=3}^{k}\left(\lambda i-3\lambda\right)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{(-3\lambda)(-2\lambda)(-\lambda)\prod_{i=3}^{k}\lambda(i-3)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{(-1)^3(3)(2)\lambda^3\prod_{i=3}^{k}\lambda(i-3)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{(-1)^3(3)(2)\lambda^3\prod_{i=3}^{k}\lambda(i-3)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{(-1)^3 3!\lambda^3(\lambda)^{k-3}\prod_{i=3}^{n}(i-3)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{(-1)^3 3!\lambda^k\prod_{i=3}^{k}(i-3)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{(-1)^2 3!\lambda^k(k-3)!}$$

Generalizing the results above, the residue at the pole $s = -[v+\lambda(n_0+k)]$ is given by;

$$Residue\left[e^{st}\overline{f}(s)\right] = \frac{e^{-[v+\lambda(n_0+k)]t}}{(-1)^n k!\lambda^k(k-k)!}$$

$$= \frac{e^{-[v+\lambda(n_0+k)]t}}{(-1)^n k!\lambda^k}$$

Therefore summing the residues at each pole yields,

$$L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+\left(n_0+i\right)\lambda+v\right]}\right\}=\sum \operatorname{Re} sidue\left(e^{st}\overline{f}(s)\right)$$

$$=\frac{e^{-[v+\lambda n_0]t}}{\lambda^k 0!k!}+\frac{e^{-[v+\lambda(n_0+1)]t}}{(-1)\,\lambda^k 1!\,(k-1)!}+\frac{e^{-[v+\lambda(n_0+2)]t}}{(-1)^2\,\lambda^k 2!\,(k-2)!}$$

$$+\frac{e^{-[v+\lambda(n_0+3)]t}}{(-1)^3\,\lambda^k 3!\,(k-3)!}+...+\frac{e^{-[v+\lambda(n_0+k)]t}}{(-1)^n\,\lambda^k k!\,(k-k)!}$$

$$=\sum_{j=0}^{k}\frac{e^{-[v+\lambda(n_0+j)]t}}{(-1)^j\,\lambda^k j!\,(n-j)!}$$

$$=\frac{1}{\lambda^k}\sum_{j=0}^{k}\frac{e^{-[v+\lambda(n_0+j)]t}}{(-1)^j\,j!\,(n-j)!}$$

But from equation (5.49) we had

$$P_n(t)=\prod_{i=0}^{k-1}\left[\left(n_0+i\right)\lambda+v\right]L^{-1}\left\{\frac{1}{\prod\limits_{i=0}^{k}\left[s+\left(n_0+i\right)\lambda+v\right]}\right\}$$

Thus

$$P_n(t)=\prod_{i=0}^{k-1}\left[\left(n_0+i\right)\lambda+v\right]\left[\frac{1}{\lambda^k}\sum_{j=0}^{k}\frac{e^{-[v+\lambda(n_0+j)]t}}{(-1)^j\,j!\,(n-j)!}\right] \tag{5.50}$$

But

$$\prod_{i=0}^{k-1}\left[\left(n_0+i\right)\lambda+v\right]=\left(n_0\lambda+v\right)\left[\left(n_0+1\right)\lambda+v\right]\left[\left(n_0+2\right)\lambda+v\right]...\left[\left(n_0+k-1\right)\lambda+v\right]$$

$$=\lambda\left(n_0+\frac{v}{\lambda}\right)\lambda\left(\left(n_0+1\right)+\frac{v}{\lambda}\right)\lambda\left(\left(n_0+2\right)+\frac{v}{\lambda}\right)...\lambda\left(\left(n_0+k-1\right)+\frac{v}{\lambda}\right)$$

$$= \lambda^k \left(n_0 + \frac{v}{\lambda}\right) \left((n_0+1) + \frac{v}{\lambda}\right) \left((n_0+2) + \frac{v}{\lambda}\right) ... \left((n_0+k-1) + \frac{v}{\lambda}\right)$$

$$= \lambda^k \left(n_0 + \frac{v}{\lambda}\right) \left(n_0 + 1 + \frac{v}{\lambda}\right) \left(n_0 + 2 + \frac{v}{\lambda}\right) ... \left(n_0 + k - 1 + \frac{v}{\lambda}\right)$$

$$= \lambda^k \left(n_0 + \frac{v}{\lambda}\right) \left[\left(n_0 + \frac{v}{\lambda}\right) + 1\right] \left[\left(n_0 + \frac{v}{\lambda}\right) + 2\right] ... \left[\left(n_0 + \frac{v}{\lambda}\right) + k - 1\right]$$

$$= \lambda^k \prod_{i=0}^{k-1} \left[\left(n_0 + \frac{v}{\lambda}\right) + i\right]$$

Letting

$$u = n_0 + \frac{v}{\lambda}$$

We have

$$\prod_{i=0}^{k-1} \left[(n_0 + i)\lambda + v\right] = \lambda^k \prod_{i=0}^{k-1} (u+i)$$

$$= \lambda^k \left[u(u+1)(u+2)(u+3)...(u+k-2)(u+k-1)\right]$$

But

$$(u+k-1)! = (u+k-1)(u+k-2)...(u+2)(u+1)u(u-1)!$$

$$\Rightarrow \frac{(u+k-1)!}{(u-1)!} = (u+k-1)(u+k-2)...(u+2)(u+1)u$$

$$= \lambda^k \left[u(u+1)(u+2)(u+3)...(u+k-2)(u+k-1)\right]$$

Thus

$$\prod_{i=0}^{k-1} \left[(n_0 + i)\lambda + v\right] = \lambda^k \frac{(u+k-1)!}{(u-1)!} \tag{5.51}$$

Therefore using equation (5.51) in equation (5.50), we get

$$P_n(t) = \prod_{i=0}^{k-1} [(n_0 + i)\lambda + v] \left[\frac{1}{\lambda^k} \sum_{j=0}^{k} \frac{e^{-[v+\lambda(n_0+j)]t}}{(-1)^j \, j! \, (n-j)!} \right]$$

$$= \lambda^k \frac{(k+u-1)!}{(u-1)!} \left[\frac{1}{\lambda^k} \sum_{j=0}^{k} \frac{e^{-[v+\lambda(n_0+j)]t}}{(-1)^j \, j! \, (n-j)!} \right]$$

$$= \frac{(u+k-1)!}{(u-1)!} \sum_{j=0}^{k} \frac{e^{-[v+\lambda(n_0+j)]t}}{(-1)^j \, j! \, (n-j)!}$$

$$= \frac{(u+k-1)! e^{-(v+\lambda n_0)t}}{(u-1)!} \sum_{j=0}^{k} \frac{e^{-\lambda t j}}{(-1)^j \, j! \, (n-j)!}$$

$$= \frac{(u+k-1)! e^{-(v+\lambda n_0)t}}{(u-1)!} \sum_{j=0}^{k} \frac{(-1)^j \, e^{-\lambda t j}}{j! \, (n-j)!}$$

$$= \frac{(u+k-1)! e^{-(v+\lambda n_0)t}}{(u-1)!} \sum_{j=0}^{k} \frac{\left(-e^{-\lambda t}\right)^j}{j! \, (n-j)!}$$

$$= \frac{(u+k-1)! e^{-(v+\lambda n_0)t}}{(u-1)!} \sum_{j=0}^{k} \frac{\left(-e^{-\lambda t}\right)^j}{j! \, (n-j)!} \times \frac{k!}{k!}$$

$$= \frac{(u+k-1)! e^{-(v+\lambda n_0)t}}{k! \, (u-1)!} \sum_{j=0}^{k} \frac{\left(-e^{-\lambda t}\right)^j k!}{j! \, (n-j)!}$$

But

$$\binom{u+k-1}{u-1} = \frac{(u+k-1)!}{k! \, (u-1)!}$$

and

$$\binom{k}{j} = \frac{k!}{(k-j)!j!}$$

But

$$P_n(t) = \frac{(u+k-1)!e^{-(v+\lambda n_0)t}}{k!\,(u-1)!} \sum_{j=0}^{k} \frac{\left(-e^{-\lambda t}\right)^j k!}{j!\,(k-j)!}$$

$$= \binom{u+k-1}{u-1} e^{-(v+\lambda n_0)t} \sum_{j=0}^{k} \binom{k}{j} \left(-e^{-\lambda t}\right)^j$$

From binomial theorem

$$\left(1-e^{-\lambda t}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(-e^{-\lambda t}\right)^j$$

Thus

$$P_n(t) = \binom{u+k-1}{u-1} e^{-(v+\lambda n_0)t} \left(1-e^{-\lambda t}\right)^k$$

But

$$u = n_0 + \frac{v}{\lambda} \Rightarrow \lambda u = \lambda n_0 + v$$

Hence

$$P_n(t) = \binom{u+k-1}{u-1} e^{-\lambda u t} \left(1-e^{-\lambda t}\right)^k$$

$$= \binom{u+k-1}{u-1} \left(e^{-\lambda t}\right)^u \left(1-e^{-\lambda t}\right)^k$$

$$= \binom{n_0+\frac{v}{\lambda}+n-n_0-1}{n_0+\frac{v}{\lambda}-1} \left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}} \left(1-e^{-\lambda t}\right)^k$$

$$= \binom{n_0+\frac{v}{\lambda}+k-1}{k} \left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}} \left(1-e^{-\lambda t}\right)^k \qquad k = 0, 1, 2, ...$$

Which is the pmf of a negative binomial distribution with parameters $p = e^{-\lambda t}$ and $q = 1 - e^{-t\lambda}$

Partial Fractions Method

By equation (5.49) we had

$$P_n(t) = \prod_{i=0}^{k-1} [(n_0 + i)\lambda + v] L^{-1} \left\{ \frac{1}{\prod_{i=0}^{k} [s + (n_0 + i)\lambda + v]} \right\}$$

But from equation (5.51)

$$\prod_{i=0}^{k-1} [(n_0 + i)\lambda + v] = \lambda^k \frac{(u + k - 1)!}{(u - 1)!}$$

Where

$$u = n_0 + \frac{v}{\lambda}$$

Therefore

$$P_n(t) = \prod_{i=0}^{k-1} [(n_0 + i)\lambda + v] L^{-1} \left\{ \frac{1}{\prod_{i=0}^{k} [s + (n_0 + i)\lambda + v]} \right\}$$

$$= \lambda^k \frac{(u + k - 1)!}{(u - 1)!} L^{-1} \left\{ \frac{1}{\prod_{i=0}^{k} [s + (n_0 + i)\lambda + v]} \right\}$$

We now determine

$$L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}\left[s+\left(n_0+i\right)\lambda+v\right]}\right\}$$

But

$$\frac{1}{\prod_{i=0}^{k}\left[s+\lambda\left(n_0+i\right)+v\right]}=\left\{\begin{array}{c}\left(\frac{1}{s+\lambda n_0+v}\right)\left(\frac{1}{s+\lambda(n_0+1)+v}\right)\cdots\left(\frac{1}{s+\lambda(j-1)+v}\right)\\ \\ \left(\frac{1}{s+\lambda j+v}\right)\left(\frac{1}{s+\lambda(j+1)+v}\right)\cdots\left(\frac{1}{s+\lambda(n_0+k)+v}\right)\end{array}\right\}$$

Using partial fractions

$$\left\{\begin{array}{c}\left(\frac{1}{s+\lambda n_0+v}\right)\left(\frac{1}{s+\lambda(n_0+1)+v}\right)\cdots\\ \\ \cdots\left(\frac{1}{s+\lambda(j-1)+v}\right)\left(\frac{1}{s+\lambda j+v}\right)\\ \\ \left(\frac{1}{s+\lambda(j+1)+v}\right)\cdots\left(\frac{1}{s+\lambda(n_0+k)+v}\right)\end{array}\right\}=\left\{\begin{array}{c}\frac{a_{n_0}}{s+\lambda n_0+v}+\frac{a_{n_0+1}}{s+\lambda(n_0+1)+v}+\\ \\ \frac{a_{n_0+1}}{s+\lambda(n_0+2)+v}+\ldots+\frac{a_{j-1}}{s+\lambda(j-1)+v}\\ \\ +\frac{a_j}{s+\lambda j+v}+\frac{a_{j+1}}{s+\lambda(j+1)+v}+\\ \\ \frac{a_{j+1}}{s+\lambda(j+2)+v}+\ldots+\frac{a_{n_0+k}}{s+\lambda(n_0+k)+v}\end{array}\right\}$$

Multiplying both sides by $s+\lambda j+v$ yields

$$\left\{\begin{array}{c} \left(\frac{1}{s+\lambda n_0+v}\right)\left(\frac{1}{s+\lambda(n_0+1)+v}\right)\cdots \\ \cdots\left(\frac{1}{s+\lambda(j-1)+v}\right)\left(\frac{s+\lambda j+v}{s+\lambda j+v}\right) \\ \left(\frac{1}{s+\lambda(j+1)+v}\right)\cdots\left(\frac{1}{s+\lambda(n_0+k)+v}\right) \end{array}\right\} = \left\{\begin{array}{c} \frac{a_{n_0}(s+\lambda j+v)}{s+\lambda n_0+v}+\frac{a_{n_0+1}(s+\lambda j+v)}{s+\lambda(n_0+1)+v}+ \\ \frac{a_{n_0+1}(s+\lambda j+v)}{s+\lambda(n_0+2)+v}+\ldots\frac{a_{j-1}(s+\lambda j+v)}{s+\lambda(j-1)+v} \\ +\frac{a_j(s+\lambda j+v)}{s+\lambda j+v}+\frac{a_{j+1}(s+\lambda j+v)}{s+\lambda(j+1)+v}+ \\ \frac{a_{j+1}(s+\lambda j+v)}{s+\lambda(j+2)+v}+\ldots+\frac{a_{n_0+k}(s+\lambda j+v)}{s+\lambda(n_0+k)+v} \end{array}\right\}$$

$$\left\{\begin{array}{c} \left(\frac{1}{s+\lambda n_0+v}\right)\left(\frac{1}{s+\lambda(n_0+1)+v}\right)\cdots \\ \left(\frac{1}{s+\lambda(n_0+2)+v}\right)\cdots\left(\frac{1}{s+\lambda(j-1)+v}\right) \\ \left(\frac{1}{s+\lambda(j+1)+v}\right)\cdots\left(\frac{1}{s+\lambda(n_0+k)+v}\right) \end{array}\right\} = \left\{\begin{array}{c} \frac{a_{n_0}(s+\lambda j+v)}{s+\lambda n_0+v}+\frac{a_{n_0+1}(s+\lambda j+v)}{s+\lambda(n_0+1)+v}+ \\ \frac{a_{n_0+1}(s+\lambda j+v)}{s+\lambda(n_0+2)+v}+\ldots \\ +\frac{a_{j-1}(s+\lambda j+v)}{s+\lambda(j-1)+v}+a_j \\ +\frac{a_{j+1}(s+\lambda j+v)}{s+\lambda(j+1)+v}+\ldots+\frac{a_{n_0+k}(s+\lambda j+v)}{s+\lambda(n_0+k)+v} \end{array}\right\}$$

Setting $s=-(\lambda j+v)$, implies

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda j-v+\lambda n_0+v}\right)\left(\frac{1}{-\lambda j-v+\lambda(n_0+1)+v}\right)\\ \left(\frac{1}{-\lambda j-v+\lambda(n_0+2)+v}\right)\cdots\left(\frac{1}{-\lambda j-v+\lambda(j-2)+v}\right)\\ \left(\frac{1}{-\lambda j-v+\lambda(j-1)+v}\right)\left(\frac{1}{-\lambda j-v+\lambda(j+1)+v}\right)\\ \left(\frac{1}{-\lambda j-v+\lambda(j+2)+v}\right)\cdots\cdots\left(\frac{1}{-\lambda j-v+\lambda(n_0+k)+v}\right)\end{array}\right\}=\left\{\begin{array}{c}\frac{a_{n_0}(-\lambda j-v+\lambda j+v)}{-\lambda j-v+\lambda n_0+v}+\\ \frac{a_{n_0+1}(-\lambda j-v+\lambda j+v)}{-\lambda j-v+\lambda(n_0+1)+v}+\ldots\\ +\frac{a_{j-1}(-\lambda j-v+\lambda j+v)}{-\lambda j-v+\lambda(j-1)+v}\\ +a_j+\\ \frac{a_{j+1}(-\lambda j-v+\lambda j+v)}{-\lambda j-v+\lambda(j+1)+v}+\ldots\\ +\frac{a_{n_0+k}(-\lambda j-v+\lambda j+v)}{-\lambda j-v+\lambda(n_0+k)+v}\end{array}\right\}$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda j-v+\lambda n_0+v}\right)\left(\frac{1}{-\lambda j-v+\lambda(n_0+1)+v}\right)\\ \left(\frac{1}{-\lambda j-v+\lambda(n_0+2)+v}\right)\cdots\left(\frac{1}{-\lambda j-v+\lambda(j-2)+v}\right)\\ \left(\frac{1}{-\lambda j-v+\lambda(j-1)+v}\right)\times\left(\frac{1}{-\lambda j-v+\lambda(j+1)+v}\right)\\ \left(\frac{1}{-\lambda j-v+\lambda(j+2)+v}\right)\cdots\cdots\left(\frac{1}{-\lambda j-v+\lambda(n_0+k)+v}\right)\end{array}\right\}=a_j$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda j-v+\lambda n_0+v}\right)\left(\frac{1}{-\lambda j-v+\lambda n_0+\lambda+v}\right)\cdots\\ \\ \left(\frac{1}{-\lambda j-v+\lambda j-2\lambda+v}\right)\left(\frac{1}{-\lambda j-v+\lambda j-\lambda+v}\right)\times\\ \\ \left(\frac{1}{-\lambda j-v+\lambda j+\lambda+v}\right)\left(\frac{1}{-\lambda j-v+\lambda j+2\lambda+v}\right)\cdots\left(\frac{1}{-\lambda j-v+\lambda n_0+\lambda k+v}\right)\end{array}\right\}=a_j$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda j+\lambda n_0}\right)\left(\frac{1}{-\lambda j+\lambda n_0+\lambda}\right)\cdots\left(\frac{1}{-2\lambda}\right)\left(\frac{1}{-\lambda}\right)\\ \\ \times\left(\frac{1}{\lambda}\right)\left(\frac{1}{2\lambda}\right)\cdots\left(\frac{1}{-\lambda j+\lambda n_0+\lambda k+}\right)\end{array}\right\}=a_j$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda(j-n_0)}\right)\left(\frac{1}{-\lambda(j-(n_0+1))}\right)\cdots\left(\frac{1}{-\lambda(2)}\right)\left(\frac{1}{-\lambda(1)}\right)\\ \\ \times\left(\frac{1}{\lambda(1)}\right)\left(\frac{1}{\lambda(2)}\right)\cdots\left(\frac{1}{\lambda((n_0+k)-j)}\right)\end{array}\right\}=a_j$$

$$\left\{\begin{array}{c}\frac{1}{(-\lambda)^{j-n_0}}\left(\frac{1}{j-n_0}\right)\left(\frac{1}{j-(n_0+1)}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{1}{1}\right)\\ \\ \times\frac{1}{\lambda^{n_0+k-j}}\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{(n_0+k)-j}\right)\end{array}\right\}=a_j$$

$$\frac{1}{(-\lambda)^{j-n_0}(j-n_0)!}\times\frac{1}{\lambda^{n_0+k-j}(n_0+k-j)!}=a_j$$

$$\Rightarrow a_j=\frac{1}{(-1)^{j-n_0}\lambda^{j-n_0}(j-n_0)!}\times\frac{1}{\lambda^{n_0+k-j}(n_0+k-j)!}$$

$$=\frac{1}{(-1)^{j-n_0}\lambda^{j-n_0}(j-n_0)!\lambda^{n_0+k-j}(n_0+k-j)!}$$

$$=\frac{(-1)^{j-n_0}}{\lambda^{j-n_0+n_0+k-j}(j-n_0)!\,(n_0+k-j)!}$$

$$=\frac{(-1)^{j-n_0}}{\lambda^{k}(j-n_0)!\,(n_0+k-j)!}$$

$$=\left(\frac{1}{\lambda}\right)^k\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}\qquad j=n_0,\ n_0+1,\ n_0+2,$$

With this it follows that

$$\frac{1}{\prod_{i=0}^{k}\left[s+\lambda(n_0+i)+v\right]}=\sum_{j=n_0}^{n_0+k}\frac{a_j}{s+j\lambda+v}$$

$$=\sum_{j=n_0}^{n_0+k}\left(\frac{1}{\lambda}\right)^k\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}\left(\frac{1}{s+j\lambda+v}\right)$$

Thus

$$P_n(t) = \lambda^k \frac{(u+k-1)!}{(u-1)!} L^{-1} \left\{ \frac{1}{\prod_{i=0}^{k} \left[s + (n_0+i)\lambda + v\right]} \right\}$$

$$= \lambda^k \frac{(u+k-1)!}{(u-1)!} L^{-1} \left\{ \sum_{j=n_0}^{n_0+k} \left(\frac{1}{\lambda}\right)^k \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} \left(\frac{1}{s+j\lambda+v}\right) \right\}$$

$$= \lambda^k \frac{(u+k-1)!}{(u-1)!} \left(\frac{1}{\lambda}\right)^k \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} L^{-1} \left\{ \frac{1}{s+j\lambda+v} \right\}$$

$$= \frac{(u+k-1)!}{(u-1)!} \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!} L^{-1} \left\{ \frac{1}{s+j\lambda+v} \right\}$$

But

$$L\left\{e^{-(\lambda j+v)t}\right\} = \int_0^\infty e^{-st} e^{-(\lambda j+v)t} dt$$

$$= \int_0^\infty e^{-(\lambda j+v+s)t} dt$$

$$= \left. \frac{e^{-(\lambda j+v+s)t}}{-(\lambda j+v+s)} \right|_0^\infty$$

$$= \frac{1}{-(\lambda j+v+s)} \left[0-1\right]$$

$$= \frac{1}{s+j\lambda+v}$$

$$\Rightarrow e^{-(\lambda j+v)t} = L^{-1}\left\{\frac{1}{s+j\lambda+v}\right\}$$

Thus

$$P_n(t) = \frac{(u+k-1)!}{(u-1)!}\sum_{j=n_0}^{n_0+k}\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}L^{-1}\left\{\frac{1}{s+j\lambda+v}\right\}$$

$$= \frac{(u+k-1)!}{(u-1)!}\sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}\right]\ e^{-(j\lambda+v)t}$$

Multiplying the above equation by

$$\frac{k!}{k!}$$

yields;

$$P_n(t) = \frac{(u+k-1)!}{(u-1)!}\sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}\,k!}{(j-n_0)!\,(n_0+k-j)!}\right]\ e^{-(j\lambda+v)t}\ \times\frac{k!}{k!}$$

$$= \frac{(u+k-1)!}{(u-1)!k!}\sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}\,k!}{(j-n_0)!\,(n_0+k-j)!}\right]\ e^{-(j\lambda+v)t}$$

$$= \binom{u+k-1}{k}\sum_{j=n_0}^{n_0+k}(-1)^{j-n_0}\binom{k}{j-n_0}\ e^{-(j\lambda+v)t}$$

$$= \binom{u+k-1}{u-1}\sum_{j=n_0}^{n_0+k}(-1)^{j-n_0}\binom{k}{j-n_0}\ e^{-(j\lambda+v)t}$$

Letting $m = j - n_0 \Rightarrow j = m + n_0$ implies

$$P_n(t) = \binom{u+k-1}{u-1} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \; e^{-[(m+n_0)\lambda+v]t}$$

$$= \binom{u+k-1}{u-1} \; e^{-(\lambda n_0+v)t} \sum_{m=0}^{k} (-1)^m \binom{k}{m} e^{-\lambda tm}$$

$$= \binom{u+k-1}{u-1} e^{-(\lambda n_0+v)t} \sum_{m=0}^{k} \binom{k}{m} \left(-e^{-\lambda t}\right)^m$$

But

$$u = n_0 + \frac{v}{\lambda},\; n = n_0 + k \Rightarrow k = n - n_0$$

implying

$$P_n(t) = \binom{n_0+\frac{v}{\lambda}+k-1}{n_0+\frac{v}{\lambda}-1} e^{-(\lambda n_0+v)t} \left(1-e^{-\lambda t}\right)^{n-n_0} \quad n = n_0,\; n_0+1,\; n_0+2, ...$$

$$= \binom{n_0+\frac{v}{\lambda}+k-1}{n_0+\frac{v}{\lambda}-1} e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)} \left(1-e^{-\lambda t}\right)^{k} \quad k = 0,\; 1,\; 2, ...$$

$$= \binom{u+k-1}{u-1} \left(e^{-\lambda t}\right)^u \left(1-e^{-\lambda t}\right)^{k} \quad k = 0,\; 1,\; 2, ...$$

Which is a negative binomial distribution with $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

Method 2: Using the pgf approach

The basic difference-differential equations for the simple process birth process as defined by equations (5.46a) and (5.46b) are;

$$P_0'(t) = -vP_0(t) \tag{5.46a}$$

$$P_n'(t) = \left[(n-1)\lambda + v\right] P_{n-1}(t) - (n\lambda+v)P_n(t)n \quad \geqslant 1 \tag{5.46b}$$

Multiplying both sides of equation (5.46b) by z^n and summing the result over n, We have

$$\left.\begin{aligned}\sum_{n=1}^{\infty} P_n'(t)z^n &= \sum_{n=1}^{\infty}\left[(n-1)\lambda+v\right]P_{n-1}(t)z^n-\sum_{n=1}^{\infty}(n\lambda+v)P_n(t)\ z^n\\ &=\underbrace{\lambda\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)z^n}_{I}+\underbrace{v\sum_{n=1}^{\infty}P_{n-1}(t)z^n}_{II}-\underbrace{\lambda\sum_{n=1}^{\infty}nP_n(t)z^n}_{III}\\ &\quad\underbrace{-v\sum_{n=1}^{\infty}P_n(t)\ z^n}_{IV}\end{aligned}\right\} \quad 5.52$$

Let $G(z,t)$ be the probability generating function of $X(t)$ defined as follows;

$$\begin{aligned}G(z,t) &= \sum_{n=0}^{\infty}P_n(t)z^n\\ &= P_0(t)+\sum_{n=1}^{\infty}P_n(t)z^n\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}G(z,t) &= \sum_{n=0}^{\infty}P_n'(t)z^n\\ &= P_0'(t)+\sum_{n=1}^{\infty}P_n'(t)z^n\end{aligned}$$

$$\sum_{n=1}^{\infty}P_n'(t)z^n = \frac{\partial}{\partial z}G(z,t)-P_0'(t)$$

$$\begin{aligned}\frac{\partial}{\partial z}G(z,t) &= \sum_{n=0}^{\infty}nP_n(t)z^{n-1}\\ &= \frac{1}{z}\sum_{n=1}^{\infty}nP_n(t)z^n\\ &= \sum_{n=1}^{\infty}(n-1)P_n(t)z^{n-2}\end{aligned}$$

We now simplify the four parts of equation (5.52) separately

Part I

$$\lambda \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) z^n = \lambda z^2 \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) z^{n-2}$$
$$= \lambda z^2 \frac{\partial}{\partial z} G(z,t)$$

Part II

$$v \sum_{n=1}^{\infty} P_{n-1}(t) z^n = vz \sum_{n=1}^{\infty} P_{n-1}(t) z^{n-1}$$
$$= vzG(z,t)$$

Part III

$$\lambda \sum_{n=1}^{\infty} n P_n(t) z^n = \lambda z \sum_{n=1}^{\infty} n P_n(t) z^{n-1}$$
$$= \lambda z \frac{\partial}{\partial z} G(z,t)$$

Part IV

$$v \sum_{n=1}^{\infty} P_n(t)\ z^n = v\left[G(z,t) - P_0(t)\right]$$
$$= vG(z,t) - vP_0(t)$$

Substituting the above in equation (5.52) yields

$$\frac{\partial}{\partial t} G(z,t) - P_0'(t) = \lambda z^2 \frac{\partial}{\partial z} G(z,t) + vzG(z,t) - \lambda z \frac{\partial}{\partial z} G(z,t) - vG(z,t) - vP_0(t)$$

But by equation (5.46a)

$$P_0'(t) = -vP_0(t)$$

Hence we now have

$$\frac{\partial}{\partial t}G(z,t) + vP_0(t) = \lambda z^2 \frac{\partial}{\partial z}G(z,t) + vzG(z,t) - \lambda z \frac{\partial}{\partial z}G(z,t)$$
$$- vG(z,t) + vP_0(t)$$

$$\frac{\partial}{\partial t}G(z,t) = \lambda z^2 \frac{\partial}{\partial z}G(z,t) - \lambda z \frac{\partial}{\partial z}G(z,t) +$$
$$vzG(z,t) - vG(z,t)$$

Grouping the like terms together yields

$$\frac{\partial}{\partial t}G(z,t) = \lambda z\,(z-1)\,\frac{\partial}{\partial z}G(z,t) + v\,(z-1)\,G(z,t)$$

$$\Rightarrow \frac{\partial}{\partial t}G(z,t) - \lambda z\,(z-1)\,\frac{\partial}{\partial z}G(z,t) - v\,(z-1)\,G(z,t) = 0 \tag{5.53}$$

Which is a partial differential equation. Appying Laplace transform to both sides yields.

$$L\left\{\frac{\partial}{\partial t}G(z,t) - \lambda z\,(z-1)\,\frac{\partial}{\partial z}G(z,t) - v\,(z-1)\,G(z,t)\right\} = L\{0\}$$

$$L\left\{\frac{\partial}{\partial t}G(z,t)\right\} - \lambda z\,(z-1)\,L\left\{\frac{\partial}{\partial z}G(z,t)\right\} - v(z-1)L\{G(z,t)\} = 0$$

But by the definition of Laplace transform

$$L\left\{\frac{\partial}{\partial t}G(z,t)\right\} = \overline{G}\,(z,s)$$
$$= \int_0^\infty e^{-st}\frac{\partial}{\partial t}G(z,t)dt$$

Let $u = e^{-st} \Rightarrow du = -se^{-st}$

and

$$dv = \frac{\partial}{\partial t}G(z,t)dt \Rightarrow v = G(z,t)$$

Thus

$$L\left\{\frac{\partial}{\partial t}G(z,t)\right\} = e^{-st}G(z,t)\Big|_0^\infty - \int\limits_0^\infty -se^{-st}G(z,t)dt$$

$$= [0 - G(z,t)] + s\underbrace{\int\limits_0^\infty e^{-st}G(z,t)}_{\overline{G}(z,s)}$$

$$= s\overline{G}(z,s) - G(z,0)$$

Similarly

$$L\left\{\frac{\partial}{\partial z}G(z,t)\right\} = \int\limits_0^\infty e^{-st}\frac{\partial}{\partial z}G(z,t)dt$$

By Leibnitz rule for differentiating under an integral, we have

$$L\left\{\frac{\partial}{\partial z}G(z,t)\right\} = \frac{d}{dz}\underbrace{\int\limits_0^\infty e^{-st}G(z,t)dt}_{\overline{G}(z,s)}$$

$$= \frac{d}{dz}\overline{G}(z,s)$$

With this now equation (5.53) becomes

$$s\overline{G}(z,s) - G(z,0) - \lambda z\,(z-1)\,\frac{d}{dz}\overline{G}(z,s) - v(z-1)\overline{G}(z,s) = 0$$

$$[s - v(z-1)]\,\overline{G}(z,s) - \lambda z\,(z-1)\,\frac{d}{dz}\overline{G}(z,s) = G(z,0)$$

$$-\lambda z\,(z-1)\,\frac{d}{dz}\overline{G}(z,s) + [s - v(z-1)]\,\overline{G}(z,s) = G(z,0)$$

$$\frac{d}{dz}\overline{G}\,(z,s) - \frac{[s - v\,(z-1)]}{\lambda z\,(z-1)}\overline{G}\,(z,s) = -\frac{G\,(z,0)}{\lambda z\,(z-1)} \tag{5.54a}$$

Which is an ordinary differential equation.

The next task is to find a solution to this differential equation. Two methods are considered.

- Dirac delta function approach
- Hyper geometric function approach

Dirac delta function approach

Equation (5.54a) is an ODE of 1^{st} order. It is of the form $y' + Py = Q$ where

$$y = \overline{G}(z,s)\,, P = -\frac{s - v(z-1)}{\lambda z\,(z-1)} \text{ and } Q = -\frac{G(z,0)}{\lambda z\,(z-1)}$$

Now the solution formula for such an ODE is given by

$$ye^{\int Pdz} = f + \int Qe^{\int Pdz}$$

Where f does not depend on the variable z but can depend on the s parameter which implies that $f = f(s)$, in our case y is a function of z and s ,that is $y = y(z,s)$.
$e^{\int Pdz}$ is called the integrating factor. We first calculate the P integral.

$$\begin{aligned}
\int Pdz &= \int -\frac{s - v(z-1)}{\lambda z\,(z-1)}dz \\
&= -\frac{1}{\lambda}\int \frac{s - v(z-1)}{z\,(z-1)}dz \\
&= -\frac{1}{\lambda}\left[\int \frac{s}{z\,(z-1)}dz - \int \frac{v(z-1)}{z\,(z-1)}dz\right] \\
&= -\frac{1}{\lambda}\left[s\int \frac{1}{z\,(z-1)}dz - v\int \frac{1}{z}dz\right] \\
&= -\frac{1}{\lambda}\left[s\int \frac{1}{z\,(z-1)}dz - v\ln|z|\right]
\end{aligned}$$

But using partial fractions

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

Multiplying both sides by $z(z-1)$ yields $1 = A(z-1) + Bz$ which holds for all values of z . Setting $z = 0$, we have

$$\begin{aligned} 1 &= A(0-1) + B(0) \\ \Rightarrow A &= -1 \end{aligned}$$

Setting $z = 1$, we have

$$\begin{aligned} 1 &= A(1-1) + B(1) \\ \Rightarrow B &= 1 \end{aligned}$$

Thus

$$= -\frac{1}{z} + \frac{1}{z-1}$$

$$\begin{aligned} \int \frac{1}{z(z-1)} dz &= \int \left[-\frac{1}{z} + \frac{1}{z-1} \right] dz \\ &= \int -\frac{1}{z} dz + \int \frac{1}{z-1} dz \\ &= -\int \frac{1}{z} dz + \int \frac{1}{z-1} dz \\ &= -\ln|z| + \ln|z-1| \\ &= \ln\left|\frac{z-1}{z}\right| \end{aligned}$$

With this P integral becomes

$$\begin{aligned} \int P dz &= -\frac{1}{\lambda}\left[s \ln\left|\frac{z-1}{z}\right| - v \ln|z| \right] \\ &= -\frac{s}{\lambda} \ln\left|\frac{z-1}{z}\right| + \frac{v}{\lambda} \ln|z| \end{aligned}$$

Thus integrating factor becomes

$$
\begin{aligned}
I.F = e^{\int P dz} &= e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right| + \frac{v}{\lambda}\ln|z|} \\
&= e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right| + \frac{v}{\lambda}\ln|z|} \\
&= e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} e^{\frac{v}{\lambda}\ln|z|} \\
&= e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} e^{\ln|z|^{\frac{v}{\lambda}}} \\
&= |z|^{\frac{v}{\lambda}} e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}
\end{aligned}
$$

From above the general solution of the ODE is given by

$$
ye^{\int P dz} = f + \int Qe^{\int P dz}
$$

So substituting y,P and Q we have

$$
\overline{G}\,(z,s)\,|z|^{\frac{v}{\lambda}}\, e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) + \int -\frac{G(z,0)}{\lambda z\,(z-1)}\,|z|^{\frac{v}{\lambda}}\, e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} dz
$$

$$
\Rightarrow \overline{G}\,(z,s)\,|z|^{\frac{v}{\lambda}}\, e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{G(z,0)}{\lambda z\,(z-1)}\,|z|^{\frac{v}{\lambda}}\, e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} dz \tag{5.54b}
$$

We consider two cases

1. When $X(0) = 1$
2. When $X(0) = n_0$

Case 1: When initial population $X(0) = 1$

Recall that

$$
\begin{aligned}
G(z,t) &= \sum_{n=0}^{\infty} P_n(t) z^n \\
\Rightarrow G(z,0) &= \sum_{n=0}^{\infty} P_n(0) z^n \\
&= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...
\end{aligned}
$$

but for the initial condition $X(0) = 1$, we have

$$P_1(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq 1$$

$$\therefore G(z, 0) = z$$

With this equation (5.54b) becomes

$$\overline{G}\,(z,s)\,|z|^{\frac{v}{\lambda}}\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{z}{\lambda z\,(z-1)}\,|z|^{\frac{v}{\lambda}}\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}dz$$

$$\overline{G}\,(z,s)\,|z|^{\frac{v}{\lambda}}\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{1}{\lambda\,(z-1)}\,|z|^{\frac{v}{\lambda}}\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}dz \qquad (5.55a)$$

Now this looks like a equation to solve, but at this juncture we can licitly apply the inverse Laplace transform to both sides.

We observe that $f(s)$ can be regarded as a Laplace transform of some unknown function$f(t)$

Applying inverse Laplace transform to both sides of equation (5.55a) we have from the table of Laplace transform in chapter 2 or examples 7 and 8 (see pages 38-39)

1. e^{-cs} is the Laplace transform of the Dirac delta function $\delta(t-c)$

2. $\overline{G}(z,s)e^{cs}$ is the Laplace transform of $G(t-c)\eta(t-c)$ where η is the Heaviside step function.

In our case $c = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$
With this equation (5.55a) can be rewritten as

$$\Rightarrow \overline{G}\,(z,s)\,|z|^{\frac{v}{\lambda}}\,e^{-cs} = f(s) - \int \frac{1}{\lambda\,(z-1)}\,|z|^{\frac{v}{\lambda}}\,e^{-cs}dz$$

So all together inversely transforming both sides the above equation, we come to

$$G(z, t-c)\,|z|^{\frac{v}{\lambda}}\,\eta(t-c) = f(t) - \int \frac{1}{\lambda\,(z-1)}\,|z|^{\frac{v}{\lambda}}\delta(t-c)dz$$

Substituting the value of c we obtain

$$G\left(z,t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)|z|^{\frac{v}{\lambda}}\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)=f(t)-\int\frac{1}{\lambda\left(z-1\right)}|z|^{\frac{v}{\lambda}}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz \tag{5.55b}$$

Now this is a pretty large equation but it can be simplified, but to do that we need to make one big detour. We can free ourselves of this terrible integral by using the Dirac delta function, but what we have is delta of function of variable z , so we need to first simplify it to a common delta of variable. To do so, we use the following property of delta function

$$\delta\left(g\left(z\right)\right)=\sum_{z_i}\frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right) \tag{5.56}$$

Where $g'\left(z\right)$ is the first derivative of $g\left(z\right)$, z_i is a simple root of $g\left(z\right)$ such that $g'\left(z_i\right)$. In our case $g\left(z\right)=t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$. To obtain the roots of $g(z)$ we solve $g\left(z\right)=0$

$$\begin{aligned}
&\Rightarrow t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|=0\\
&\Rightarrow t=\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\\
&\Rightarrow t\lambda=\ln\left|\frac{z-1}{z}\right|\\
&\Rightarrow e^{t\lambda}=\left|\frac{z-1}{z}\right|\\
&\Rightarrow e^{t\lambda}=\pm\frac{z-1}{z}
\end{aligned}$$

Case 1

$$\begin{aligned}
e^{t\lambda}&=\frac{z-1}{z}\\
ze^{t\lambda}&=z-1\\
1&=z-ze^{t\lambda}\\
1&=z\left(1-e^{t\lambda}\right)\\
z_1&=\frac{1}{1-e^{t\lambda}}
\end{aligned}$$

Case 2

$$\begin{aligned}
e^{t\lambda} &= -\frac{z-1}{z} \\
ze^{t\lambda} &= -\left(z-1\right) \\
1 &= z + ze^{t\lambda} \\
1 &= z\left(1+e^{t\lambda}\right) \\
z_2 &= \frac{1}{1+e^{t\lambda}}
\end{aligned}$$

Therefore

$$z_i = \frac{1}{1 \pm e^{t\lambda}}$$

The next step is to determine

$$\begin{aligned}
g'\left(z\right) &= \frac{d}{dz}\left[t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right] \\
&= -\frac{1}{\lambda}\frac{d}{dz}\ln\left|\frac{z-1}{z}\right|
\end{aligned}$$

Using the property $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

$$g'\left(z\right) = -\frac{1}{\lambda}\left(\frac{\frac{d}{dz}\left|\frac{z-1}{z}\right|}{\left|\frac{z-1}{z}\right|}\right)$$

$$g'\left(z\right) = -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \tag{5.57}$$

Now remembering that $|x| = x\,\mathrm{sgn}\left(x\right)$ and $\mathrm{sgn}\left(x\right) = 2\eta(x) - 1$, it follows that

$$\left|\frac{z}{z-1}\right| = \frac{z}{z-1}\,\mathrm{sgn}\left(\frac{z}{z-1}\right) \tag{5.58}$$

Also

$$\left|\frac{z-1}{z}\right| = \frac{z-1}{z}\operatorname{sgn}\left(\frac{z-1}{z}\right)$$
$$= \frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right)-1\right]$$

Where sgn is the sign distribution and eta is the Heaviside distribution .Hence

$$\left|\frac{z-1}{z}\right|' = \frac{d}{dz}\left|\frac{z-1}{z}\right|$$
$$= \frac{d}{dz}\left\{\frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right)-1\right]\right\} \tag{5.59}$$

At this step we need to recall that

$$\eta'_{h(z)} = h'(z)\frac{\partial\eta(z)}{\partial h}$$

Using product rule of differentiation, we simplify equation (5.59) as follows Let

$$U = \frac{z-1}{z} = 1-\frac{1}{z} \Rightarrow U' = \frac{1}{z^2}$$
$$V = 2\eta\left(\frac{z-1}{z}\right)-1$$
$$= 2\eta\left(1-\frac{1}{z}\right)-1 \Rightarrow V' = \frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)$$

We now have

$$\left|\frac{z-1}{z}\right|' = U'V + V'U$$
$$= \frac{1}{z^2}\underbrace{\left[2\eta\left(\frac{z-1}{z}\right)-1\right]}_{\operatorname{sgn}\left(\frac{z-1}{z}\right)} + \frac{z-1}{z}\left[\frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)\right]$$
$$= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\eta'\left(1-\frac{1}{z}\right)$$
$$= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(1-\frac{1}{z}\right)$$

Since $\eta'(x) = \delta(x)$
But

$$\delta(g(z)) = \sum_{z_i} \frac{1}{|g'(z_i)|} \delta(z - z_i)$$

Let

$$g(z) = 1 - \frac{1}{z}$$
$$g'(z) = \frac{1}{z^2}$$

Solving for roots of $g(z)$, we have

$$g(z) = 0$$
$$\Rightarrow 1 - \frac{1}{z} = 0$$
$$1 = \frac{1}{z} \Rightarrow z = 1$$
$$g'(z) = \frac{1}{1^2}$$
$$\therefore \delta\left(1 - \frac{1}{z}\right) = \delta(z - 1)$$

Hence equation (5.59) becomes

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\,\mathrm{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta(z-1) \tag{5.60}$$

Substituting equations (5.58) and (5.60) in equation (5.57) we get

$$g'(z) = -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right)$$

$$= -\frac{1}{\lambda}\left\{\frac{z}{z-1}\,\mathrm{sgn}\left(\frac{z}{z-1}\right)\left[\frac{1}{z^2}\,\mathrm{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta(z-1)\right]\right\}$$

$$= -\frac{1}{\lambda}\left[\frac{1}{z^2}\left(\frac{z}{z-1}\right)\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)+2\left(\frac{z}{z-1}\right)\left(\frac{z-1}{z^3}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)+\frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

Recalling that we obtained the roots of $g\left(z\right)$ as

$$z_1 = \frac{1}{1+e^{t\lambda}}, z_2 = \frac{1}{1-e^{t\lambda}}$$

And the Dirac delta function is zero everywhere except when its argument is zero.

$$\arg\delta\left(z-1\right) = z-1$$

For both roots, the delta term in

$$g'\left(z\right)$$

vanishes because for it to exist it requires that argument of delta function must be zero.

That is

$$0 = z_i - 1 = \frac{1}{1\pm e^{t\lambda}} - 1$$

which is equivalent to saying $e^{t\lambda} = 0$ where both λ and t are greater than zero. This can never be true in Mathematics so only one term remains after neglecting delta in $g'\left(z_i\right)$. Therefore

$$g'\left(z_1\right) = -\frac{1}{\lambda}\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)+\frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

But

$$\operatorname{sgn}\left(x\right)\operatorname{sgn}\left(y\right) = \operatorname{sgn}\left(xy\right)$$

$$\begin{aligned}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) &= \operatorname{sgn}\left[\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right)\right]\\ &= \operatorname{sgn}\left(1\right)\\ &= 1\end{aligned}$$

This implies that

$$g'(z_1) = -\frac{1}{\lambda}\left[\frac{1}{z_1(z_1-1)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1+e^{\lambda t}}\left(\frac{1}{1+e^{\lambda t}}-1\right)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1}{1+e^{\lambda t}}-1}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1-\left(1+e^{\lambda t}\right)}{1+e^{\lambda t}}}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{1-(1+e^{\lambda t})}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{-e^{\lambda t}}\right]$$

$$= \frac{\left(1+e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}$$

Similarly

$$g'(z_2) = -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1-e^{\lambda t}}\left(\frac{1}{1-e^{\lambda t}}-1\right)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1}{1-e^{\lambda t}}-1}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1-\left(1-e^{\lambda t}\right)}{1-e^{\lambda t}}}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{1-\left(1-e^{\lambda t}\right)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{e^{\lambda t}}\right]$$

$$= -\frac{\left(1-e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}$$

Therefore

$$g'\left(z_i\right) = \pm\left(1 \pm e^{\lambda t}\right)^2 \frac{1}{\lambda e^{\lambda t}}$$

But $g'\left(z_i\right)$ must not be zero, and for the initial condition that is at $t = 0$ the negative root makes $g'\left(z_i\right)$ to be singular $[g'\left(z_i\right) = 0]$ so that root needs to be discarded in the summation only one root of z_i is left and will be from now on $z_i = z_1$

It is now time to go back to equation (5.56)

$$\begin{aligned}
\delta\left(g\left(z\right)\right) &= \sum_{z_i} \frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z - z_i\right) \\
&= \frac{1}{\left|g'\left(z_1\right)\right|}\delta\left(z - z_1\right) \\
&= \frac{1}{\left|(1+e^{\lambda t})^2 \frac{1}{\lambda e^{\lambda t}}\right|}\delta\left(z - \frac{1}{1+e^{\lambda t}}\right) \\
&= \frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\delta\left[z - \left(1+e^{\lambda t}\right)^{-1}\right]
\end{aligned}$$

And this is the formula we searched all this time. We can now use the delta function to eliminate the integral in equation (5.55b). To do so we need to recall that

$$\int f(x)\delta\left(x-a\right)dx = f(a)$$

That is the integral of a function multiplied by a Dirac delta shifted by a units is the value of that function at a . With this we now have

$$\int \frac{1}{\lambda(z-1)}|z|^{\frac{v}{\lambda}}\,\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz = \left\{\begin{array}{c} \int \frac{1}{\lambda(z-1)}|z|^{\frac{v}{\lambda}}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2} \\ \\ \delta\left[z-\left(1+e^{\lambda t}\right)^{-1}\right]dz \end{array}\right\}$$

Integral disappears and instead we get value of integrand for $z=\frac{1}{1+e^{t\lambda}}$

$$\begin{aligned}
&= \frac{1}{\lambda(z-1)}|z|^{\frac{v}{\lambda}}\frac{\lambda e^{\lambda t}}{(1+e^{\lambda t})^2}\Big|_{z=\frac{1}{1+e^{t\lambda}}} \\
&= \frac{1}{\lambda\left(\frac{1}{1+e^{t\lambda}}-1\right)}\frac{\lambda e^{\lambda t}}{(1+e^{\lambda t})^2}\left(\frac{1}{1+e^{t\lambda}}\right)^{\frac{v}{\lambda}} \\
&= \frac{1}{\left[\frac{1-\left(1+e^{t\lambda}\right)}{1+e^{t\lambda}}\right]}\frac{e^{\lambda t}}{(1+e^{\lambda t})^2}\left(\frac{1}{1+e^{t\lambda}}\right)^{\frac{v}{\lambda}} \\
&= \left[\frac{e^{\lambda t}}{(1+e^{\lambda t})^2}\right]\left[\frac{\left(1+e^{\lambda t}\right)}{-e^{\lambda t}}\right]\left(\frac{1}{1+e^{t\lambda}}\right)^{\frac{v}{\lambda}} \\
&= -\left(\frac{1}{1+e^{t\lambda}}\right)^{1+\frac{v}{\lambda}} \\
&= -\left(1+e^{t\lambda}\right)^{-1-\frac{v}{\lambda}}
\end{aligned}$$

$$\therefore -\int \frac{1}{\lambda(z-1)}|z|^{-\frac{v}{\lambda}}\,\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz = -\left(1+e^{t\lambda}\right)^{-1-\frac{v}{\lambda}}$$

But this is not the end of our trouble, with the integral solved, our problem as in equation (5.55b) is reduced to

$$G\left(z,t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)|z|^{\frac{v}{\lambda}}\,\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t)+\left(1+e^{t\lambda}\right)^{-1-\frac{v}{\lambda}}$$

Multiplying both sides of the above equation by $|z|^{-\frac{v}{\lambda}}$ we have

$$G\left(z,t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t)\,|z|^{-\frac{v}{\lambda}}+|z|^{-\frac{v}{\lambda}}\left(1+e^{t\lambda}\right)^{-1-\frac{v}{\lambda}}$$

There is no harm if we make another substitution. Thus the final step towards the solution is to change the variable from t to

$$T = t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$$

Multiplying both sides by λ yields

$$\begin{aligned}\lambda T &= \lambda t - \ln\left|\frac{z-1}{z}\right| \\ \lambda t &= \lambda T + \ln\left|\frac{z-1}{z}\right|\end{aligned}$$

Exponentiating both sides we get

$$\begin{aligned}e^{\lambda t} &= e^{\lambda T + \ln\left|\frac{z-1}{z}\right|} \\ &= e^{\lambda T} e^{\ln\left|\frac{z-1}{z}\right|} \\ &= e^{\lambda T}\left|\frac{z-1}{z}\right|\end{aligned}$$

Remembering that for our root

$$\begin{aligned}z &= \frac{1}{1+e^{t\lambda}} \\ \Rightarrow \frac{z-1}{z} &= 1 - \frac{1}{z} \\ &= 1 - \frac{1}{\left[\frac{1}{1+e^{t\lambda}}\right]} \\ &= 1 - \left[1+e^{t\lambda}\right] \\ &= -e^{t\lambda} \\ \therefore \left|-e^{t\lambda}\right| &= -\left(-e^{t\lambda}\right) \\ \Rightarrow \left|\frac{z-1}{z}\right| &= -\frac{z-1}{z}\end{aligned}$$

$$\therefore e^{t\lambda} = e^{T\lambda}\left|\frac{z-1}{z}\right|$$
$$= e^{T\lambda}\left[-\frac{z-1}{z}\right]$$
$$\Rightarrow 1+e^{t\lambda} = 1-e^{T\lambda}\left[\frac{z-1}{z}\right]$$

Multiplying the right hand side by $1=\frac{e^{-\lambda T}}{e^{-\lambda T}}$ we have

$$\Rightarrow 1+e^{t\lambda} = 1-e^{T\lambda}\left[\frac{z-1}{z}\right]\left(\frac{e^{-\lambda T}}{e^{-\lambda T}}\right)$$

$$= 1-\left[\frac{z-1}{ze^{-\lambda T}}\right]$$

$$= \frac{ze^{-\lambda T}-(z-1)}{ze^{-\lambda T}}$$

$$= \frac{ze^{-\lambda T}-z+1}{ze^{-\lambda T}}$$

$$= \frac{1-z\left[1-e^{-\lambda T}\right]}{ze^{-\lambda T}}$$

$$\therefore 1+e^{t\lambda} = \frac{1-z\left[1-e^{-\lambda T}\right]}{ze^{-\lambda T}}$$

Described by$T$ variable, equation for G becomes

$$G(z,T)\,\eta(T) = f\left(T+\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)|z|^{-\frac{v}{\lambda}} + |z|^{-\frac{v}{\lambda}}\left(1+e^{t\lambda}\right)^{-1-\frac{v}{\lambda}}$$
$$= |z|^{-\frac{v}{\lambda}} f\left(T+\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1-\frac{v}{\lambda}}$$

But we know that f does not depend on z

$$G(z,T)\,\eta(T) = |z|^{-\frac{v}{\lambda}} f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1-\frac{v}{\lambda}}$$

When $T=0$ it follows that

$$G(z,0)\,\eta(0) = |z|^{-\frac{v}{\lambda}} f\left(0 + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda(0)}\right)}{ze^{-\lambda(0)}}\right]^{-1-\frac{v}{\lambda}}$$

But $\eta(0)=1$ hence

$$\begin{aligned}G(z,0) &= |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1-z(1-1)}{z}\right]^{-1-\frac{v}{\lambda}}\\ &= |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1}{z}\right]^{-1-\frac{v}{\lambda}}\\ &= |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}} z^{1+\frac{v}{\lambda}}\end{aligned}$$

But knowing that

$$z = G(z,0) = |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + z$$

It follows that

$$|z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = 0$$

So we have finally eliminated f. Thus the solution is

$$G(z,T)\,\eta(T) = z^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-1-\frac{v}{\lambda}}$$

Since $T \geqslant 0$ it follows by definition of tau that

$$\eta(T) = \begin{cases}1 & T \geqslant 0\\ 0 & T \leqslant 0\end{cases}$$

Thus we have $\eta(T) = 1$ hence

$$G(z,T)\,\eta(T) = z^{-\frac{v}{\lambda}} \left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{z e^{-\lambda T}}\right]^{-1-\frac{v}{\lambda}}$$

$$G(z,T) = z^{-\frac{v}{\lambda}} \left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{z e^{-\lambda T}}\right]^{-1-\frac{v}{\lambda}}$$

$$= z^{-\frac{v}{\lambda}} \left[\frac{z e^{-\lambda T}}{1 - z\left(1 - e^{-\lambda T}\right)}\right]^{1+\frac{v}{\lambda}}$$

$$= z^{-\frac{v}{\lambda}+1+\frac{v}{\lambda}} \left[\frac{e^{-\lambda T}}{1 - z\left(1 - e^{-\lambda T}\right)}\right]^{1+\frac{v}{\lambda}}$$

$$= z \left[\frac{e^{-\lambda T}}{1 - z\left(1 - e^{-\lambda T}\right)}\right]^{1+\frac{v}{\lambda}}$$

We can rewrite it as

$$G(z,t) = z \left[\frac{e^{-\lambda t}}{1 - z\left(1 - e^{-\lambda t}\right)}\right]^{1+\frac{v}{\lambda}}$$

Which by identification is the pgf of a negative binomial distribution with parameters $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

$G(z,t)$ can be expressed as

$$G(z,t) = z \left[\frac{p}{1 - qz}\right]^{\frac{v}{\lambda}+1}$$

$P_n(t)$ is the coefficient of z^n in $G(z,t)$. But

$$G(z,t) = z p^{\frac{v}{\lambda}+1} (1 - qz)^{-\left(\frac{v}{\lambda}+1\right)}$$

$$= z p^{\frac{v}{\lambda}+1} \sum_{n=0}^{\infty} \binom{-\left[\frac{v}{\lambda}+1\right]}{n} (-qz)^n$$

$$= zp^{\frac{v}{\lambda}+1}\sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}(-1)^n(-qz)^n$$

$$= zp^{\frac{v}{\lambda}+1}\sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}(-1)^n(-1)^n(qz)^n$$

$$= zp^{\frac{v}{\lambda}+1}\sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}(qz)^n$$

$$= \sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}p^{\frac{v}{\lambda}+1}q^n z^{n+1}$$

$$= \sum_{n=1}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+(n-1)-1}{n-1}p^{\frac{v}{\lambda}+1}q^{n-1}z^n$$

$$= \sum_{n=1}^{\infty}\binom{n+\frac{v}{\lambda}-1}{n-1}p^{\frac{v}{\lambda}+1}q^{n-1}z^n$$

Therefore the coefficient of z^n is

$$\binom{n+\frac{v}{\lambda}-1}{n-1}p^{\frac{v}{\lambda}+1}q^{n-1} = \binom{n+\frac{v}{\lambda}-1}{n-1}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{n-1}$$

Hence

$$P_n\left(t\right) = \binom{n+\frac{v}{\lambda}-1}{n-1}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{n-1} \qquad n = 1, 2, 3, ...$$

$$= \binom{\frac{v}{\lambda}+1+k-1}{k}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{k} \qquad k = 0,\ 1,\ 2, 3, ...$$

where $k = n-1$

This is the pmf of a negative binomial distribution with parameters $r = \frac{v}{\lambda}+1$ and $p = e^{-\lambda t}$

Case 2: When initial population $X(0) = n_0$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$
$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$
$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = n_0$, we have

$$P_{n_0}(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq n_0$$

$$\therefore G(z,0) = z^{n_0}$$

With this equation (5.54b) becomes

$$\overline{G}\,(z,s)\,|z|^{\frac{v}{\lambda}}\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{z^{n_0}}{\lambda z\,(z-1)}\,|z|^{\frac{v}{\lambda}}\,e^{-\frac{s}{\lambda}\ln\left|\frac{z-1}{z}\right|}dz \qquad (5.62)$$

Now this looks like a complicated equation to solve, but at this juncture we can licitly apply the inverse Laplace transform to both sides.

We observe that $f(s)$ can be regarded as a Laplace transform of some unknown function $f(t)$

Applying inverse Laplace transform to both sides of equation (5.62) we have from the table of transform pairs in chapter 2 or examples 7 and 8 (see pages 38-39)

1. e^{-cs} is the Laplace transform of the Dirac delta function $\delta(t-c)$

2. $\overline{G}(z,s)e^{cs}$ is the Laplace transform of $G(t-c)\eta(t-c)$ where η is the Heaviside step function.

In our case $c = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$

With this equation (5.62) can be rewritten as

$$\Rightarrow \overline{G}\left(z,s\right)\left|z\right|^{\frac{v}{\lambda}}e^{-cs} = f(s) - \int \frac{z^{n_0}}{\lambda z\left(z-1\right)}\left|z\right|^{\frac{v}{\lambda}}e^{-cs}dz$$

So all together inversely transforming both sides the above equation, we come to

$$G(z,t-c)\left|z\right|^{\frac{v}{\lambda}}\eta(t-c) = f(t) - \int \frac{z^{n_0}}{\lambda z\left(z-1\right)}\left|z\right|^{\frac{v}{\lambda}}\delta(t-c)dz$$

Substituting the value of a we obtain

$$G\left(z,t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t) - \int \frac{z^{n_0}}{\lambda z\left(z-1\right)}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz \tag{5.63}$$

Now this is a pretty large equation but it can be simplified, but to do that we need to make one big detour. We can free ourselves of this terrible integral by using the Dirac delta function, but what we have is delta of function of variable z , so we need to first simplify it to a common delta of variable. To do so, we use the following property of delta function

$$\delta\left(g\left(z\right)\right) = \sum_{z_i}\frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right) \tag{5.64}$$

Where $g'\left(z\right)$ is the first derivative of $g\left(z\right)$, z_i is a simple root of $g\left(z\right)$ such that $g'\left(z_i\right)$. In our case $g\left(z\right) = t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$. To obtain the roots of $g(z)$ we solve $g\left(z\right) = 0$

$$\begin{aligned}
&\Rightarrow t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right| = 0\\
&\Rightarrow t = \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\\
&\Rightarrow t\lambda = \ln\left|\frac{z-1}{z}\right|\\
&\Rightarrow e^{t\lambda} = \left|\frac{z-1}{z}\right|\\
&\Rightarrow e^{t\lambda} = \pm\frac{z-1}{z}
\end{aligned}$$

Case 1

$$\begin{aligned}
e^{t\lambda} &= \frac{z-1}{z} \\
ze^{t\lambda} &= z-1 \\
1 &= z - ze^{t\lambda} \\
1 &= z\left(1-e^{t\lambda}\right) \\
z_1 &= \frac{1}{1-e^{t\lambda}}
\end{aligned}$$

Case 2

$$\begin{aligned}
e^{t\lambda} &= -\frac{z-1}{z} \\
ze^{t\lambda} &= -\left(z-1\right) \\
1 &= z + ze^{t\lambda} \\
1 &= z\left(1+e^{t\lambda}\right) \\
z_2 &= \frac{1}{1+e^{t\lambda}}
\end{aligned}$$

Therefore

$$z_i = \frac{1}{1 \pm e^{t\lambda}}$$

The next step is to determine

$$\begin{aligned}
g'\left(z\right) &= \frac{d}{dz}\left[t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right] \\
&= -\frac{1}{\lambda}\frac{d}{dz}\ln\left|\frac{z-1}{z}\right|
\end{aligned}$$

Using the property $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

$$g'(z) = -\frac{1}{\lambda}\left(\frac{\frac{d}{dz}\left|\frac{z-1}{z}\right|}{\left|\frac{z-1}{z}\right|}\right)$$

$$g'(z) = -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \tag{5.65}$$

Now remembering that $|x| = x\,\mathrm{sgn}\,(x)$ and $\mathrm{sgn}\,(x) = 2\eta(x) - 1$, therefore

$$\left|\frac{z}{z-1}\right| = \frac{z}{z-1}\,\mathrm{sgn}\left(\frac{z}{z-1}\right) \tag{5.66}$$

Also

$$\begin{aligned}\left|\frac{z-1}{z}\right| &= \frac{z-1}{z}\,\mathrm{sgn}\left(\frac{z-1}{z}\right)\\ &= \frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\end{aligned}$$

Where sgn is the sign distribution and eta is the Heaviside distribution .Hence

$$\begin{aligned}\left|\frac{z-1}{z}\right|' &= \frac{d}{dz}\left|\frac{z-1}{z}\right|\\ &= \frac{d}{dz}\left\{\frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\right\}\end{aligned} \tag{5.67}$$

At this step we need to recall that

$$\eta'_{h(z)} = h'(z)\frac{\partial\eta(z)}{\partial h}$$

Use product rule of differentiation, we simplify equation (5.67) as follows Let

$$\begin{aligned}U &= \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow U' = \frac{1}{z^2}\\ V &= 2\eta\left(\frac{z-1}{z}\right) - 1\\ &= 2\eta\left(1 - \frac{1}{z}\right) - 1 \Rightarrow V' = \frac{2}{z^2}\eta'\left(1 - \frac{1}{z}\right)\end{aligned}$$

We now have

$$\begin{aligned}
\left|\frac{z-1}{z}\right|' &= U'V + V'U \\
&= \frac{1}{z^2}\underbrace{\left[2\eta\left(\frac{z-1}{z}\right)-1\right]}_{\operatorname{sgn}\left(\frac{z-1}{z}\right)} + \frac{z-1}{z}\left[\frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)\right] \\
&= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\eta'\left(1-\frac{1}{z}\right) \\
\left|\frac{z-1}{z}\right|' &= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(1-\frac{1}{z}\right) \qquad (5.68)
\end{aligned}$$

Since $\eta'(x) = \delta(x)$

But

$$\delta(g(z)) = \sum_{z_i} \frac{1}{|g'(z_i)|}\delta(z - z_i)$$

Let

$$\begin{aligned}
g(z) &= 1 - \frac{1}{z} \\
g'(z) &= \frac{1}{z^2}
\end{aligned}$$

Solving for roots of $g(z)$, we have

$$\begin{aligned}
&g(z) = 0 \\
&\Rightarrow 1 - \frac{1}{z} = 0 \\
&1 = \frac{1}{z} \Rightarrow z = 1 \\
&g'(z) = \frac{1}{1^2} \\
&\therefore \delta\left(1-\frac{1}{z}\right) = \delta(z-1)
\end{aligned}$$

Hence equation (5.68) becomes

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(z-1\right) \tag{5.69}$$

Substituting equations (5.66) and (5.69) in equation (5.65) yields

$$g'\left(z\right) = -\frac{1}{\lambda}\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right)$$

$$= -\frac{1}{\lambda}\left\{\frac{z}{z-1}\operatorname{sgn}\left(\frac{z}{z-1}\right)\left[\frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(z-1\right)\right]\right\}$$

$$= -\frac{1}{\lambda}\left[\frac{1}{z^2}\left(\frac{z}{z-1}\right)\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + 2\left(\frac{z}{z-1}\right)\left(\frac{z-1}{z^3}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

Recalling that we obtained the roots of $g\left(z\right)$ as

$$z_1 = \frac{1}{1+e^{t\lambda}},\, z_2 = \frac{1}{1-e^{t\lambda}}$$

And the Dirac delta function is zero everywhere except when its argument is zero.

$$\arg\delta\left(z-1\right) = z-1$$

For both roots, the delta term in $g'\left(z\right)$ vanishes because for it to exist it requires that argument of delta function must be zero. That is

$$0 = z_i - 1 = \frac{1}{1\pm e^{t\lambda}} - 1$$

which is equivalent to $e^{t\lambda} = 0$ and that is a Mathematical lie since both λ and t are greater than zero, so only one term remains after neglecting delta in $g'\left(z_i\right)$.

Therefore

$$g'(z_1) = -\frac{1}{\lambda}\left[\frac{1}{z(z-1)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\right]$$

But

$$\operatorname{sgn}(x)\operatorname{sgn}(y) = \operatorname{sgn}(xy)$$

$$\begin{aligned}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) &= \operatorname{sgn}\left[\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right)\right]\\ &= \operatorname{sgn}(1)\\ &= 1\end{aligned}$$

This implies that

$$\begin{aligned}g'(z_1) &= -\frac{1}{\lambda}\left[\frac{1}{z_1(z_1-1)}\right]\\ &= -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1+e^{\lambda t}}\left(\frac{1}{1+e^{\lambda t}}-1\right)}\right]\\ &= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1}{1+e^{\lambda t}}-1}\right]\\ &= -\frac{1}{\lambda}\left[\frac{1+e^{\lambda t}}{\frac{1-\left(1+e^{\lambda t}\right)}{1+e^{\lambda t}}}\right]\\ &= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{1-(1+e^{\lambda t})}\right]\end{aligned}$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1+e^{\lambda t}\right)^2}{-e^{\lambda t}}\right]$$

$$= \frac{\left(1+e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}$$

Similarly

$$g'\left(z_2\right) = -\frac{1}{\lambda}\left[\frac{1}{\frac{1}{1-e^{\lambda t}}\left(\frac{1}{1-e^{\lambda t}}-1\right)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1}{1-e^{\lambda t}}-1}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{1-e^{\lambda t}}{\frac{1-\left(1-e^{\lambda t}\right)}{1-e^{\lambda t}}}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{1-\left(1-e^{\lambda t}\right)}\right]$$

$$= -\frac{1}{\lambda}\left[\frac{\left(1-e^{\lambda t}\right)^2}{e^{\lambda t}}\right]$$

$$= -\frac{\left(1-e^{\lambda t}\right)^2}{\lambda e^{\lambda t}}$$

Therefore

$$g'\left(z_i\right) = \pm\left(1 \pm e^{\lambda t}\right)^2 \frac{1}{\lambda e^{\lambda t}}$$

But $g'\left(z_i\right)$ must not be zero, and for the initial condition that is at $t = 0$ the negative root makes $g'\left(z_i\right)$ to be singular $[g'\left(z_i\right) = 0]$ so that root needs to be discarded in the summation only one root of z_i is left and will be from now on $z_i = z_1$

It is now time to go back to equation (5.64)

$$\delta\left(g\left(z\right)\right)=\sum_{z_i}\frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right)$$

$$=\frac{1}{\left|g'\left(z_1\right)\right|}\delta\left(z-z_1\right)$$

$$=\frac{1}{\left|\left(1+e^{\lambda t}\right)^2\frac{1}{\lambda e^{\lambda t}}\right|}\delta\left(z-\frac{1}{1+e^{\lambda t}}\right)$$

$$=\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\delta\left[z-\left(1+e^{\lambda t}\right)^{-1}\right]$$

And this is the formula we searched all this time. We can now use the delta function to eliminate the integral in equation (5.63) In doing so we need to recall that

$$\int f(x)\delta\left(x-a\right)dx=f(a)$$

That is the integral of a function multiplied by a Dirac delta shifted by a units is the value of that function at a . With this we have

$$\int\frac{z^{n_0}}{\lambda z\left(z-1\right)}\left|z\right|^{\frac{v}{\lambda}}\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz=\int\frac{z^{n_0}}{\lambda z\left(z-1\right)}\left|z\right|^{\frac{v}{\lambda}}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\delta\left[z-\left(1+e^{\lambda t}\right)^{-1}\right]dz$$

Integral disappears and instead we get value of integrand for $z=\frac{1}{1+e^{t\lambda}}$

$$=\frac{z^{n_0}}{\lambda z\left(z-1\right)}\left|z\right|^{\frac{v}{\lambda}}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\Big|_{z=\frac{1}{1+e^{t\lambda}}}$$

$$=\frac{\left(\frac{1}{1+e^{t\lambda}}\right)^{n_0}}{\lambda\left(\frac{1}{1+e^{t\lambda}}\right)\left(\frac{1}{1+e^{t\lambda}}-1\right)}\frac{\lambda e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\left(\frac{1}{1+e^{t\lambda}}\right)^{\frac{v}{\lambda}}$$

$$=\frac{\left(\frac{1}{1+e^{t\lambda}}\right)^{n_0}}{\left(\frac{1}{1+e^{t\lambda}}\right)\left[\frac{1-\left(1+e^{t\lambda}\right)}{1+e^{t\lambda}}\right]}\frac{e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\left(\frac{1}{1+e^{t\lambda}}\right)^{\frac{v}{\lambda}}$$

$$= \left(\frac{1}{1+e^{t\lambda}}\right)^{n_0}\left[\frac{e^{\lambda t}}{\left(1+e^{\lambda t}\right)^2}\right]\left[\frac{\left(1+e^{\lambda t}\right)^2}{-e^{\lambda t}}\right]\left(\frac{1}{1+e^{t\lambda}}\right)^{\frac{v}{\lambda}}$$

$$= -\left(\frac{1}{1+e^{t\lambda}}\right)^{n_0+\frac{v}{\lambda}}$$

$$= -\left(1+e^{t\lambda}\right)^{-n_0-\frac{v}{\lambda}}$$

$$\therefore -\int \frac{z^{n_0}}{\lambda z\,(z-1)}\,|z|^{-\frac{v}{\lambda}}\,\delta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)dz = \left(1+e^{t\lambda}\right)^{-n_0-\frac{v}{\lambda}}$$

But this is not the end of our trouble, with the integral solved, our problem as in equation (5.63) is reduced to

$$G\left(z, t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)|z|^{\frac{v}{\lambda}}\,\eta\left(t-\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = f(t) + \left(1+e^{t\lambda}\right)^{-n_0-\frac{v}{\lambda}}$$

There is no harm if we make another substitution. Thus the final step towards the solution is to change the variable from t to

$$T = t - \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|$$

Multiplying both sides by λ yields

$$\lambda T = \lambda t - \ln\left|\frac{z-1}{z}\right|$$
$$\lambda t = \lambda T + \ln\left|\frac{z-1}{z}\right|$$

Exponentiating both sides we get

$$\begin{aligned} e^{\lambda t} &= e^{\lambda T+\ln\left|\frac{z-1}{z}\right|} \\ &= e^{\lambda T}e^{\ln\left|\frac{z-1}{z}\right|} \\ &= e^{\lambda T}\left|\frac{z-1}{z}\right| \end{aligned}$$

Remembering that for our root

$$\begin{aligned}
z &= \frac{1}{1+e^{t\lambda}} \\
\Rightarrow \frac{z-1}{z} &= 1 - \frac{1}{z} \\
&= 1 - \frac{1}{\left[\frac{1}{1+e^{t\lambda}}\right]} \\
&= 1 - \left[1 + e^{t\lambda}\right] \\
&= -e^{t\lambda}
\end{aligned}$$

Thus

$$\begin{aligned}
\left|-e^{t\lambda}\right| &= -\left(-e^{t\lambda}\right) \\
\Rightarrow \left|\frac{z-1}{z}\right| &= -\frac{z-1}{z} \\
\therefore e^{t\lambda} &= e^{T\lambda}\left|\frac{z-1}{z}\right| \\
&= e^{T\lambda}\left[-\frac{z-1}{z}\right] \\
\Rightarrow 1 + e^{t\lambda} &= 1 - e^{T\lambda}\left[\frac{z-1}{z}\right]
\end{aligned}$$

Multiplying the right hand side by $1 = \frac{e^{-\lambda T}}{e^{-\lambda T}}$ we get

$$1 + e^{t\lambda} = 1 - e^{T\lambda}\left[\frac{z-1}{z}\right]\left(\frac{e^{-\lambda T}}{e^{-\lambda T}}\right)$$

$$= 1 - \left[\frac{z-1}{ze^{-\lambda T}}\right]$$

$$= \frac{ze^{-\lambda T} - (z-1)}{ze^{-\lambda T}}$$

$$= \frac{ze^{-\lambda T} - z + 1}{ze^{-\lambda T}}$$

$$= \frac{1 - z\left[1 - e^{-\lambda T}\right]}{ze^{-\lambda T}}$$

$$\therefore 1 + e^{t\lambda} = \frac{1 - z\left[1 - e^{-\lambda T}\right]}{ze^{-\lambda T}}$$

Described by T variable, equation for G becomes

$$\begin{aligned} G(z,T)\,\eta(T) &= f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right)|z|^{-\frac{v}{\lambda}} + |z|^{-\frac{v}{\lambda}}\left(1 + e^{t\lambda}\right)^{-n_0-\frac{v}{\lambda}} \\ &= |z|^{-\frac{v}{\lambda}} f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0-\frac{v}{\lambda}} \end{aligned}$$

But we know that f does not depend on z

$$G(z,T)\,\eta(T) = |z|^{-\frac{v}{\lambda}} f\left(T + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1 - z\left(1 - e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0-\frac{v}{\lambda}}$$

When $T = 0$ it follows that

$$G(z,0)\,\eta(0) = |z|^{-\frac{v}{\lambda}} f\left(0 + \frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1 - z\left(1 - e^{-\lambda(0)}\right)}{ze^{-\lambda(0)}}\right]^{-n_0-\frac{v}{\lambda}}$$

But $\eta(0) = 1$ hence

$$\begin{aligned} G(z,0) &= |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1 - z(1-1)}{z}\right]^{-n_0-\frac{v}{\lambda}} \\ &= |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\left[\frac{1}{z}\right]^{-n_0-\frac{v}{\lambda}} \\ &= |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{v}{\lambda}}\, z^{n_0+\frac{v}{\lambda}} \end{aligned}$$

But knowing that

$$z^{n_0} = G(z,0) = |z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) + z^{n_0}$$

It follows that

$$|z|^{-\frac{v}{\lambda}} f\left(\frac{1}{\lambda}\ln\left|\frac{z-1}{z}\right|\right) = 0$$

So we have finally eliminated f. Thus the solution is

$$G(z,T)\,\eta(T) = z^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0-\frac{v}{\lambda}}$$

Since $T \geqslant 0$ it follows by definition of eta that $\eta(T) = 1$ hence

$$G(z,T)\,\eta(T) = z^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0-\frac{v}{\lambda}}$$

$$G(z,T) = z^{-\frac{v}{\lambda}}\left[\frac{1-z\left(1-e^{-\lambda T}\right)}{ze^{-\lambda T}}\right]^{-n_0-\frac{v}{\lambda}}$$

$$= z^{-\frac{v}{\lambda}}\left[\frac{ze^{-\lambda T}}{1-z\left(1-e^{-\lambda T}\right)}\right]^{n_0+\frac{v}{\lambda}}$$

$$= z^{-\frac{v}{\lambda}+n_0+\frac{v}{\lambda}}\left[\frac{e^{-\lambda T}}{1-z\left(1-e^{-\lambda T}\right)}\right]^{n_0+\frac{v}{\lambda}}$$

$$= z^{n_0}\left[\frac{e^{-\lambda T}}{1-z\left(1-e^{-\lambda T}\right)}\right]^{n_0+\frac{v}{\lambda}}$$

We can rewrite it as

$$G(z,t) = z^{n_0}\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0+\frac{v}{\lambda}}$$

By identification, this is the pgf of a negative binomial distribution with $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

$P_n(t)$ is the coefficient of z^n in $G(z,t)$

But $G(z,t)$ is of the form

$$\begin{aligned}
G(z,t) &= z^{n_0}\left(\frac{p}{1-qz}\right)^{n_0+\frac{v}{\lambda}} \\
&= z^{n_0}p^{n_0+\frac{v}{\lambda}}(1-qz)^{-\left(n_0+\frac{v}{\lambda}\right)} \\
&= z^{n_0}p^{n_0+\frac{v}{\lambda}}\sum_{n=0}^{\infty}\binom{-\left[n_0+\frac{v}{\lambda}\right]}{n}(-qz)^n \\
&= z^{n_0}p^{n_0+\frac{v}{\lambda}}\sum_{n=0}^{\infty}\binom{\left[n_0+\frac{v}{\lambda}\right]+n-1}{n}(-1)^n(-1)^n(zq)^n \\
&= z^{n_0}p^{n_0+\frac{v}{\lambda}}\sum_{n=0}^{\infty}\binom{\left[n_0+\frac{v}{\lambda}\right]+n-1}{n}(zq)^n \\
&= \sum_{n=0}^{\infty}\binom{\left[n_0+\frac{v}{\lambda}\right]+n-1}{n}p^{n_0+\frac{v}{\lambda}}q^n z^{n+n_0} \\
&= \sum_{n=n_0}^{\infty}\binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0}p^{n_0+\frac{v}{\lambda}}q^{n-n_0}z^{n_0} \\
&= \sum_{n=n_0}^{\infty}\binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0}p^{n_0+\frac{v}{\lambda}}q^{n-n_0}z^{n_0}
\end{aligned}$$

Thus the coefficient of z^n is

$$\binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0}p^{n_0+\frac{v}{\lambda}}q^{n-n_0}$$

But $p=e^{-\lambda t}$ and $q=e^{-\lambda t}$ implying that

$$P_n(t) = \binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0}p^{n_0+\frac{v}{\lambda}}q^{n-n_0} \qquad n=n_0,\ n_0+1,\ n_0+2,...$$

$$= \binom{\left[n_0+\frac{v}{\lambda}\right]+(k)-1}{k}p^{n_0+\frac{v}{\lambda}}q^{k} \qquad k=0,\ 1,\ 2,...$$

Which is the pmf of a negative binomial distribution

Hyper geometric function approach

From equation (5.54a) we had

$$\frac{d}{dz}\overline{G}\left(z,s\right)-\frac{\left[s-v\left(z-1\right)\right]}{\lambda z\left(z-1\right)}\overline{G}\left(z,s\right)=-\frac{G\left(z,0\right)}{\lambda z\left(z-1\right)}$$

This can be re-expressed as

$$\frac{d}{dz}\overline{G}\left(z,s\right)+\frac{\left[s+v\left(1-z\right)\right]}{\lambda z\left(1-z\right)}\overline{G}\left(z,s\right)=\frac{G\left(z,0\right)}{\lambda z\left(1-z\right)} \tag{5.70a}$$

which is an ODE of 1^{st} order. It is of the form $y'+Py=Q$ where

$$y=\overline{G}\left(z,s\right),\quad P=\frac{s+v(1-z)}{\lambda z\left(1-z\right)}\text{ and }Q=\frac{G(z,0)}{\lambda z\left(1-z\right)}$$

Using the Integrating Factor technique it follows that

$$\begin{aligned}I.F&=e^{\int\frac{s+v(1-z)}{\lambda z(1-z)}dz}\\&=e^{\frac{1}{\lambda}\int\frac{s+v(1-z)}{z(1-z)}dz}\end{aligned}$$

But

$$\begin{aligned}\int\frac{s+v\left(1-z\right)}{z\left(1-z\right)}dz&=\int\frac{s}{z\left(1-z\right)}dz+\int\frac{v}{z}dz\\&=s\int\frac{1}{z\left(1-z\right)}dz+v\int\frac{1}{z}dz\\&=s\int\frac{1}{z\left(1-z\right)}dz+v\ln z\end{aligned}$$

By Partial Fractions, We have

$$\frac{1}{z\left(1-z\right)}=\frac{A}{z}+\frac{B}{1-z}$$

Multiplying both sides by $z(1-z)$ yields

$$1=A\left(1-z\right)+Bz$$

Which holds true for all values of s

Setting $s = 0$ we have

$$1 = A(1-0) + B(0) \Rightarrow A = 1$$

Setting $s = 1$ We have

$$1 = A(1-1) + B(1) \Rightarrow B = 1$$

Thus

$$\begin{aligned}\int \frac{1}{z(1-z)}dz &= \int \frac{1}{z}dz + \int \frac{1}{1-z}dz \\ &= \ln z - \ln(1-z) \\ &= \ln\left(\frac{z}{1-z}\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow \frac{1}{\lambda}\int \frac{s + v(1-z)}{z(1-z)}dz &= \frac{s}{\lambda}\ln\left(\frac{z}{1-z}\right) + \frac{v}{\lambda}\ln z \\ &= \ln\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} + \ln z^{\frac{v}{\lambda}} \\ &= \ln\left[\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} z^{\frac{v}{\lambda}}\right]\end{aligned}$$

Therefore

$$\begin{aligned}I.F &= e^{\frac{1}{\lambda}\int \frac{s+v(1-z)}{z(1-z)}} \\ &= e^{\ln\left[\left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} z^{\frac{v}{\lambda}}\right]} \\ &= \left(\frac{z}{1-z}\right)^{\frac{s}{\lambda}} z^{\frac{v}{\lambda}} \\ &= z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\end{aligned}$$

Multiplying both sides of equation (5.70a) by the integrating factor yields

$$z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\frac{d}{dz}\overline{G}(z,s)+z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\frac{s+v(1-z)}{\lambda z(1-z)}\overline{G}(z,s)=z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\frac{G(z,0)}{\lambda z(1-z)}$$

Simplifying this we get

$$\frac{d}{dz}\left[\overline{G}(z,s)\,z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\right]=\frac{z^{\frac{s+v}{\lambda}}G(z,0)}{\lambda z(1-z)^{\frac{s}{\lambda}+1}}$$

Integrating both sides with respect to z we have

$$\int\frac{d}{dz}\left[(1-z)^{-\frac{s}{\lambda}}z^{\frac{s+v}{\lambda}}\overline{G}(z,s)\right]dz=\int\frac{z^{\frac{s+v}{\lambda}}G(z,0)}{\lambda z(1-z)^{\frac{s}{\lambda}+1}}$$

$$\int d\left[(1-z)^{-\frac{s}{\lambda}}z^{\frac{s+v}{\lambda}}\overline{G}(z,s)\right]=\int\frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}}G(z,0)}{\lambda z(1-z)^{\frac{s}{\lambda}+1}}dz$$

$$(1-z)^{-\frac{s}{\lambda}}z^{\frac{s+v}{\lambda}}\overline{G}(z,s)=\frac{1}{\lambda}\int\frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}}G(z,0)}{z(1-z)^{\frac{s}{\lambda}+1}}dz \qquad (5.70b)$$

We consider two cases

1. When $X(0)=1$
2. When $X(0)=n_0$

Case 1: When initial population $X(0)=1$

Recall that

$$G(z,t)=\sum_{n=0}^{\infty}P_n(t)z^n$$
$$\Rightarrow G(z,0)=\sum_{n=0}^{\infty}P_n(0)z^n$$
$$=P_0(0)+P_1(0)z+P_2(0)z^2+...+P_{n_0}(0)z^{n_0}+...$$

but for the initial condition $X(0) = 1$, we have

$$P_1(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq 1$$

$$\therefore G(z, 0) = z$$

With this equation (5.70b) becomes

$$(1-z)^{-\frac{s}{\lambda}}\, z^{\frac{s+v}{\lambda}}\overline{G}\,(z,s) = \frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}}z}{z\,(1-z)^{\frac{s}{\lambda}+1}}dz$$

$$(1-z)^{-\frac{s}{\lambda}}\, z^{\frac{s+v}{\lambda}}\overline{G}\,(z,s) = \frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}}}{(1-z)^{\frac{s}{\lambda}+1}}dz \tag{5.71}$$

We first simplify the right hand side as follows

Recall that

$$\int \frac{x^{a+b}}{(1-x)^{a+1}}dx = \frac{x^{a+b+1}\quad {}_2F_1\,(a+1, a+b+1; a+b+2; x)}{a+b+1} + \text{constant}$$

Where ${}_2F_1\,(a+1, a+1; a+2; x)$ is the gauss hyper geometric function.

Therefore

$$\frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}}}{(1-z)^{\frac{s}{\lambda}+1}}dz = \frac{1}{\lambda}\left[\frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}+1}\quad {}_2F_1\left(\frac{s}{\lambda}+1, \frac{s}{\lambda}+\frac{v}{\lambda}+1; \frac{s}{\lambda}+\frac{v}{\lambda}+2; z\right)}{\frac{s}{\lambda}+\frac{v}{\lambda}+1}\right] + \frac{d_1}{\lambda}$$

Where d_1 is a constant of integration

Using this in equation (5.71) We get

$$(1-z)^{-\frac{s}{\lambda}} z^{\frac{s+v}{\lambda}} \overline{G}(z,s) = \frac{1}{\lambda}\left[\frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}+1} \;\; {}_2F_1\left(\frac{s}{\lambda}+1, \frac{s}{\lambda}+\frac{v}{\lambda}+1; \frac{s}{\lambda}+\frac{v}{\lambda}+2; z\right)}{\frac{s}{\lambda}+\frac{v}{\lambda}+1}\right] + \frac{d_1}{\lambda}$$

$$= \frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}+1} \;\; {}_2F_1\left(\frac{s}{\lambda}+1, \frac{s}{\lambda}+\frac{v}{\lambda}+1; \frac{s}{\lambda}+\frac{v}{\lambda}+2; z\right)}{\lambda\left(\frac{s}{\lambda}+\frac{v}{\lambda}+1\right)} + \frac{d_1}{\lambda} \qquad (5.72)$$

But

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b} \;\; {}_2F_1(c-a, c-b; c; z)$$

We now use this property to simplify the term

$${}_2F_1\left(\frac{s}{\lambda}+1, \frac{s}{\lambda}+\frac{v}{\lambda}+1; \frac{s}{\lambda}+\frac{v}{\lambda}+2; z\right)$$

Here

$$a = \frac{s}{\lambda}+1, \quad b = \frac{s}{\lambda}+\frac{v}{\lambda}+1, \quad c = \frac{s}{\lambda}+\frac{v}{\lambda}+2$$

$$\begin{aligned} c-a &= \frac{s}{\lambda}+\frac{v}{\lambda}+2-\left(\frac{s}{\lambda}+1\right) \\ &= \frac{s}{\lambda}+\frac{v}{\lambda}+2-\frac{s}{\lambda}-1 \\ &= \frac{v}{\lambda}+1 \end{aligned}$$

$$\begin{aligned} c-b &= \frac{s}{\lambda}+\frac{v}{\lambda}+2-\left(\frac{s}{\lambda}+\frac{v}{\lambda}+1\right) \\ &= \frac{s}{\lambda}+2+\frac{v}{\lambda}-\frac{s}{\lambda}-\frac{v}{\lambda}-1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} c-a-b &= \frac{s}{\lambda}+\frac{v}{\lambda}+2-\left(\frac{s}{\lambda}+\frac{v}{\lambda}+1\right)-\left(\frac{s}{\lambda}+1\right) \\ &= \frac{s}{\lambda}+\frac{v}{\lambda}+2-\frac{s}{\lambda}-\frac{v}{\lambda}-1-\frac{s}{\lambda}-1 \\ &= -\frac{s}{\lambda} \end{aligned}$$

Thus

$$ {}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+\frac{v}{\lambda}+1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)=(1-z)^{-\frac{s}{\lambda}}\quad {}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right) $$

With this equation (5.72) becomes

$$ (1-z)^{-\frac{s}{\lambda}}\,z^{\frac{s+v}{\lambda}}\overline{G}(z,s)=\frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}+1}\quad(1-z)^{-\frac{s}{\lambda}}\quad {}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)}{\lambda\left(\frac{s}{\lambda}+\frac{v}{\lambda}+1\right)}+\frac{d_1}{\lambda} $$

$$ \therefore\overline{G}(z,s)=\frac{1}{(1-z)^{-\frac{s}{\lambda}}\,z^{\frac{s+v}{\lambda}}}\frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}+1}(1-z)^{-\frac{s}{\lambda}}\quad {}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)}{\lambda\left(\frac{s}{\lambda}+\frac{v}{\lambda}+1\right)}+\frac{1}{(1-z)^{-\frac{s}{\lambda}}\,z^{\frac{s+v}{\lambda}}}\frac{d_1}{\lambda} $$

$$ =\frac{z}{\lambda\left(\frac{s}{\lambda}+\frac{v}{\lambda}+1\right)}\,{}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)+\frac{d_1}{\lambda}\frac{(1-z)^{\frac{s}{\lambda}}}{z^{\frac{s+v}{\lambda}}} $$

$$ =\frac{z}{s+v+\lambda}\,{}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)+\frac{d_1}{\lambda}\left(\frac{1-z}{z}\right)^{\frac{s}{\lambda}} $$

But since for all $t,\quad z\leqslant 1$ we have $G(z,t)\leqslant 1$ it follows that $d_1=0$

Thus

$$ \overline{G}(z,s)=\frac{z}{s+v+\lambda}\,{}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right) \tag{5.73} $$

But according to Euler the Gauss hyper geometric series can be expressed as

$$ \begin{aligned} {}_2F_1(a,b;c;x)&=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}\\ &=1+\frac{ab}{c}\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!}+... \end{aligned} $$

Where $a,\ b$ and c are complex numbers and

$$(a)_k = \prod_{i=0}^{k-i} (a+i)$$
$$= a\,(a+1)\,(a+2)\,(a+3)\ldots(a+k-1)$$

Thus Letting $a = \frac{v}{\lambda} + 1, b = 1$ and $c = \frac{s}{\lambda} + \frac{v}{\lambda} + 2$ We have

$$F\left(\frac{v}{\lambda} + 1, 1; \frac{s}{\lambda} + \frac{v}{\lambda} + 2; z\right) = F\,(a, b; c; z)$$

$$= \left\{ \begin{array}{c} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 1 + \frac{\left(\frac{v}{\lambda}+1\right)z}{\left(\frac{s}{\lambda}+\frac{v}{\lambda}+2\right)} + \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)2!}{\left(\frac{s}{\lambda}+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+\frac{v}{\lambda}+3\right)}\frac{z^2}{2!} + \\ \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+3\right)3!}{\left(\frac{s}{\lambda}+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+\frac{v}{\lambda}+3\right)\left(\frac{s}{\lambda}+\frac{v}{\lambda}+4\right)}\frac{z^2}{3!} + \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\left(\frac{v}{\lambda}+1\right)z}{\left(\frac{s}{\lambda}+\frac{v}{\lambda}+2\right)} + \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)z^2}{\left(\frac{s}{\lambda}+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+\frac{v}{\lambda}+3\right)} + \\ \\ \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+3\right)z^3}{\left(\frac{s}{\lambda}+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+\frac{v}{\lambda}+3\right)\left(\frac{s}{\lambda}+\frac{v}{\lambda}+4\right)} + \dots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\frac{1}{\lambda}(v+\lambda)z}{\frac{1}{\lambda}(s+v+2\lambda)} + \frac{\frac{1}{\lambda^2}(v+\lambda)(v+2\lambda)z^2}{\frac{1}{\lambda^2}(s+v+2\lambda)(s+v+3\lambda)} + \\ \\ \frac{\frac{1}{\lambda^3}(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{\frac{1}{\lambda^3}(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)} + \dots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{(v+\lambda)z}{(s+v+2\lambda)} + \frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+2\lambda)(s+v+3\lambda)} + \\ \\ \frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)} + \dots \end{array} \right\}$$

With this equation (5.73) becomes

$$\overline{G}\left(z,s\right)=\frac{z}{s+v+\lambda}\quad {}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)$$

$$=\frac{z}{s+v+\lambda}\left\{\begin{array}{c}1+\frac{(v+\lambda)z}{(s+v+2\lambda)}+\frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+2\lambda)(s+v+3\lambda)}+\\ \frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}+...\end{array}\right\}$$

$$\therefore\overline{G}\left(z,s\right)=z\left\{\begin{array}{c}\frac{1}{s+v+\lambda}+\frac{(v+\lambda)z}{(s+v+\lambda)(s+v+2\lambda)}+\frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\\ +\frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}+...\end{array}\right\}$$

Applying inverse Laplace transform to both sides We get

$$L^{-1}\left\{\overline{G}\left(z,s\right)\right\}=zL^{-1}\left\{\begin{array}{c}\frac{1}{s+v+\lambda}+\frac{(v+\lambda)z}{(s+v+\lambda)(s+v+2\lambda)}+\frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\\ +\frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}+...\end{array}\right\}$$

$$\therefore G\left(z,t\right)=zL^{-1}\left(D\right)\tag{5.74}$$

Where

$$D = \left\{ \begin{array}{c} \frac{1}{s+v+\lambda} + \frac{(v+\lambda)z}{(s+v+\lambda)(s+v+2\lambda)} + \frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)} \\ \\ +\frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)} + \ldots \end{array} \right\}$$

Thus

$$L^{-1}(D) = \left\{ \begin{array}{c} \underbrace{L^{-1}\left\{\frac{1}{s+v+\lambda}\right\}}_{PartI} + \underbrace{L^{-1}\left\{\frac{(v+\lambda)z}{(s+v+\lambda)(s+v+2\lambda)}\right\}}_{PartII} \\ + \underbrace{L^{-1}\left\{\frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\right\}}_{PartIII} \\ + \underbrace{L^{-1}\left\{\frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}\right\}}_{PartIV} \\ \\ +\ldots \end{array} \right\}$$

The next step is to simplify the four Parts of the above equation

Part I

$$L^{-1}\left\{\frac{z}{s+\lambda}\right\} = zL^{-1}\left\{\frac{1}{s+\lambda}\right\}$$

Miscellaneous Method

By the first shifting property

$$L\left\{e^{-at}f(t)\right\} = \overline{f}(s+a)$$

and

$$L\{1\} = \frac{1}{s}$$

$$\Rightarrow L\left\{e^{-at}\right\} = \frac{1}{s+a}$$

Thus Letting $a = v + \lambda$, we have

$$L\left\{e^{-(v+\lambda)t}\right\} = \frac{1}{s+v+\lambda}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s+v+\lambda}\right\} = e^{-(v+\lambda)t}$$

$$\therefore L^{-1}\left\{\frac{z}{s+v+\lambda}\right\} = ze^{-(v+\lambda)t}$$

Use of tables

The Laplace transform of $f(t)$ is of the form

$$L\left\{f(t)\right\} = \frac{k}{s+a}$$

Which from the table of transform pairs yields

$$f(t) = ke^{-at}$$

In our case $k = z,\ a = v + \lambda$

$$\Rightarrow L^{-1}\left\{\frac{z}{s+v+\lambda}\right\} = ze^{-(v+\lambda)t}$$

Part II

$$L^{-1}\left\{\frac{(v+\lambda)\,z}{(s+v+\lambda)\,(s+v+2\lambda)}\right\} = (v+\lambda)\,zL^{-1}\left\{\frac{1}{(s+v+\lambda)\,(s+v+2\lambda)}\right\}$$

The function

$$\frac{1}{(s+v+\lambda)\,(s+v+2\lambda)}$$

has simple poles at $s = -\,(v+\lambda)$ and $s = -\,(v+2\lambda)$

Its residues are given by

At the pole $s = -(v + \lambda)$

$$a_{-i} = \lim_{s \to -(v+\lambda)} \frac{[s + (v + \lambda)]\, e^{st}}{(s + v + \lambda)(s + v + 2\lambda)}$$

$$= \lim_{s \to -(v+\lambda)} \frac{e^{st}}{[s + v + 2\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{[-(v + \lambda) + v + 2\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{[-v - \lambda + v + 2\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{\lambda}$$

Similarly at $s = -(v + 2\lambda)$

$$a_{-i} = \lim_{s \to -(v+2\lambda)} \frac{[s + (v + 2\lambda)]\, e^{st}}{(s + v + \lambda)(s + v + 2\lambda)}$$

$$= \lim_{s \to -(v+\lambda)} \frac{e^{st}}{[s + v + \lambda]}$$

$$= \frac{e^{-(v+2\lambda)t}}{[-(v + 2\lambda) + v + 2\lambda]}$$

$$= \frac{e^{-(v+2\lambda)t}}{[-v - 2\lambda + v + \lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{-\lambda}$$

Thus

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s+v+\lambda)(s+v+2\lambda)}\right\} &= \sum a_{-i} \\ &= \frac{e^{-(v+\lambda)t}}{\lambda} - \frac{e^{-(v+2\lambda)t}}{\lambda} \\ &= \frac{e^{-(v+\lambda)t}}{\lambda}\left(1-e^{-\lambda t}\right) \end{aligned}$$

$$\begin{aligned} \therefore L^{-1}\left\{\frac{(v+\lambda)z}{(s+v+\lambda)(s+v+2\lambda)}\right\} &= (v+\lambda)z\frac{e^{-(v+\lambda)t}}{\lambda}\left(1-e^{-\lambda t}\right) \\ &= (v+\lambda)e^{-(v+\lambda)t}\frac{z}{\lambda}\left(1-e^{-\lambda t}\right) \end{aligned}$$

Part III

$$L^{-1}\left\{\frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\right\} = (v+\lambda)(v+2\lambda)z^2L^{-1}\left\{\frac{1}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\right\}$$

The function

$$\frac{1}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}$$

has simple poles at $s=-(v+\lambda)$, $s=-(v+2\lambda)$ and $s=-(v+3\lambda)$

Its residues are given by

At the pole $s = -(v + \lambda)$

$$a_{-i} = \lim_{s \to -(v+\lambda)} \frac{(s + v + \lambda) e^{st}}{(s + v + \lambda)(s + v + 2\lambda)(s + v + 3\lambda)}$$

$$= \lim_{s \to -(v+\lambda)} \frac{e^{st}}{(s + v + 2\lambda)(s + v + 3\lambda)}$$

$$= \frac{e^{-(v+\lambda)t}}{[-(v + \lambda) + v + 2\lambda][-(v + \lambda) + v + 3\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{[-v - \lambda + v + 2\lambda][-v - \lambda + v + 3\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{\lambda (2\lambda)}$$

$$= \frac{e^{-(v+\lambda)t}}{\lambda^2 2!}$$

Similarly at the pole $s = -(v + 2\lambda)$

$$a_{-i} = \lim_{s \to -(v+2\lambda)} \frac{(s + v + 2\lambda) e^{st}}{(s + v + \lambda)(s + v + 2\lambda)(s + v + 3\lambda)}$$

$$= \lim_{s \to -2\lambda} \frac{e^{st}}{(s + v + \lambda)(s + v + 3\lambda)}$$

$$= \frac{e^{-(v+2\lambda)t}}{[-(v + 2\lambda) + v + \lambda][-(v + 2\lambda) + v + 3\lambda]}$$

$$= \frac{e^{-(v+2\lambda)t}}{[-v - 2\lambda + v + \lambda][-v - 2\lambda + v + 3\lambda]}$$

$$= \frac{e^{-(v+2\lambda)t}}{-\lambda(\lambda)}$$

$$= \frac{e^{-(v+2\lambda)t}}{-\lambda^2}$$

Similarly at $s = -(v+3\lambda)$

$$a_{-i} = \lim_{s\to-(v+3\lambda)} \frac{(s+v+3\lambda)e^{st}}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}$$

$$= \lim_{s\to-(v+3\lambda)} \frac{e^{st}}{(s+v+\lambda)(s+v+2\lambda)}$$

$$= \frac{e^{-(v+3\lambda)t}}{[-(v+3\lambda)+v+\lambda][-(v+3\lambda)+v+2\lambda]}$$

$$= \frac{e^{-(v+3\lambda)t}}{[-v-3\lambda+v+\lambda][-v-3\lambda+v+2\lambda]}$$

$$= \frac{e^{-(v+3\lambda)t}}{-2\lambda(-\lambda)}$$

$$= \frac{e^{-(v+3\lambda)t}}{2!\lambda^2}$$

Thus

$$L^{-1}\left\{\frac{1}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\right\} = \sum a_{-i}$$

$$= \frac{e^{-(v+\lambda)t}}{2!\lambda^2} - \frac{e^{-(v+\lambda)t}}{\lambda^2} + \frac{e^{-(v+\lambda)t}}{2!\lambda^2}$$

$$= \frac{e^{-(v+\lambda)t}}{2!\lambda^2}\left(1-2e^{-\lambda t}+e^{-2\lambda t}\right)$$

$$= \frac{e^{-(v+\lambda)t}}{2!\lambda^2}\left(1-e^{-\lambda t}\right)^2$$

$$\therefore L^{-1}\left\{\frac{(v+\lambda)(v+2\lambda)z^2}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)}\right\}=\left\{\begin{array}{l}(v+\lambda)(v+2\lambda)z^2\times \\ \\ \frac{e^{-(v+\lambda)t}}{2!\lambda^2}\left(1-e^{-\lambda t}\right)^2\end{array}\right\}$$

$$=(v+\lambda)(v+2\lambda)e^{-(v+\lambda)t}\frac{z^2}{2!\lambda^2}\left(1-e^{-\lambda t}\right)^2$$

Part IV

$$L^{-1}\left\{\frac{\begin{array}{c}(v+\lambda)(v+2\lambda)\\ (v+3\lambda)z^3\end{array}}{\begin{array}{c}(s+v+\lambda)(s+v+2\lambda)\\ (s+v+3\lambda)(s+v+4\lambda)\end{array}}\right\}=\left\{\begin{array}{c}(v+\lambda)(v+2\lambda)(v+3\lambda)z^3\times \\ \\ L^{-1}\left\{\frac{1}{\begin{array}{c}(s+v+\lambda)(s+v+2\lambda)\\ (s+v+3\lambda)(s+v+4\lambda)\end{array}}\right\}\end{array}\right\}$$

The function

$$\frac{1}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}$$

has simple poles at $s=-(v+\lambda)$, $s=-(v+2\lambda)$, $s=-(v+3\lambda)$ and $s=-(v+4\lambda)$

Its residues at each are given by

At the pole $s=-(v+\lambda)$

$$a_{-i}=\lim_{s\to-(v+\lambda)}\frac{(s+v+\lambda)e^{st}}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}$$

$$=\lim_{s\to-(v+\lambda)}\frac{e^{st}}{(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}$$

$$= \frac{e^{-(v+\lambda)t}}{[-(v+\lambda)+v+2\lambda][-(v+\lambda)+v+3\lambda][-(v+\lambda)+v+4\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{[-v-\lambda+v+2\lambda][-v-\lambda+v+3\lambda][-v-\lambda+v+4\lambda]}$$

$$= \frac{e^{-(v+\lambda)t}}{\lambda(2\lambda)(3\lambda)}$$

$$= \frac{e^{-(v+\lambda)t}}{\lambda^3 3!}$$

Similarly at the pole $s = -(v+2\lambda)$

$$a_{-i} = \lim_{s \to -(v+2\lambda)} \frac{(s+v+2\lambda)e^{st}}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}$$

$$= \lim_{s \to -2\lambda} \frac{e^{st}}{(s+v+\lambda)(s+v+3\lambda)(s+v+4\lambda)}$$

$$= \frac{e^{-(v+2\lambda)t}}{[-(v+2\lambda)+v+\lambda][-(v+2\lambda)+v+3\lambda][-(v+2\lambda)+v+4\lambda]}$$

$$= \frac{e^{-(v+2\lambda)t}}{[-v-2\lambda+v+\lambda][-v-2\lambda+v+3\lambda][-v-2\lambda+v+4\lambda]}$$

$$= \frac{e^{-(v+2\lambda)t}}{-\lambda(\lambda)(2\lambda)}$$

$$= \frac{e^{-(v+2\lambda)t}}{-\lambda^3 2!}$$

Similarly at the pole $s = -(v + 3\lambda)$

$$a_{-i} = \lim_{s \to -(v+3\lambda)} \frac{(s + v + 3\lambda)\, e^{st}}{(s + v + \lambda)(s + v + 2\lambda)(s + v + 3\lambda)(s + v + 4\lambda)}$$

$$= \lim_{s \to -(v+3\lambda)} \frac{e^{st}}{(s + v + \lambda)(s + v + 2\lambda)(s + v + 4\lambda)}$$

$$= \frac{e^{-(v+3\lambda)t}}{[-(v + 3\lambda) + v + \lambda]\,[-(v + 3\lambda) + v + 2\lambda]\,[-(v + 3\lambda) + v + 4\lambda]}$$

$$= \frac{e^{-(v+3\lambda)t}}{[-v - 3\lambda + v + \lambda]\,[-v - 3\lambda + v + 2\lambda]\,[-v - 3\lambda + v + 4\lambda]}$$

$$= \frac{e^{-(v+3\lambda)t}}{-2\lambda\,(-\lambda)\,(\lambda)}$$

$$= \frac{e^{-(v+3\lambda)t}}{\lambda^3 2!}$$

Similarly at the pole $s = -(v + 4\lambda)$

$$a_{-i} = \lim_{s \to -(v+4\lambda)} \frac{(s + v + 4\lambda)\, e^{st}}{(s + v + \lambda)(s + v + 2\lambda)(s + v + 3\lambda)(s + v + 4\lambda)}$$

$$= \lim_{s \to -(v+4\lambda)} \frac{e^{st}}{(s + v + \lambda)(s + v + 2\lambda)(s + v + 3\lambda)}$$

$$= \frac{e^{-(v+4\lambda)t}}{[-(v + 4\lambda) + v + \lambda]\,[-(v + 4\lambda) + v + 2\lambda]\,[-(v + 4\lambda) + v + 3\lambda]}$$

$$= \frac{e^{-(v+4\lambda)t}}{[-v - 4\lambda + v + \lambda]\,[-v - 4\lambda + v + 2\lambda]\,[-v - 4\lambda + v + 3\lambda]}$$

$$= \frac{e^{-(v+4\lambda)t}}{-3\lambda(-2\lambda)(-\lambda)}$$

$$= \frac{e^{-(v+4\lambda)t}}{-\lambda^3 3!}$$

Thus

$$L^{-1}\left\{\frac{1}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}\right\} = \sum a_{-i}$$

$$= \frac{e^{-(v+\lambda)t}}{3!\lambda^3} - \frac{e^{-(v+2\lambda)t}}{2!\lambda^3} + \frac{e^{-(v+3\lambda)t}}{2!\lambda^3} - \frac{e^{-(v+4\lambda)t}}{3!\lambda^3}$$
$$= \frac{e^{-(v+\lambda)t}}{3!\lambda^3}\left(1 - 3e^{-\lambda t} + 3e^{-2\lambda t} - e^{-3\lambda t}\right)$$
$$= \frac{e^{-(v+\lambda)t}}{3!\lambda^3}\left(1 - e^{-\lambda t}\right)^3$$

Therefore

$$L^{-1}\left\{\frac{(v+\lambda)(v+2\lambda)(v+3\lambda)z^3}{(s+v+\lambda)(s+v+2\lambda)(s+v+3\lambda)(s+v+4\lambda)}\right\} = (v+\lambda)(v+2\lambda)(v+3\lambda)z^3\frac{e^{-(v+\lambda)t}}{\lambda^3 3!}\left(1 - e^{-\lambda t}\right)^3$$

$$= (v+\lambda)(v+2\lambda)(v+3\lambda)e^{-(v+\lambda)t}\frac{z^3}{\lambda^3 3!}\left(1 - e^{-\lambda t}\right)^3$$

Consolidating the above results we get

$$L^{-1}(D) = \left\{\begin{array}{c} e^{-(v+\lambda)t} + (v+\lambda)e^{-(v+\lambda)t}\frac{z}{\lambda}\left(1 - e^{-\lambda t}\right) + \\ (v+\lambda)(v+2\lambda)e^{-(v+\lambda)t}\frac{z^2}{\lambda^2 2!}\left(1 - e^{-\lambda t}\right)^2 + \\ (v+\lambda)(v+2\lambda)(v+3\lambda)e^{-(v+\lambda)t}\frac{z^3}{\lambda^3 3!}\left(1 - e^{-\lambda t}\right)^3 + ... \end{array}\right\}$$

But by equation (5.74)

$$G(z,t) = zL^{-1}(D)$$

$$= z\left\{\begin{array}{c} e^{-(v+\lambda)t} + (v+\lambda)\, e^{-(v+\lambda)t}\frac{z}{\lambda}\left(1-e^{-\lambda t}\right) + \\ (v+\lambda)(v+2\lambda)\, e^{-(v+\lambda)t}\frac{z^2}{\lambda^2 2!}\left(1-e^{-\lambda t}\right)^2 + \\ (v+\lambda)(v+2\lambda)(v+3\lambda)\, e^{-(v+\lambda)t}\frac{z^3}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3 + ... \end{array}\right\}$$

$$= ze^{-(v+\lambda)t}\left\{\begin{array}{c} 1 + \left(\frac{v}{\lambda}+1\right) z\left(1-e^{-\lambda t}\right) + \left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2 + \\ \left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+3\right)\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3 + ... \end{array}\right\}$$

Multiplying the RHS by $\frac{\left(\frac{v}{\lambda}\right)!}{\left(\frac{v}{\lambda}\right)!}$ yields

$$G(z,t) = ze^{-(v+\lambda)t}\frac{\frac{v}{\lambda}!}{\frac{v}{\lambda}!}\left\{\begin{array}{c} 1 + \left(\frac{v}{\lambda}+1\right) z\left(1-e^{-\lambda t}\right) + \\ \left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2 + \\ \left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+3\right)\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3 + ... \end{array}\right\}$$

$$G(z,t) = ze^{-(v+\lambda)t}\left\{\begin{array}{c} \frac{\frac{v}{\lambda}!}{\frac{v}{\lambda}!} + \frac{\left(\frac{v}{\lambda}+1\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!} z\left(1-e^{-\lambda t}\right) + \\ \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!}\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2 + \\ \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+3\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3 + ... \end{array}\right\}$$

$$= ze^{-(v+\lambda)t} \left\{ \begin{array}{c} 1 + \frac{\left(\frac{v}{\lambda}+1\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!1!} z\left(1 - e^{-\lambda t}\right) + \\ \\ \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!2!} \left[z\left(1 - e^{-\lambda t}\right)\right]^2 + \\ \\ \frac{\left(\frac{v}{\lambda}+1\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+3\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!2!} \left[z\left(1 - e^{-\lambda t}\right)\right]^3 + ... \end{array} \right\}$$

Since

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We have

$$\binom{\frac{v}{\lambda}+0}{0} = 1$$

$$\begin{aligned} \binom{\frac{v}{\lambda}+1}{1} &= \frac{\left(\frac{v}{\lambda}+1\right)!}{\frac{v}{\lambda}!1!} \\ &= \frac{\left(\frac{v}{\lambda}+1\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!1!} \end{aligned}$$

$$\begin{aligned} \binom{\frac{v}{\lambda}+2}{2} &= \frac{\left(\frac{v}{\lambda}+2\right)!}{\frac{v}{\lambda}!2!} \\ &= \frac{\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+1\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!2!} \end{aligned}$$

$$\begin{aligned} \binom{\frac{v}{\lambda}+3}{3} &= \frac{\left(\frac{v}{\lambda}+3\right)!}{\frac{v}{\lambda}!3!} \\ &= \frac{\left(\frac{v}{\lambda}+3\right)\left(\frac{v}{\lambda}+2\right)\left(\frac{v}{\lambda}+1\right)\frac{v}{\lambda}!}{\frac{v}{\lambda}!2!} \end{aligned}$$

And so on

With this we have

$$G(z,t) = ze^{-(v+\lambda)t} \left\{ \begin{array}{c} 1 + \binom{\frac{v}{\lambda}+1}{1} z\left(1-e^{-\lambda t}\right) + \\ \binom{\frac{v}{\lambda}+2}{2} \left[z\left(1-e^{-\lambda t}\right)\right]^2 + \\ \binom{\frac{v}{\lambda}+3}{3} \left[z\left(1-e^{-\lambda t}\right)\right]^3 + \ldots \end{array} \right\}$$

$$= ze^{-(v+\lambda)t} \sum_{j=0}^{\infty} \binom{\frac{v}{\lambda}+j}{j} \left[z\left(1-e^{-\lambda t}\right)\right]^j$$

But

$$\binom{\frac{v}{\lambda}+j}{j} = \binom{\left[\frac{v}{\lambda}+1\right]+j-1}{j}$$

Therefore

$$G(z,t) = ze^{-(v+\lambda)t} \sum_{j=0}^{\infty} \binom{\left[\frac{v}{\lambda}+1\right]+j-1}{j} (-1)^j \left[z\left(1-e^{-\lambda t}\right)\right]^j$$

But

$$\binom{-r}{j}(-1)^j = \binom{r+j-1}{j}$$

$$\Rightarrow \binom{-\left[\frac{v}{\lambda}+1\right]}{j}(-1)^j = \binom{\left[\frac{v}{\lambda}+1\right]+j-1}{j}$$

We have

$$G(z,t) = ze^{-(v+\lambda)t} \sum_{j=0}^{\infty} \binom{-\left[\frac{v}{\lambda}+1\right]}{j} (-1)^j \left[z\left(1-e^{-\lambda t}\right)\right]^j$$

$$= ze^{-(v+\lambda)t} \sum_{j=0}^{\infty} \binom{-\left[\frac{v}{\lambda}+1\right]}{j} \left[-z\left(1-e^{-\lambda t}\right)\right]^j$$

$$= ze^{-(v+\lambda)t} \left[1-z\left(1-e^{-\lambda t}\right)\right]^{-\left(\frac{v}{\lambda}+1\right)}$$

$$= ze^{-\lambda t\left(\frac{v}{\lambda}+1\right)} \left[1-z\left(1-e^{-\lambda t}\right)\right]^{-\left(\frac{v}{\lambda}+1\right)}$$

$$= z\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{\frac{v}{\lambda}+1}$$

$$= z^{-\frac{v}{\lambda}}\left[\frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{\frac{v}{\lambda}+1}$$

Which by identification is the pgf of a negative binomial distribution with $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

$G(z,t)$ can be expressed as

$$G(z,t) = z\left[\frac{p}{1-qz}\right]^{\frac{v}{\lambda}+1}$$

$P_n(t)$ is the coefficient of z^n in $G(z,t)$. But

$$G(z,t) = zp^{\frac{v}{\lambda}+1}(1-qz)^{-\left(\frac{v}{\lambda}+1\right)}$$
$$= zp^{\frac{v}{\lambda}+1} \sum_{n=0}^{\infty} \binom{-\left[\frac{v}{\lambda}+1\right]}{n} (-qz)^n$$

$$= zp^{\frac{v}{\lambda}+1} \sum_{n=0}^{\infty} \binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n} (-1)^n(-qz)^n$$

$$= zp^{\frac{v}{\lambda}+1}\sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}(-1)^n(-1)^n(qz)^n$$

$$= zp^{\frac{v}{\lambda}+1}\sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}(qz)^n$$

$$= \sum_{n=0}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+n-1}{n}p^{\frac{v}{\lambda}+1}q^n z^{n+1}$$

$$= \sum_{n=1}^{\infty}\binom{\left[\frac{v}{\lambda}+1\right]+(n-1)-1}{n-1}p^{\frac{v}{\lambda}+1}q^{n-1}z^n$$

$$= \sum_{n=1}^{\infty}\binom{n+\frac{v}{\lambda}-1}{n-1}p^{\frac{v}{\lambda}+1}q^{n-1}z^n$$

Therefore the coefficient of z^n is

$$\binom{n+\frac{v}{\lambda}-1}{n-1}p^{\frac{v}{\lambda}+1}q^{n-1} = \binom{n+\frac{v}{\lambda}-1}{n-1}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{n-1}$$

Hence

$$\begin{aligned} P_n(t) &= \binom{n+\frac{v}{\lambda}-1}{n-1}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{n-1} \qquad n = 1, 2, 3, ... \\ &= \binom{\left[\frac{v}{\lambda}+1\right]+k-1}{k}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{k} \qquad k = 0, 1, 2, 3, ... \end{aligned}$$

where $k = n - 1$.

This is the pmf of a negative binomial distribution with parameters $r = \frac{v}{\lambda} + 1$ and $p = e^{-\lambda t}$

Case 2: When initial population $X(0) = n_0$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$
$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$
$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = n_0$, we have

$$P_{n_0}(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq n_0$$

$$\therefore G(z,0) = z^{n_0}$$

With this equation (5.70b) becomes

$$(1-z)^{-\frac{s}{\lambda}} z^{\frac{s+v}{\lambda}} \overline{G}(z,s) = \frac{1}{\lambda} \int \frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}} z^{n_0}}{z(1-z)^{\frac{s}{\lambda}+1}} dz$$

$$(1-z)^{-\frac{s}{\lambda}} z^{\frac{s+v}{\lambda}} \overline{G}(z,s) = \frac{1}{\lambda} \int \frac{z^{\frac{s}{\lambda}+\frac{v}{\lambda}+n_0-1}}{(1-z)^{\frac{s}{\lambda}+1}} dz \tag{5.76}$$

We first simplify the right hand side as follows;

Recall that

$$\int \frac{x^{a+r-1}}{(1-x)^{a+1}} dx = x^{a+r} \left\{ \begin{array}{l} \frac{{}_2F_1(a,a+r;a+r+1;x)}{a+r} + \\ \\ x \ \frac{{}_2F_1(a+1,a+r+1;a+r+2;x)}{a+r+1} \end{array} \right\} + \text{constant}$$

$$\therefore \frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}+\left(n_0+\frac{v}{\lambda}\right)-1}}{(1-z)^{\frac{s}{\lambda}+1}}dz = \frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}}{\lambda}\left\{\begin{array}{c} \frac{{}_2F_1\left(\frac{s}{\lambda},\frac{s}{\lambda}+n_0+\frac{v}{\lambda};\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}+ \\ \\ z\frac{{}_2F_1\left(\frac{s}{\lambda}+1,\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1} \end{array}\right\} + \frac{d_2}{\lambda}$$

But

$$ {}_2F_1\left(a,b;c;z\right) = (1-z)^{c-a-b} \quad {}_2F_1\left(c-a,c-b;c;z\right)$$

We now use this property to simplify the above integral, We start with

$$ {}_2F_1\left(\frac{s}{\lambda},\frac{s}{\lambda}+n_0+\frac{v}{\lambda};\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)$$

Here

$$a=\frac{s}{\lambda}, \quad b=\frac{s}{\lambda}+n_0+\frac{v}{\lambda}, \quad c=\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1$$

Thus

$$\begin{aligned} c-a &= \frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1-\frac{s}{\lambda} \\ &= n_0+\frac{v}{\lambda}+1 \end{aligned}$$

$$\begin{aligned} c-b &= \frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1-\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}\right) \\ &= \frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1-\frac{s}{\lambda}-n_0-\frac{v}{\lambda} \\ &= 1 \end{aligned}$$

$$\begin{aligned} c-a-b &= \frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1-\frac{s}{\lambda}-\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}\right) \\ &= \frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1-\frac{s}{\lambda}-\frac{s}{\lambda}-n_0-\frac{v}{\lambda} \\ &= 1-\frac{s}{\lambda} \end{aligned}$$

Thus

$$ {}_2F_1\left(\frac{s}{\lambda}, \frac{s}{\lambda} + n_0 + \frac{v}{\lambda}; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1; z\right) = (1-z)^{1-\frac{s}{\lambda}} \;\; {}_2F_1\left(n_0 + \frac{v}{\lambda} + 1, 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1; z\right) $$

Similarly for

$$ {}_2F_1\left(\frac{s}{\lambda} + 1, \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2; z\right) $$

We have

$$ a = \frac{s}{\lambda} + 1, \quad b = \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1, \quad c = \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2 $$

implying that

$$ \begin{aligned} c - a &= \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2 - \left(\frac{s}{\lambda} + 1\right) \\ &= \frac{s}{\lambda} + n_0 + 2 - \frac{s}{\lambda} - 1 \\ &= n_0 + \frac{v}{\lambda} + 1 \end{aligned} $$

$$ \begin{aligned} c - b &= \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2 - \left(\frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1\right) \\ &= \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2 - \frac{s}{\lambda} - n_0 - \frac{v}{\lambda} - 1 \\ &= 1 \end{aligned} $$

$$ \begin{aligned} c - a - b &= \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2 - \left(\frac{s}{\lambda} + 1\right) - \left(\frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1\right) \\ &= \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2 - \frac{s}{\lambda} - 1 - \frac{s}{\lambda} - n_0 - \frac{v}{\lambda} - 1 \\ &= -\frac{s}{\lambda} \end{aligned} $$

Thus

$$ \begin{aligned} {}_2F_1\left(\frac{s}{\lambda} + 1, \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2; z\right) = (1-z)^{-\frac{s}{\lambda}} \times \\ {}_2F_1\left(n_0 + \frac{v}{\lambda} + 1, 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2; z\right) \end{aligned} $$

consolidating the above results, We get

$$\frac{1}{\lambda}\int \frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}-1}}{(1-z)^{\frac{s}{\lambda}+1}}dz = \frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}}{\lambda}\left\{\begin{array}{c} \frac{(1-z)^{1-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}+ \\ \\ z\frac{(1-z)^{-\frac{s}{\lambda}}\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1} \end{array}\right\}+\frac{d_2}{\lambda}$$

$$= \frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}}{\lambda}\left\{\begin{array}{c} \frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}+ \\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1} \end{array}\right\}+\frac{d_2}{\lambda}$$

$$= \frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}}{\lambda}\left\{\begin{array}{c} \frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{\frac{1}{\lambda}(s+v+\lambda n_0)}+ \\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{\frac{1}{\lambda}[s+v+\lambda(n_0+1)]} \end{array}\right\}+\frac{d_2}{\lambda}$$

$$= \frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\lambda}{\lambda}\left\{\begin{array}{c} \frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+ \\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]} \end{array}\right\}+\frac{d_2}{\lambda}$$

$$= z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\left\{\begin{array}{c} \frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+ \\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]} \end{array}\right\}+\frac{d_2}{\lambda}$$

Therefore

$$z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\,\overline{G}(z,s)=\frac{1}{\lambda}\int\frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}-1}}{(1-z)^{\frac{s}{\lambda}+1}}dz$$

$$=z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]}\end{array}\right\}+\frac{d_2}{\lambda}$$

Thus

$$\overline{G}(z,s)=\frac{z^{\frac{s}{\lambda}+n_0+\frac{v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}}{z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]}\end{array}\right\}+\frac{d_2}{\lambda z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}}$$

$$=z^{n_0}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]}\end{array}\right\}+\frac{d_2}{\lambda z^{\frac{s+v}{\lambda}}(1-z)^{-\frac{s}{\lambda}}}$$

But since for all $t, z \leqslant 1$ we have $G(z,t) \leqslant 1$, It follows that $d_2 = 0$

Therefore

$$\overline{G}(z,s)=z^{n_0}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]}\end{array}\right\}$$

But according to Euler the Gauss hyper geometric series can be expressed as

$$ {}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\frac{x^k}{k!} $$

$$ = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a\left(a+1\right)b\left(b+1\right)}{c\left(c+1\right)}\frac{x^2}{2!} + \frac{a\left(a+1\right)\left(a+2\right)b\left(b+1\right)\left(b+2\right)}{c\left(c+1\right)\left(c+2\right)}\frac{x^3}{3!} + ... $$

Where $a,\ b$ and c are complex numbers and

$$ (a)_k = \prod_{i=0}^{k-i}\left(a+i\right) $$
$$ = a\left(a+1\right)\left(a+2\right)\left(a+3\right)...\left(a+k-1\right) $$

We shall use this property to simplify the terms in the RHS

For

$$ {}_2F_1\left(n_0 + \frac{v}{\lambda} + 1, 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1; z\right) $$

we let $a = n_0 + \frac{v}{\lambda} + 1,\quad b = 1$ and $c = \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1$

Thus

$$ F\left(n_0 + \frac{v}{\lambda} + 1, 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 1; z\right) = F\left(a,b;c;z\right) $$

$$ = \left\{ \begin{array}{c} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + ... \end{array} \right\} $$

$$= \left\{ \begin{matrix} 1 + \frac{\left(n_0+\frac{v}{\lambda}+1\right)z}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1\right)} + \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)2!}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)}\frac{z^2}{2!}+ \\ \\ \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)\left(n_0+\frac{v}{\lambda}+3\right)3!}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+3\right)}\frac{z^3}{3!} + \ldots \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} 1 + \frac{\left(n_0+\frac{v}{\lambda}+1\right)z}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1\right)} + \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)z^2}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)}+ \\ \\ \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)\left(n_0+\frac{v}{\lambda}+3\right)z^3}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+3\right)} + \ldots \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} 1 + \frac{\frac{1}{\lambda}(\lambda n_0+v+\lambda)z}{\frac{1}{\lambda}(s+\lambda n_0+v+\lambda)} + \frac{\frac{1}{\lambda^2}(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)z^2}{\frac{1}{\lambda^2}(s+\lambda n_0+v+\lambda)(s+\lambda n_0+v+2\lambda)}+ \\ \\ \frac{\frac{1}{\lambda^3}(\lambda n_0+v+\lambda)(\lambda n_0+v+\lambda)(\lambda n_0+v+\lambda)z^3}{\frac{1}{\lambda^3}(s+\lambda n_0+v+\lambda)(s+\lambda n_0+v+2\lambda)(s+\lambda n_0+v+3\lambda)} + \ldots \end{matrix} \right\}$$

Therefore

$$F\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right) = \left\{ \begin{matrix} 1 + \frac{(\lambda n_0+v+\lambda)z}{(s+\lambda n_0+v+\lambda)} + \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)z^2}{(s+\lambda n_0+v+\lambda)(s+\lambda n_0+v+2\lambda)}+ \\ \\ \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+\lambda)(\lambda n_0+v+\lambda)z^3}{(s+\lambda n_0+v+\lambda)(s+\lambda n_0+v+2\lambda)(s+\lambda n_0+v+3\lambda)} + \ldots \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} 1 + \frac{(\lambda n_0+v+\lambda)z}{[s+v+\lambda(n_0+1)]} + \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)z^2}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}+ \\ \\ \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+\lambda)(\lambda n_0+v+\lambda)z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]} + \ldots \end{matrix} \right\}$$

Similarly for

$${}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)$$

Letting $a = n_0 + \frac{v}{\lambda} + 1, \quad b = 1$ and implies that $c = \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2$

$$F\left(n_0 + \frac{v}{\lambda} + 1, 1; \frac{s}{\lambda} + n_0 + \frac{v}{\lambda} + 2; z\right) = F(a, b; c; z)$$

$$= \left\{ \begin{array}{l} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \\ \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\left(n_0+\frac{v}{\lambda}+1\right)z}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)} + \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)2!}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+3\right)}\frac{z^2}{2!} + \\ \\ \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)\left(n_0+\frac{v}{\lambda}+3\right)3!}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+3\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+4\right)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\left(n_0+\frac{v}{\lambda}+1\right)z}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)} + \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)z^2}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+3\right)} + \\ \\ \frac{\left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}+2\right)\left(n_0+\frac{v}{\lambda}+3\right)z^3}{\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+3\right)\left(\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+4\right)} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\frac{1}{\lambda}(\lambda n_0+v+\lambda)z}{\frac{1}{\lambda}[s+\lambda n_0+v+2\lambda]} + \frac{\frac{1}{\lambda^2}(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)z^2}{\frac{1}{\lambda^2}[s+\lambda n_0+v+2\lambda][s+\lambda n_0+v+3\lambda]} + \\ \\ \frac{\frac{1}{\lambda^3}(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)(\lambda n_0+v+3\lambda)z^3}{\frac{1}{\lambda^3}[s+\lambda n_0+v+2\lambda][s+\lambda n_0+v+3\lambda][s+\lambda n_0+v+4\lambda]} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{(\lambda n_0+v+\lambda)z}{[s+\lambda n_0+v+2\lambda]} + \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)z^2}{[s+\lambda n_0+v+2\lambda][s+\lambda n_0+v+3\lambda]} + \\ \\ \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)(\lambda n_0+v+3\lambda)z^3}{[s+\lambda n_0+v+2\lambda][s+\lambda n_0+v+3\lambda][s+\lambda n_0+v+4\lambda]} + \ldots \end{array} \right\}$$

Therefore

$$F\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)=\left\{\begin{array}{c}1+\frac{[v+\lambda(n_0+1)]z}{[s+v+\lambda(n_0+2)]}+\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{[s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}+\\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]}+\ldots\end{array}\right\}$$

Using the above results, We get

$$\overline{G}(z,s)=z^{n_0}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{[s+v+\lambda(n_0+1)]}\end{array}\right\}$$

$$=z^{n_0}\left\{\begin{array}{c}\frac{(1-z)}{(s+v+\lambda n_0)}\left[\begin{array}{c}1+\frac{[v+\lambda(n_0+1)]z}{[s+v+\lambda(n_0+1)]}+\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}+\\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}+\ldots\end{array}\right]\\ \\ \frac{z}{[s+v+\lambda(n_0+1)]}\left[\begin{array}{c}1+\frac{(\lambda n_0+v+\lambda)z}{[s+v+\lambda(n_0+2)]}+\frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)z^2}{[s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}+\\ \\ \frac{(\lambda n_0+v+\lambda)(\lambda n_0+v+2\lambda)(\lambda n_0+v+3\lambda)z^3}{[s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]}+\ldots\end{array}\right]\end{array}\right\}$$

$$= z^{n_0} \left\{ \begin{array}{c} (1-z) \left[\begin{array}{c} \frac{1}{(s+v+\lambda n_0)} + \frac{[v+\lambda(n_0+1)]z}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)]} + \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]} + \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]} + \ldots \end{array} \right] \\ z \left[\begin{array}{c} \frac{1}{[s+v+\lambda(n_0+1)]} + \frac{[v+\lambda(n_0+1)]z}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]} + \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]} + \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]} + \ldots \end{array} \right] \end{array} \right\}$$

From this it follows that $\overline{G}(z,s)$ is of the form

$$\overline{G}(z,s) = z^{n_0}\left[(1-z)E + zF\right] \tag{5.77}$$

Where

$$E = \left\{ \begin{array}{c} \frac{1}{(s+v+\lambda n_0)} + \frac{[v+\lambda(n_0+1)]z}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)]} + \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]} + \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]} + \ldots \end{array} \right\}$$

and

$$F=\left\{\begin{array}{c}\frac{1}{[s+v+\lambda(n_0+1)]}+\frac{[v+\lambda(n_0+1)]z}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}+\\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}+\\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]}+\ldots\end{array}\right\}$$

The next step is to apply inverse Laplace transform to both sides of equation (5.77).

$$\begin{aligned}L^{-1}\left\{\overline{G}\left(z,s\right)\right\}&=L^{-1}\left\{z^{n_0}\left[\left(1-z\right)E+zF\right]\right\}\\&=z^{n_0}\left[\left(1-z\right)L^{-1}\left\{E\right\}+zL^{-1}\left\{F\right\}\right]\end{aligned}$$

$$\therefore G\left(z,t\right)=z^{n_0}\left[\left(1-z\right)L^{-1}\left\{E\right\}+zL^{-1}\left\{F\right\}\right]$$

Various methods can be used, We however focus on the complex inversion formula . For ease of computations We deal with both E and F separately.

Dealing with E

$$L^{-1}\left\{E\right\}=L^{-1}\left\{\begin{array}{c}\frac{1}{(s+v+\lambda n_0)}+\frac{[v+\lambda(n_0+1)]z}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)]}+\\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}+\\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}+\ldots\end{array}\right\}$$

$$
= \left[\begin{array}{c} L^{-1}\left\{\frac{1}{(s+v+\lambda n_0)}\right\} + L^{-1}\left\{\frac{[v+\lambda(n_0+1)]z}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)]}\right\} + \\ L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}\right\} + \\ L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}\right\} + ... \end{array} \right]
$$

We now simplify the above first four terms separately

First term

$$
L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}
$$

$$
\Rightarrow L^{-1}\left\{\frac{1}{(s+v+\lambda n_0)}\right\} = e^{-(v+\lambda n_0)t}
$$

Second term

$$
L^{-1}\left\{\frac{[v+\lambda(n_0+1)]\,z}{(s+v+\lambda n_0)\,[s+v+\lambda(n_0+1)]}\right\} = \left\{ \begin{array}{c} [v+\lambda(n_0+1)]\,z\,* \\ L^{-1}\left\{\frac{1}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)]}\right\} \end{array} \right\}
$$

The function

$$
\frac{1}{(s+v+\lambda n_0)\,[s+v+\lambda(n_0+1)]}
$$

has simple poles at $s = -(v+\lambda n_0)$ and $s = -[v+\lambda(n_0+1)]$

Thus its residue at each pole is obtained as follows

At $s = -(v + \lambda n_0)$

$$a_{-i} = \lim_{s \to -(v+\lambda n_0)} \frac{[s + v + \lambda n_0]\, e^{st}}{(s + v + \lambda n_0)\,[s + v + \lambda (n_0 + 1)]}$$

$$= \lim_{s \to -(v+\lambda n_0)} \frac{e^{st}}{[s + v + \lambda (n_0 + 1)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{[-(v + \lambda n_0) + v + \lambda (n_0 + 1)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{[-v - \lambda n_0 + v + \lambda n_0 + \lambda]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda}$$

At $s = -[v + \lambda (n_0 + 1)]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+1)]} \frac{[s + v + \lambda (n_0 + 1)]\, e^{st}}{(s + v + \lambda n_0)\,[s + v + \lambda (n_0 + 1)]}$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \frac{e^{st}}{(s + v + \lambda n_0)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-[v + \lambda (n_0 + 1)] + v + \lambda n_0]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-v - \lambda n_0 - \lambda + v + \lambda n_0]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda}$$

Thus

$$L^{-1}\left\{\frac{1}{(s+v+\lambda n_0)\left[s+v+\lambda(n_0+1)\right]}\right\}=\sum a_{-i}$$
$$=\frac{e^{-(v+\lambda n_0)t}}{\lambda}-\frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda}$$
$$=\frac{e^{-(v+\lambda n_0)t}}{\lambda}\left(1-e^{-\lambda t}\right)$$

Therefore

$$L^{-1}\left\{\frac{\left[v+\lambda(n_0+1)\right]z}{(s+v+\lambda n_0)\left[s+v+\lambda(n_0+1)\right]}\right\}=\left[v+\lambda(n_0+1)\right]z\frac{e^{-(v+\lambda n_0)t}}{\lambda}\left(1-e^{-\lambda t}\right)$$

$$=\left[v+\lambda(n_0+1)\right]e^{-(v+\lambda n_0)t}\frac{z}{\lambda}\left(1-e^{-\lambda t}\right)$$
$$=\lambda\left[\frac{v}{\lambda}+(n_0+1)\right]e^{-(v+\lambda n_0)t}\frac{z}{\lambda}\left(1-e^{-\lambda t}\right)$$
$$=\left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}z\left(1-e^{-\lambda t}\right)$$

Third term

$$L^{-1}\left\{\frac{\left[v+\lambda(n_0+1)\right]\left[v+\lambda(n_0+2)\right]z^2}{(s+v+\lambda n_0)\left[s+v+\lambda(n_0+1)\right]\left[s+v+\lambda(n_0+2)\right]}\right\}$$

$$=\left\{\begin{array}{c}\left[v+\lambda(n_0+1)\right]\left[v+\lambda(n_0+2)\right]z^2*\\ \\ L^{-1}\left\{\frac{1}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}\right\}\end{array}\right\}$$

The function

$$\frac{1}{(s+v+\lambda n_0)\left[s+v+\lambda(n_0+1)\right]\left[s+v+\lambda(n_0+2)\right]}$$

has simple poles at $s=-(v+\lambda n_0)$, $s=-\left[v+\lambda(n_0+1)\right]$ and $s=-\left[v+\lambda(n_0+2)\right]$

Thus its residue at each pole is obtained as follows

At $s = -(v + \lambda n_0)$

$$a_{-i} = \lim_{s \to -(v+\lambda n_0)} \frac{(s + v + \lambda n_0)\, e^{st}}{(s + v + \lambda n_0)\,[s + v + \lambda (n_0 + 1)]\,[s + v + \lambda (n_0 + 2)]}$$

$$= \lim_{s \to -(v+\lambda n_0)} \frac{e^{st}}{[s + v + \lambda (n_0 + 1)]\,[s + v + \lambda (n_0 + 2)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{[-(v + \lambda n_0) + v + \lambda (n_0 + 1)]\,[-(v + \lambda n_0) + v + \lambda (n_0 + 2)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{[-v - \lambda n_0 + v + \lambda n_0 + \lambda]\,[-v - \lambda n_0 + v + \lambda n_0 + 2\lambda]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda (2\lambda)}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda^2 2!}$$

At $s = -[v + \lambda (n_0 + 1)]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+1)]} \frac{[s + v + \lambda (n_0 + 1)]\, e^{st}}{(s + v + \lambda n_0)\,[s + v + \lambda (n_0 + 1)]\,[s + v + \lambda (n_0 + 2)]}$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \frac{e^{st}}{(s + v + \lambda n_0)\,[s + v + \lambda (n_0 + 2)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-[v + \lambda (n_0 + 1)] + v + \lambda n_0]\,[-[v + \lambda (n_0 + 1)] + v + \lambda (n_0 + 2)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-v - \lambda n_0 - \lambda + v + \lambda n_0]\,[-v - \lambda n_0 - \lambda + v + \lambda n_0 + 2\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda(\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda^2}$$

At $s = -[v + \lambda(n_0 + 2)]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+2)]} \frac{[s + v + \lambda(n_0 + 2)] e^{st}}{(s + v + \lambda n_0)[s + v + \lambda(n_0 + 1)][s + v + \lambda(n_0 + 2)]}$$

$$= \lim_{s \to -[v+\lambda(n_0+2)]} \frac{e^{st}}{(s + v + \lambda n_0)[s + v + \lambda(n_0 + 1)]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{[-[v + \lambda(n_0 + 2)] + v + \lambda n_0][-[v + \lambda(n_0 + 2)] + v + \lambda(n_0 + 1)]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{[-v - \lambda n_0 - 2\lambda + v + \lambda n_0][-v - \lambda n_0 - 2\lambda + v + \lambda n_0 + \lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-2\lambda(-\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{\lambda^2 2!}$$

Thus

$$L^{-1}\left\{\frac{1}{(s + v + \lambda n_0)[s + v + \lambda(n_0 + 1)][s + v + \lambda(n_0 + 2)]}\right\} = \sum a_{-i}$$

But

$$\sum a_{-i} = \frac{e^{-(v+\lambda n_0)t}}{\lambda^2 2!} - \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^2} + \frac{e^{-[v+\lambda(n_0+2)]t}}{\lambda^2 2!}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda^2 2!}\left(1 - 2e^{-\lambda t} + e^{-2\lambda t}\right)$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda^2 2!}\left(1 - e^{-\lambda t}\right)^2$$

Therefore

$$L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}\right\}$$

$$= [v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2\frac{e^{-(v+\lambda n_0)t}}{\lambda^2 2!}\left(1-e^{-\lambda t}\right)^2$$

$$= [v+\lambda(n_0+1)][v+\lambda(n_0+2)]e^{-(v+\lambda n_0)t}\frac{z^2}{\lambda^2 2!}\left(1-e^{-\lambda t}\right)^2$$

$$= \lambda\left[\frac{v}{\lambda}+n_0+1\right]\lambda\left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}\frac{z^2}{\lambda^2 2!}\left(1-e^{-\lambda t}\right)^2$$

$$= \left[\frac{v}{\lambda}+n_0+1\right]\left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2$$

Fourth term

$$L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}\right\}$$

$$= \left\{\begin{array}{l}[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3 * \\ \\ L^{-1}\left\{\frac{1}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}\right\}\end{array}\right\}$$

The function

$$\frac{1}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}$$

has simple poles at $s = -(v + \lambda n_0)$, $s = -[v + \lambda(n_0 + 1)]$, $s = -[v + \lambda(n_0 + 2)]$ and $s = -[v + \lambda(n_0 + 3)]$

Thus its residue at each pole is obtained as follows

At $s = -(v + \lambda n_0)$

$$a_{-i} = \lim_{s \to -(v+\lambda n_0)} \frac{(s + v + \lambda n_0) e^{st}}{(s + v + \lambda n_0)[s + v + \lambda(n_0 + 1)][s + v + \lambda(n_0 + 2)][s + v + \lambda(n_0 + 3)]}$$

$$= \lim_{s \to -(v+\lambda n_0)} \frac{e^{st}}{[s + v + \lambda(n_0 + 1)][s + v + \lambda(n_0 + 2)][s + v + \lambda(n_0 + 3)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{[-(v + \lambda n_0) + v + \lambda(n_0 + 1)][-(v + \lambda n_0) + v + \lambda(n_0 + 2)][-(v + \lambda n_0) + v + \lambda(n_0 + 3)]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{[-v - \lambda n_0 + v + \lambda n_0 + \lambda][-v - \lambda n_0 + v + \lambda n_0 + 2\lambda][-v - \lambda n_0 + v + \lambda n_0 + 3\lambda]}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda(2\lambda)(3\lambda)}$$

$$= \frac{e^{-(v+\lambda n_0)t}}{\lambda^3 3!}$$

At $s = -[v + \lambda(n_0 + 1)]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+1)]} \frac{[s + v + \lambda(n_0 + 1)] e^{st}}{(s + v + \lambda n_0)[s + v + \lambda(n_0 + 1)][s + v + \lambda(n_0 + 2)][s + v + \lambda(n_0 + 3)]}$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \frac{e^{st}}{(s + v + \lambda n_0)[s + v + \lambda(n_0 + 2)][s + v + \lambda(n_0 + 3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-[v+\lambda(n_0+1)]+v+\lambda n_0][-[v+\lambda(n_0+1)]+v+\lambda(n_0+2)] \\ [-[v+\lambda(n_0+1)]+v+\lambda(n_0+3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-v-\lambda n_0-\lambda+v+\lambda n_0][-v-\lambda n_0-\lambda+v+\lambda n_0+2\lambda][-v-\lambda n_0-\lambda+v+\lambda n_0+3\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda(\lambda)(2\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{-\lambda^3 2!}$$

At $s = -[v+\lambda(n_0+2)]$

$$a_{-i} = \lim_{s\to-[v+\lambda(n_0+2)]} \frac{[s+v+\lambda(n_0+2)]e^{st}}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}$$

$$= \lim_{s\to-[v+\lambda(n_0+2)]} \frac{e^{st}}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{[-[v+\lambda(n_0+2)]+v+\lambda n_0][-[v+\lambda(n_0+2)]+v+\lambda(n_0+1)] \\ [-[v+\lambda(n_0+2)]+v+\lambda(n_0+3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{[-v-\lambda n_0-2\lambda+v+\lambda n_0][-v-\lambda n_0-2\lambda+v+\lambda n_0+\lambda] \\ [-v-\lambda n_0-2\lambda+v+\lambda n_0+3\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-2\lambda(-\lambda)\lambda}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{\lambda^3 2!}$$

At $s = -\left[v + \lambda\left(n_0 + 3\right)\right]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+3)]} \frac{\left[s + v + \lambda\left(n_0 + 3\right)\right] e^{st}}{\left(s + v + \lambda n_0\right)\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]}$$

$$= \lim_{s \to -[v+\lambda(n_0+3)]} \frac{e^{st}}{\left(s + v + \lambda n_0\right)\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{\left[-\left[v + \lambda\left(n_0 + 3\right)\right] + v + \lambda n_0\right]\left[-\left[v + \lambda\left(n_0 + 3\right)\right] + v + \lambda\left(n_0 + 1\right)\right]\left[-\left[v + \lambda\left(n_0 + 3\right)\right] + v + \lambda\left(n_0 + 2\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{\left[-\lambda n_0 - 3\lambda + \lambda n_0\right]\left[-\lambda n_0 - 3\lambda + \lambda n_0 + \lambda\right]\left[-\lambda n_0 - 3\lambda + \lambda n_0 + 2\lambda\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{-3\lambda\left(-2\lambda\right)\left(-\lambda\right)}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{\lambda^3 3!}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s + v + \lambda n_0\right)\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]}\right\} = \sum a_{-i}$$

But

$$\begin{aligned}
\sum a_{-i} &= \frac{e^{-[v+\lambda n_0]t}}{\lambda^3 3!} - \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 2!} + \frac{e^{-[v+\lambda(n_0+2)]t}}{\lambda^3 2!} - \frac{e^{-[v+\lambda(n_0+3)]t}}{\lambda^3 3!} \\
&= \frac{e^{-[v+\lambda n_0]t}}{\lambda^3 3!}\left(1 - \frac{3!}{2!}e^{-\lambda t} + \frac{3!}{2!}e^{-2\lambda t} - e^{-3\lambda t}\right) \\
&= \frac{e^{-[v+\lambda n_0]t}}{\lambda^3 3!}\left(1 - 3e^{-\lambda t} + 3e^{-2\lambda t} - e^{-3\lambda t}\right) \\
&= \frac{e^{-[v+\lambda n_0]t}}{\lambda^3 3!}\left(1 - e^{-\lambda t}\right)^3
\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{(s+v+\lambda n_0)[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}\right\}$$

$$=[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3\frac{e^{-[v+\lambda n_0]t}}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$

$$=[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]e^{-[v+\lambda n_0]t}\frac{z^3}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$

$$=\left[\frac{v}{\lambda}+(n_0+1)\right]\left[\frac{v}{\lambda}+(n_0+2)\right]\left[\frac{v}{\lambda}+(n_0+3)\right]e^{-[v+\lambda n_0]t}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3$$

Consolidating the above results We get

$$L^{-1}\{E\}=\left\{\begin{array}{c} e^{-[v+\lambda n_0 t]}+\left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}z\left(1-e^{-\lambda t}\right) \\ +\left[\frac{v}{\lambda}+n_0+1\right]\left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2+ \\ \left[\begin{array}{c}\left[\frac{v}{\lambda}+(n_0+1)\right]* \\ \left[\frac{v}{\lambda}+(n_0+2)\right]* \\ \left[\frac{v}{\lambda}+(n_0+3)\right]\end{array}\right]e^{-[v+\lambda n_0]t}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3+\ldots \end{array}\right\}$$

Similarly

Dealing with F

$$L^{-1}\left\{F\right\} = L^{-1}\left\{\begin{array}{c} \frac{1}{[s+v+\lambda(n_0+1)]} + \frac{[v+\lambda(n_0+1)]z}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]} + \\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]} + \\ \\ \frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]} + ... \end{array}\right\}$$

$$= \left[\begin{array}{c} L^{-1}\left\{\frac{1}{[s+v+\lambda(n_0+1)]}\right\} + L^{-1}\left\{\frac{[v+\lambda(n_0+1)]z}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]}\right\} + \\ \\ L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)]z^2}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]}\right\} + \\ \\ L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]}\right\} + ... \end{array}\right]$$

The next step is to simplify the above first four terms of F separately

First term

From the table of transform pairs

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{[s+v+\lambda\,(n_0+1)]}\right\} = e^{-[v+\lambda(n_0+1)]t}$$

Second term

$$L^{-1}\left\{\frac{[v+\lambda\,(n_0+1)]\,z}{[s+v+\lambda\,(n_0+1)]\,[s+v+\lambda\,(n_0+2)]}\right\}$$

$$= \begin{bmatrix} [v + \lambda (n_0 + 1)] \, z* \\ \\ L^{-1} \left\{ \frac{1}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)]} \right\} \end{bmatrix}$$

The function

$$\frac{1}{[s + v + \lambda (n_0 + 1)] [s + v + \lambda (n_0 + 2)]}$$

has simple poles at $s = -[v + \lambda (n_0 + 1)]$ and $s = -[v + \lambda (n_0 + 2)]$

Thus its residue at each pole is obtained as follows

At $s = -[v + \lambda (n_0 + 1)]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+1)]} \frac{[s + v + \lambda (n_0 + 1)] \, e^{st}}{[s + v + \lambda (n_0 + 1)] [s + v + \lambda (n_0 + 2)]}$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \frac{e^{st}}{[s + v + \lambda (n_0 + 2)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-[v + \lambda (n_0 + 1)] + v + \lambda (n_0 + 2)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-v - \lambda n_0 - \lambda + v + \lambda n_0 + 2\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda}$$

At $s = -\left[v + \lambda\left(n_0 + 2\right)\right]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+2)]} \frac{\left[s + v + \lambda\left(n_0 + 2\right)\right] e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]}$$

$$= \lim_{s \to -[v+\lambda(n_0+2)]} \frac{e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{\left[-\left[v + \lambda\left(n_0 + 2\right)\right] + v + \lambda\left(n_0 + 1\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{\left[-v - \lambda n_0 - 2\lambda + v + \lambda n_0 + \lambda\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-\lambda}$$

Thus

$$L^{-1}\left\{\frac{1}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]}\right\} = \sum a_{-i}$$
$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda} - \frac{e^{-[v+\lambda(n_0+2)]t}}{\lambda}$$
$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda}\left(1 - e^{-\lambda t}\right)$$

Therefore

$$L^{-1}\left\{\frac{\left[v + \lambda\left(n_0 + 1\right)\right] z}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]}\right\}$$

$$= \left[v + \lambda\left(n_0 + 1\right)\right] z \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda}\left(1 - e^{-\lambda t}\right)$$

$$= \left[v + \lambda\left(n_0 + 1\right)\right] e^{-[v+\lambda(n_0+1)]t} \frac{z}{\lambda}\left(1 - e^{-\lambda t}\right)$$

$$= \lambda \left[\frac{v}{\lambda} + (n_0 + 1) \right] e^{-[v+\lambda(n_0+1)]t} \frac{z}{\lambda} \left(1 - e^{-\lambda t}\right)$$

$$= \left[\frac{v}{\lambda} + (n_0 + 1) \right] e^{-[v+\lambda(n_0+1)]t} z \left(1 - e^{-\lambda t}\right)$$

Third term

$$L^{-1} \left\{ \frac{[v + \lambda (n_0 + 1)] [v + \lambda (n_0 + 2)] z^2}{[s + v + \lambda (n_0 + 1)] [s + v + \lambda (n_0 + 2)] [s + v + \lambda (n_0 + 3)]} \right\}$$

$$= \left\{ \begin{array}{c} [v + \lambda (n_0 + 1)] [v + \lambda (n_0 + 2)] z^2 * \\ \\ L^{-1} \left\{ \frac{1}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)]} \right\} \end{array} \right\}$$

The function

$$\frac{1}{[s + v + \lambda (n_0 + 1)] [s + v + \lambda (n_0 + 2)] [s + v + \lambda (n_0 + 3)]}$$

has simple poles at $s = -[v + \lambda (n_0 + 1)]$, $s = -[v + \lambda (n_0 + 2)]$ and $s = -[v + \lambda (n_0 + 3)]$

Thus its residue at each pole is given by

At $s = -[v + \lambda (n_0 + 1)]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+1)]} \frac{[s + v + \lambda (n_0 + 1)] e^{st}}{[s + v + \lambda (n_0 + 1)] [s + v + \lambda (n_0 + 2)] [s + v + \lambda (n_0 + 3)]}$$

$$= \lim_{s \to -[v+\lambda(n_0+1)]} \frac{e^{st}}{[s + v + \lambda (n_0 + 2)] [s + v + \lambda (n_0 + 3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-[v + \lambda (n_0 + 1)] + v + \lambda (n_0 + 2)] [-[v + \lambda (n_0 + 1)] + v + \lambda (n_0 + 3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{[-v - \lambda n_0 - \lambda + v + \lambda n_0 + 2\lambda]\,[-v - \lambda n_0 - \lambda + v + \lambda n_0 + 3\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda\,(2\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^2 2!}$$

At $s = -\left[v + \lambda\left(n_0 + 2\right)\right]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+2)]} \frac{\left[s + v + \lambda\left(n_0 + 2\right)\right] e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]}$$

$$= \lim_{s \to -[v+\lambda(n_0+2)]} \frac{e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{\left[-\left[v + \lambda\left(n_0 + 2\right)\right] + v + \lambda\left(n_0 + 1\right)\right]\left[-\left[v + \lambda\left(n_0 + 2\right)\right] + v + \lambda\left(n_0 + 3\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{[-v - \lambda n_0 - 2\lambda + v + \lambda n_0 + \lambda]\,[-v - \lambda n_0 - 2\lambda + v + \lambda n_0 + 3\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-\lambda\,(\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+2)]t}}{-\lambda^2}$$

At $s = -\left[v + \lambda\left(n_0 + 3\right)\right]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+3)]} \frac{\left[s + v + \lambda\left(n_0 + 3\right)\right] e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]\left[s + v + \lambda\left(n_0 + 4\right)\right]}$$

$$= \lim_{s \to -[v+\lambda(n_0+3)]} \frac{e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 4\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{\left[-\left[v + \lambda\left(n_0 + 3\right)\right] + v + \lambda\left(n_0 + 1\right)\right]\left[-\left[v + \lambda\left(n_0 + 3\right)\right] + v + \lambda\left(n_0 + 2\right)\right]\left[-\left[v + \lambda\left(n_0 + 3\right)\right] + v + \lambda\left(n_0 + 4\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{\left[-v - \lambda n_0 - 3\lambda + v + \lambda n_0 + \lambda\right]\left[-v - \lambda n_0 - 3\lambda + \lambda n_0 + 2\lambda\right]\left[-v - \lambda n_0 - 3\lambda + v + \lambda n_0 + 4\lambda\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{-2\lambda\left(-\lambda\right)\lambda}$$

$$= \frac{e^{-[v+\lambda(n_0+3)]t}}{\lambda^3 2!}$$

At $s = -\left[v + \lambda\left(n_0 + 4\right)\right]$

$$a_{-i} = \lim_{s \to -[v+\lambda(n_0+4)]} \frac{\left[s + v + \lambda\left(n_0 + 4\right)\right] e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]\left[s + v + \lambda\left(n_0 + 4\right)\right]}$$

$$= \lim_{s \to -[v+\lambda(n_0+4)]} \frac{e^{st}}{\left[s + v + \lambda\left(n_0 + 1\right)\right]\left[s + v + \lambda\left(n_0 + 2\right)\right]\left[s + v + \lambda\left(n_0 + 3\right)\right]}$$

$$= \frac{e^{-[v+\lambda(n_0+4)]t}}{[-[v+\lambda(n_0+4)]+v+\lambda(n_0+1)][-[v+\lambda(n_0+4)]+v+\lambda(n_0+2)][-\lambda(n_0+4)+v+\lambda(n_0+3)]}$$

$$= \frac{e^{-[v+\lambda(n_0+4)]t}}{[-v-\lambda n_0-4\lambda+v+\lambda n_0+\lambda][-v-\lambda n_0-4\lambda+v+\lambda n_0+2\lambda][-v-\lambda n_0-4\lambda+v+\lambda n_0+3\lambda]}$$

$$= \frac{e^{-[v+\lambda(n_0+4)]t}}{-3\lambda(-2\lambda)(-\lambda)}$$

$$= \frac{e^{-[v+\lambda(n_0+4)]t}}{-\lambda^3 3!}$$

Thus

$$L^{-1}\left\{\frac{1}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]}\right\}$$

$$= \sum a_{-i}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 3!} - \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 2!} + \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 2!} - \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 3!}$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 3!}\left(1 - \frac{3!}{2!}e^{-\lambda t} + \frac{3!}{2!}e^{-2\lambda t} - e^{-3\lambda t}\right)$$

$$= \frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 3!}\left(1 - 3e^{-\lambda t} + 3e^{-2\lambda t} - e^{-3\lambda t}\right)$$

$$= \frac{e^{-[v+\lambda(n_0+1)]}}{\lambda^3 3!}\left(1 - e^{-\lambda t}\right)^3$$

Therefore

$$L^{-1}\left\{\frac{[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3}{[s+v+\lambda(n_0+1)][s+v+\lambda(n_0+2)][s+v+\lambda(n_0+3)][s+v+\lambda(n_0+4)]}\right\}$$

$$=[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]z^3\frac{e^{-[v+\lambda(n_0+1)]t}}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$

$$=[v+\lambda(n_0+1)][v+\lambda(n_0+2)][v+\lambda(n_0+3)]e^{-[v+\lambda(n_0+1)]t}\frac{z^3}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$

$$=\lambda\left[\frac{v}{\lambda}+(n_0+1)\right]\lambda\left[\frac{v}{\lambda}+(n_0+2)\right]\lambda\left[\frac{v}{\lambda}+(n_0+3)\right]e^{-[v+\lambda(n_0+1)]t}\frac{z^3}{\lambda^3 3!}\left(1-e^{-\lambda t}\right)^3$$

$$=\left[\frac{v}{\lambda}+(n_0+1)\right]\left[\frac{v}{\lambda}+(n_0+2)\right]\left[\frac{v}{\lambda}+(n_0+3)\right]e^{-[v+\lambda(n_0+1)]t}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3$$

Consolidating the above results We get

$$L^{-1}\{F\}=\left\{\begin{array}{c} e^{-[v+\lambda(n_0+1)]t}+\left[\frac{v}{\lambda}+(n_0+1)\right]e^{-[v+\lambda(n_0+1)]t}z\left(1-e^{-\lambda t}\right)+ \\ \left[\frac{v}{\lambda}+(n_0+1)\right]\left[\frac{v}{\lambda}+(n_0+2)\right]e^{-[v+\lambda(n_0+1)]t}\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2+ \\ \left[\begin{array}{c}\left[\frac{v}{\lambda}+(n_0+1)\right]* \\ \left[\frac{v}{\lambda}+(n_0+2)\right]* \\ \left[\frac{v}{\lambda}+(n_0+3)\right]\end{array}\right]e^{-[v+\lambda(n_0+1)]t}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3+... \end{array}\right\}$$

We now have

$G(z,t) = z^{n_0}\left[(1-z)\,L^{-1}\{E\} + zL^{-1}\{F\}\right]$

$$
= z^{n_0}\left\{\begin{array}{l}
(1-z)\left[\begin{array}{c}
e^{-[v+\lambda n_0 t]} + \left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}z\left(1-e^{-\lambda t}\right)+\\
\left[\frac{v}{\lambda}+n_0+1\right]\left[\frac{v}{\lambda}+n_0+1\right]e^{-(v+\lambda n_0)t}\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2+\\
\left[\begin{array}{c}
\left[\frac{v}{\lambda}+(n_0+1)\right]*\\
\left[\frac{v}{\lambda}+(n_0+2)\right]*\\
\left[\frac{v}{\lambda}+(n_0+3)\right]
\end{array}\right]e^{-[v+\lambda n_0]t}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3+...
\end{array}\right]\\
z\left[\begin{array}{c}
e^{-[v+\lambda(n_0+1)]t} + \left[\frac{v}{\lambda}+(n_0+1)\right]e^{-[v+\lambda(n_0+1)]t}z\left(1-e^{-\lambda t}\right)+\\
\left[\frac{v}{\lambda}+(n_0+1)\right]\left[\frac{v}{\lambda}+(n_0+2)\right]e^{-[v+\lambda(n_0+1)]t}\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2+\\
\left[\begin{array}{c}
\left[\frac{v}{\lambda}+(n_0+1)\right]*\\
\left[\frac{v}{\lambda}+(n_0+2)\right]*\\
\left[\frac{v}{\lambda}+(n_0+3)\right]
\end{array}\right]e^{-[v+\lambda(n_0+1)]t}\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3+...
\end{array}\right]
\end{array}\right\}
$$

Thus

$$G(z,t) = z^{n_0} \left\{ \begin{array}{c} e^{-[v+\lambda n_0 t]} + \left[\frac{v}{\lambda} + n_0 + 1\right] e^{-(v+\lambda n_0)t} z \left(1 - e^{-\lambda t}\right) + \\ \left[\frac{v}{\lambda} + n_0 + 1\right] \left[\frac{v}{\lambda} + n_0 + 2\right] e^{-(v+\lambda n_0)t} \frac{z^2}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0+1)\right] * \\ \left[\frac{v}{\lambda} + (n_0+2)\right] * \\ \left[\frac{v}{\lambda} + (n_0+3)\right] \end{array}\right] e^{-[v+\lambda n_0]t} \frac{z^3}{3!} \left(1 - e^{-\lambda t}\right)^3 - \\ z e^{-[v+\lambda n_0 t]} - \left[\frac{v}{\lambda} + n_0 + 1\right] e^{-(v+\lambda n_0)t} z^2 \left(1 - e^{-\lambda t}\right) - \\ \left[\frac{v}{\lambda} + n_0 + 1\right] \left[\frac{v}{\lambda} + n_0 + 2\right] e^{-(v+\lambda n_0)t} \frac{z^3}{2!} \left(1 - e^{-\lambda t}\right)^2 - \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0+1)\right] * \\ \left[\frac{v}{\lambda} + (n_0+2)\right] * \\ \left[\frac{v}{\lambda} + (n_0+3)\right] \end{array}\right] e^{-[v+\lambda n_0]t} \frac{z^4}{3!} \left(1 - e^{-\lambda t}\right)^3 + \\ z e^{-[v+\lambda(n_0+1)]t} + \left[\frac{v}{\lambda} + (n_0+1)\right] e^{-[v+\lambda(n_0+1)]t} z^2 \left(1 - e^{-\lambda t}\right) + \\ \left[\frac{v}{\lambda} + (n_0+1)\right] \left[\frac{v}{\lambda} + (n_0+2)\right] e^{-[v+\lambda(n_0+1)]t} \frac{z^3}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0+1)\right] * \\ \left[\frac{v}{\lambda} + (n_0+2)\right] * \\ \left[\frac{v}{\lambda} + (n_0+3)\right] \end{array}\right] e^{-[v+\lambda(n_0+1)]t} \frac{z^4}{3!} \left(1 - e^{-\lambda t}\right)^3 + .. \end{array} \right\}$$

We can rewrite the above equation as $G(z,t) = z^{n_0} H$, where

$$H = \left\{ \begin{array}{c} e^{-[v+\lambda n_0 t]} + \left[\frac{v}{\lambda} + n_0 + 1\right] e^{-(v+\lambda n_0)t} z \left(1 - e^{-\lambda t}\right) + \\ \left[\frac{v}{\lambda} + n_0 + 1\right] \left[\frac{v}{\lambda} + n_0 + 2\right] e^{-(v+\lambda n_0)t} \frac{z^2}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0+1)\right] * \\ \left[\frac{v}{\lambda} + (n_0+2)\right] * \\ \left[\frac{v}{\lambda} + (n_0+3)\right] \end{array}\right] e^{-[v+\lambda n_0]t} \frac{z^3}{3!} \left(1 - e^{-\lambda t}\right)^3 - \\ z e^{-[v+\lambda n_0 t]} - \left[\frac{v}{\lambda} + n_0 + 1\right] e^{-(v+\lambda n_0)t} z^2 \left(1 - e^{-\lambda t}\right) - \\ \left[\frac{v}{\lambda} + n_0 + 1\right] \left[\frac{v}{\lambda} + n_0 + 2\right] e^{-(v+\lambda n_0)t} \frac{z^3}{2!} \left(1 - e^{-\lambda t}\right)^2 - \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0+1)\right] * \\ \left[\frac{v}{\lambda} + (n_0+2)\right] * \\ \left[\frac{v}{\lambda} + (n_0+3)\right] \end{array}\right] e^{-[v+\lambda n_0]t} \frac{z^4}{3!} \left(1 - e^{-\lambda t}\right)^3 + \\ z e^{-[v+\lambda(n_0+1)]t} + \left[\frac{v}{\lambda} + (n_0+1)\right] e^{-[v+\lambda(n_0+1)]t} z^2 \left(1 - e^{-\lambda t}\right) + \\ \left[\frac{v}{\lambda} + (n_0+1)\right] \left[\frac{v}{\lambda} + (n_0+2)\right] e^{-[v+\lambda(n_0+1)]t} \frac{z^3}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0+1)\right] * \\ \left[\frac{v}{\lambda} + (n_0+2)\right] * \\ \left[\frac{v}{\lambda} + (n_0+3)\right] \end{array}\right] e^{-[v+\lambda(n_0+1)]t} \frac{z^4}{3!} \left(1 - e^{-\lambda t}\right)^3 + .. \end{array} \right\}$$

$$H = e^{-[v+\lambda n_0 t]} \left\{ \begin{array}{c} 1 + \left[\frac{v}{\lambda} + n_0 + 1\right] z \left(1 - e^{-\lambda t}\right) + \\ \left[\frac{v}{\lambda} + n_0 + 1\right] \left[\frac{v}{\lambda} + n_0 + 2\right] \frac{z^2}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0 + 1)\right] * \\ \left[\frac{v}{\lambda} + (n_0 + 2)\right] * \\ \left[\frac{v}{\lambda} + (n_0 + 3)\right] \end{array} \right] \frac{z^3}{3!} \left(1 - e^{-\lambda t}\right)^3 - \\ z - \left[\frac{v}{\lambda} + n_0 + 1\right] z^2 \left(1 - e^{-\lambda t}\right) - \\ \left[\frac{v}{\lambda} + n_0 + 1\right] \left[\frac{v}{\lambda} + n_0 + 2\right] \frac{z^3}{2!} \left(1 - e^{-\lambda t}\right)^2 - \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0 + 1)\right] * \\ \left[\frac{v}{\lambda} + (n_0 + 2)\right] * \\ \left[\frac{v}{\lambda} + (n_0 + 3)\right] \end{array} \right] \frac{z^4}{3!} \left(1 - e^{-\lambda t}\right)^3 + \\ z e^{-\lambda t} + \left[\frac{v}{\lambda} + (n_0 + 1)\right] e^{-\lambda t} z^2 \left(1 - e^{-\lambda t}\right) + \\ \left[\frac{v}{\lambda} + (n_0 + 1)\right] \left[\frac{v}{\lambda} + (n_0 + 2)\right] e^{-\lambda t} \frac{z^3}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left[\frac{v}{\lambda} + (n_0 + 1)\right] * \\ \left[\frac{v}{\lambda} + (n_0 + 2)\right] * \\ \left[\frac{v}{\lambda} + (n_0 + 3)\right] \end{array} \right] e^{-\lambda t} \frac{z^4}{3!} \left(1 - e^{-\lambda t}\right)^3 + .. \end{array} \right\}$$

To simplify H we group together the coefficients of corresponding powers of $\left(1-e^{-\lambda t}\right)$ To make our life easier, we express this in a table as shown below where the terms of each power of $\left(1-e^{-\lambda t}\right)$ are specified in the corresponding columns.

Power of $\left(1-e^{-\lambda t}\right)$	Coefficient
0	$1-z+ze^{-\lambda t}$
1	$\left(\frac{v}{\lambda}+n_0+1\right)z-\left(\frac{v}{\lambda}+n_0+1\right)z^2+\left(\frac{v}{\lambda}+n_0+1\right)z^2e^{-\lambda t}$
2	$\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\frac{z^2}{2!}-\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\frac{z^3}{2!}+\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\frac{z^3}{2!}e^{-\lambda t}$
3	$\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\left(\frac{v}{\lambda}+n_0+3\right)\frac{z^3}{3!}-\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\left(\frac{v}{\lambda}+n_0+3\right)\frac{z^4}{3!}+\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\left(+\frac{v}{\lambda}n_0+3\right)\frac{z^4}{3!}e^{-\lambda t}$

Therefore

$$H = e^{-(v+\lambda n_0)t} \left\{ \begin{array}{c} 1 - z + ze^{-\lambda t} + \\ \left[\begin{array}{c} \left(\frac{v}{\lambda} + n_0 + 1\right) z - \\ \left(\frac{v}{\lambda} + n_0 + 1\right) z^2 + \\ \left(\frac{v}{\lambda} + n_0 + 1\right) z^2 e^{-\lambda t} \end{array} \right] \left(1 - e^{-\lambda t}\right) + \\ \left[\begin{array}{c} \left(\frac{v}{\lambda} + n_0 + 1\right) \left(\frac{v}{\lambda} + n_0 + 2\right) \frac{z^2}{2!} - \\ \left(\frac{v}{\lambda} + n_0 + 1\right) \left(\frac{v}{\lambda} + n_0 + 2\right) \frac{z^3}{2!} + \\ \left(\frac{v}{\lambda} + n_0 + 1\right) \left(\frac{v}{\lambda} + n_0 + 2\right) \frac{z^3}{2!} e^{-\lambda t} \end{array} \right] \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left(\frac{v}{\lambda} + n_0 + 1\right) \left(\frac{v}{\lambda} + n_0 + 2\right) \left(\frac{v}{\lambda} + n_0 + 3\right) \frac{z^3}{3!} - \\ \left(\frac{v}{\lambda} + n_0 + 1\right) \left(\frac{v}{\lambda} + n_0 + 2\right) \left(\frac{v}{\lambda} + n_0 + 3\right) \frac{z^4}{3!} + \\ \left(\frac{v}{\lambda} + n_0 + 1\right) \left(\frac{v}{\lambda} + n_0 + 2\right) \left(\frac{v}{\lambda} + n_0 + 3\right) \frac{z^4}{3!} e^{-\lambda t} \end{array} \right] \left(1 - e^{-\lambda t}\right)^3 + ... \end{array} \right\}$$

$$H = e^{-(v+\lambda n_0)t} \left\{ \begin{array}{c} 1 - z + ze^{-\lambda t} + \left(\frac{v}{\lambda} + n_0 + 1\right) z \left[1 - z + ze^{-\lambda t}\right] \left(1 - e^{-\lambda t}\right) + \\ \left(\frac{v}{\lambda} + n_0 + 1\right)\left(\frac{v}{\lambda} + n_0 + 2\right) \frac{z^2}{2!} \left[1 - z + ze^{-\lambda t}\right] \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left(\frac{v}{\lambda} + n_0 + 1\right)\left(\frac{v}{\lambda} + n_0 + 2\right) * \\ \left(\frac{v}{\lambda} + n_0 + 3\right) \end{array}\right] \frac{z^3}{3!} \left[1 - z + ze^{-\lambda t}\right] \left(1 - e^{-\lambda t}\right)^3 + ... \end{array} \right\}$$

Factoring out $1 - z + ze^{-\lambda t}$ in the RHS implies that

$$H = e^{-(v+\lambda n_0)t} \left[1 - z + ze^{-\lambda t}\right] \left\{ \begin{array}{c} 1 + \left(\frac{v}{\lambda} + n_0 + 1\right) z \left(1 - e^{-\lambda t}\right) + \\ \left(\frac{v}{\lambda} + n_0 + 1\right)\left(\frac{v}{\lambda} + n_0 + 2\right) \frac{z^2}{2!} \left(1 - e^{-\lambda t}\right)^2 + \\ \left[\begin{array}{c} \left(\frac{v}{\lambda} + n_0 + 1\right)\left(\frac{v}{\lambda} + n_0 + 2\right) * \\ \left(\frac{v}{\lambda} + n_0 + 3\right) \end{array}\right] \frac{z^3}{3!} \left(1 - e^{-\lambda t}\right)^3 + ... \end{array} \right\}$$

$$= e^{-(v+\lambda n_0)t} \left[1 - z + ze^{-\lambda t}\right] \left\{ \begin{array}{c} 1 + \frac{\left(\frac{v}{\lambda} + n_0 + 1\right)}{1!} \left[z \left(1 - e^{-\lambda t}\right)\right] + \\ \frac{\left(\frac{v}{\lambda} + n_0 + 1\right)\left(\frac{v}{\lambda} + n_0 + 2\right)}{2!} \left[z \left(1 - e^{-\lambda t}\right)\right]^2 + \\ \left[\begin{array}{c} \left(\frac{v}{\lambda} + n_0 + 1\right)\left(\frac{v}{\lambda} + n_0 + 2\right) * \\ \frac{\left(\frac{v}{\lambda} + n_0 + 3\right)}{3!} \end{array}\right] \left[z \left(1 - e^{-\lambda t}\right)\right]^3 + ... \end{array} \right\}$$

To simplify this, We use the binomial expansion properties, Let us multiply the RHS by 1 in a nice way that is

$$\frac{\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!}$$

This yields

$$H=\left\{\begin{array}{c} e^{-(v+\lambda n_0)t}* \\ \\ \left[1-z\left(1-e^{-\lambda t}\right)\right]\frac{\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!}\end{array}\right\}\left\{\begin{array}{c} 1+\left(\frac{v}{\lambda}+n_0+1\right)z\left(1-e^{-\lambda t}\right)+ \\ \\ \left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\frac{z^2}{2!}\left(1-e^{-\lambda t}\right)^2+ \\ \\ \left[\begin{array}{c}\left(\frac{v}{\lambda}+n_0+1\right)* \\ \\ \left(\frac{v}{\lambda}+n_0+2\right)* \\ \\ \left(\frac{v}{\lambda}+n_0+3\right)\end{array}\right]\frac{z^3}{3!}\left(1-e^{-\lambda t}\right)^3+...\end{array}\right\}$$

$$=\left\{\begin{array}{c} e^{-(v+\lambda n_0)t}* \\ \\ \left[1-z\left(1-e^{-\lambda t}\right)\right]\end{array}\right\}\left\{\begin{array}{c} \frac{\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!}+\frac{(n_0+1)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!1!}\left[z\left(1-e^{-\lambda t}\right)\right]+ \\ \\ \frac{\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!2!}\left[z\left(1-e^{-\lambda t}\right)\right]^2+ \\ \\ \left[\begin{array}{c}\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)* \\ \\ \frac{\left(\frac{v}{\lambda}+n_0+3\right)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!3!}\end{array}\right]\left[z\left(1-e^{-\lambda t}\right)\right]^3+...\end{array}\right\}$$

Therefore

$$H = \left\{ \begin{array}{c} e^{-(v+\lambda n_0)t} * \\ \\ \left[1 - z\left(1 - e^{-\lambda t}\right)\right] \end{array} \right\} \left\{ \begin{array}{c} 1 + \frac{(n_0+1)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!1!}\left[z\left(1-e^{-\lambda t}\right)\right] + \\ \\ \frac{\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!2!}\left[z\left(1-e^{-\lambda t}\right)\right]^2 + \\ \\ \left[\begin{array}{c} \left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0+2\right) * \\ \\ \frac{\left(\frac{v}{\lambda}+n_0+3\right)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!3!} \end{array} \right] \left[z\left(1-e^{-\lambda t}\right)\right]^3 + ... \end{array} \right\}$$

Using the fact that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We have

$$\binom{\frac{v}{\lambda}+n_0+0}{0} = 1$$

$$\begin{aligned} \binom{\frac{v}{\lambda}+n_0+1}{1} &= \frac{\left(\frac{v}{\lambda}+n_0+1\right)!}{\left(\frac{v}{\lambda}+n_0\right)!1!} \\ &= \frac{\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n\right)_0!}{\left(\frac{v}{\lambda}+n_0\right)!1!} \end{aligned}$$

$$\begin{aligned} \binom{\frac{v}{\lambda}+n_0+2}{2} &= \frac{\left(\frac{v}{\lambda}+n_0+2\right)!}{\left(\frac{v}{\lambda}+n_0\right)!2!} \\ &= \frac{\left(\frac{v}{\lambda}+n_0+2\right)!\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!2!} \end{aligned}$$

$$\binom{\frac{v}{\lambda}+n_0+3}{3} = \frac{\left(\frac{v}{\lambda}+n_0+3\right)!}{\left(\frac{v}{\lambda}+n_0\right)!3!}$$
$$= \frac{\left(\frac{v}{\lambda}+n_0+3\right)\left(\frac{v}{\lambda}+n_0+2\right)!\left(\frac{v}{\lambda}+n_0+1\right)\left(\frac{v}{\lambda}+n_0\right)!}{\left(\frac{v}{\lambda}+n_0\right)!3!}$$

And so on, with this equation H can be expressed as

$$H = \left\{\begin{array}{c} e^{-(v+\lambda n_0)t} * \\ \\ \left[1-z\left(1-e^{-\lambda t}\right)\right] \end{array}\right\} \left\{\begin{array}{c} 1+\binom{\frac{v}{\lambda}+n_0+1}{1}\left[z\left(1-e^{-\lambda t}\right)\right]+ \\ \\ \binom{\frac{v}{\lambda}+n_0+2}{2}\left[z\left(1-e^{-\lambda t}\right)\right]^2+ \\ \\ \binom{\frac{v}{\lambda}+n_0+3}{3}\left[z\left(1-e^{-\lambda t}\right)\right]^2+\ldots \end{array}\right\}$$

$$= e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]\sum_{j=0}^{\infty}\binom{\frac{v}{\lambda}+n_0+j}{j}\left[z\left(1-e^{-\lambda t}\right)\right]^j$$

It's now time to go back to $G(z,\ t)$. Remember we had

$$G(z,t) = z^{n_0}H$$

Thus

$$G(z,t) = z^{n_0}e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]\sum_{j=0}^{\infty}\binom{\frac{v}{\lambda}+n_0+j}{j}\left[z\left(1-e^{-\lambda t}\right)\right]^j$$

But

$$\binom{\frac{v}{\lambda}+n_0+j}{j} = \binom{\left[\frac{v}{\lambda}+n_0+1\right]+j-1}{j}$$

Thus

$$G(z,t) = z^{n_0}e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]\sum_{j=0}^{\infty}\binom{\left[\frac{v}{\lambda}+n_0+1\right]+j-1}{j}\left[z\left(1-e^{-\lambda t}\right)\right]^j$$

Also

$$\binom{-r}{j}(-1)^j = \binom{r+j-1}{j}$$

$$\Rightarrow \binom{-\left[\frac{v}{\lambda}+n_0+1\right]}{j}(-1)^j = \binom{\left[\frac{v}{\lambda}+n_0+1\right]+j-1}{j}$$

Therefore

$$G(z,t) = z^{n_0}e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]\sum_{j=0}^{\infty}\binom{-\left[\frac{v}{\lambda}+n_0+1\right]}{j}(-1)^j\left[z\left(1-e^{-\lambda t}\right)\right]^j$$

$$= z^{n_0}e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]\sum_{j=0}^{\infty}\binom{-\left[\frac{v}{\lambda}+n_0+1\right]}{j}\left[-z\left(1-e^{-\lambda t}\right)\right]^j$$

$$= z^{n_0}e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]\left[1-z\left(1-e^{-\lambda t}\right)\right]^{-\left(\frac{v}{\lambda}+n_0+1\right)}$$

$$= z^{n_0}e^{-(v+\lambda n_0)t}\left[1-z\left(1-e^{-\lambda t}\right)\right]^{-\left(\frac{v}{\lambda}+n_0\right)}$$

$$= z^{n_0}e^{-\lambda t\left(\frac{v}{\lambda}+n_0\right)}\left[1-z\left(1-e^{-\lambda t}\right)\right]^{-\left(\frac{v}{\lambda}+n_0\right)}$$

$$\therefore G(z,t) = z^{n_0}\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0+\frac{v}{\lambda}}$$

By identification, this is the pgf of a negative binomial distribution with $p = e^{-\lambda t}$ and $q = 1-e^{-\lambda t}$

$P_n(t)$ is the coefficient of z^n in $G(z,t)$

But $G(z,t)$ is of the form

$$G(z,t) = z^{n_0}\left(\frac{p}{1-qz}\right)^{n_0+\frac{v}{\lambda}}$$

$$= z^{n_0}p^{n_0+\frac{v}{\lambda}}(1-qz)^{-\left(n_0+\frac{v}{\lambda}\right)}$$

$$= z^{n_0} p^{n_0+\frac{v}{\lambda}} \sum_{n=0}^{\infty} \binom{-\left[n_0+\frac{v}{\lambda}\right]}{n} (-qz)^n$$

$$= z^{n_0} p^{n_0+\frac{v}{\lambda}} \sum_{n=0}^{\infty} \binom{\left[n_0+\frac{v}{\lambda}\right]+n-1}{n} (-1)^n (-1)^n (zq)^n$$

$$= z^{n_0} p^{n_0+\frac{v}{\lambda}} \sum_{n=0}^{\infty} \binom{\left[n_0+\frac{v}{\lambda}\right]+n-1}{n} (zq)^n$$

$$= \sum_{n=0}^{\infty} \binom{\left[n_0+\frac{v}{\lambda}\right]+n-1}{n} p^{n_0+\frac{v}{\lambda}} q^n z^{n+n_0}$$

$$= \sum_{n=n_0}^{\infty} \binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0} p^{n_0+\frac{v}{\lambda}} q^{n-n_0} z^{n_0}$$

$$= \sum_{n=n_0}^{\infty} \binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0} p^{n_0+\frac{v}{\lambda}} q^{n-n_0} z^{n_0}$$

Thus the coefficient of z^n is

$$\binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0} p^{n_0+\frac{v}{\lambda}} q^{n-n_0}$$

But $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$ implying that

$$P_n(t) = \binom{\left[n_0+\frac{v}{\lambda}\right]+(n-n_0)-1}{n-n_0} \left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}} \left(1-e^{-\lambda t}\right)^{n-n_0} \qquad n = n_0,\ n_0+1,\ n_0+2, ...$$

$$= \binom{\left[n_0+\frac{v}{\lambda}\right]+(k)-1}{k} \left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}} \left(1-e^{-\lambda t}\right)^{k} \qquad k = 0,\ 1,\ 2, ...$$

Which is the pmf of a negative binomial distribution

5.6 Polya Process

The basic difference-differential equations for this process are obtained from the basic difference-differential equations for the general birth process by letting

$$\lambda_n = \lambda \left[\frac{1 + an}{1 + \lambda at} \right] \forall n$$

These equations are

$$P_0'(t) = -\left(\frac{\lambda}{1 + \lambda at} \right) P_0(t) \tag{5.78a}$$

$$P_n'(t) = -\lambda \left(\frac{1 + an}{1 + \lambda at} \right) P_n(t) + \lambda \left(\frac{1 + a(n-1)}{1 + \lambda at} \right) P_{n-1}(t) \quad n = 1, 2, 3, ... \tag{5.78b}$$

We now solve these equations;

Method 1: By Iteration

Taking the Laplace transform of both sides of equation (5.78b) yields;

$$L\{P_n'(t)\} = L\left\{ -\lambda \left(\frac{1 + an}{1 + \lambda at} \right) P_n(t) + \lambda \left(\frac{1 + a(n-1)}{1 + \lambda at} \right) P_{n-1}(t) \right\}$$

But by the linearity property of Laplace transform (Property 1)

$$L\{P_n'(t)\} = -\lambda \left(\frac{1 + an}{1 + \lambda at} \right) L\{P_n(t)\} + \lambda \left(\frac{1 + a(n-1)}{1 + \lambda at} \right) L\{P_{n-1}(t)\}$$

And by the Laplace transform of derivatives (Property 6)

If

$$L\{f(t)\} = \overline{f}(s)$$

Then

$$L\{f'(t)\} = s\overline{f}(s) - f(0)$$

This implies that

$$L\{P_n'(t)\} = sL\{P_n(t)\} - P_n(0)$$

Hence we have;

$$L\left\{P_n'(t)\right\} = -\lambda\left(\frac{1+an}{1+\lambda at}\right)L\left\{P_n(t)\right\} + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right)L\left\{P_{n-1}(t)\right\}$$

$$sL\left\{P_n(t)\right\} - P_n(0) = -\lambda\left(\frac{1+an}{1+\lambda at}\right)L\left\{P_n(t)\right\} + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right)L\left\{P_{n-1}(t)\right\}$$

$$\left[s+\lambda\left(\frac{1+an}{1+\lambda at}\right)\right]L\left\{P_n(t)\right\} = P_n(0) + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right)L\left\{P_{n-1}(t)\right\} \tag{5.79}$$

Remark

We assume that the initial population at time $t = 0$ is n_0 i.e. $X(0) = n_0$ thus $P_{n_0}(0) = 1$ and $P_n(0) = 0 \quad \forall n \neq n_0$. With this initial condition we solve equation (5.79) iteratively as follows.

For $n = n_0$ equation (5.79) becomes;

$$\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]L\left\{P_{n_0}(t)\right\} = P_{n0}(0) + \lambda\left(\frac{1+a(n_0-1)}{1+\lambda at}\right)L\left\{P_{n_0-1}(t)\right\}$$

But

$$P_{n_0}(0) = 1$$

and

$$P_{n_0-1}(t) = 0 \Rightarrow L\left\{P_{n_0-1}(t)\right\} = L\left\{0\right\} = 0$$

We thus have

$$\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]L\left\{P_{n_0}(t)\right\} = 1 + \lambda\left(\frac{1+a(n_0-1)}{1+\lambda at}\right)\underbrace{L\left\{P_{n_0-1}(t)\right\}}_{0}$$

$$\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]L\left\{P_{n_0}(t)\right\} = 1$$

$$\Rightarrow L\left\{P_{n_0}(t)\right\} = \frac{1}{\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]}$$

For $n = n_0 + 1$ equation (5.79) becomes;

$$\left[s + \lambda\left(\frac{1 + a(n_0 + 1)}{1 + \lambda at}\right)\right] L\left\{P_{n_0+1}(t)\right\} = P_{n_0+1}(0) + \lambda\left(\frac{1 + a(n_0 + 1 - 1)}{1 + \lambda at}\right) L\left\{P_{n_0+1-1}(t)\right\}$$

$$\left[s + \lambda\left(\frac{1 + a(n_0 + 1)}{1 + \lambda at}\right)\right] L\left\{P_{n_0+1}(t)\right\} = P_{n_0+1}(0) + \lambda\left(\frac{1 + an_0}{1 + \lambda at}\right) L\left\{P_{n_0}(t)\right\}$$

But

$$P_{n_0+1}(0) = 0$$

and

$$L\left\{P_{n_0}(t)\right\} = \frac{1}{\left[s + \lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]}$$

$$\left[s + \lambda\left(\frac{1 + a(n_0 + 1)}{1 + \lambda at}\right)\right] L\left\{P_{n_0+1}(t)\right\} = 0 + \lambda\left(\frac{1 + an_0}{1 + \lambda at}\right)\left[\frac{1}{s + \lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right]$$

$$\left[s + \lambda\left(\frac{1 + a(n_0 + 1)}{1 + \lambda at}\right)\right] L\left\{P_{n_0+1}(t)\right\} = \frac{\lambda(1 + an_0)}{(1 + \lambda at)\left[s + \lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]}$$

$$\Rightarrow L\left\{P_{n_0+1}(t)\right\} = \frac{\lambda(1 + an_0)}{(1 + \lambda at)\left[s + \lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]\left[s + \lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]}$$

Similarly for $n = n_0 + 2$ equation (5.79) becomes

$$\left[s + \lambda\left(\frac{1 + a(n_0 + 2)}{1 + \lambda at}\right)\right] L\left\{P_{n_0+2}(t)\right\} = P_{n_0+2}(0) + \lambda\left(\frac{1 + a(n_0 + 2 - 1)}{1 + \lambda at}\right) L\left\{P_{n_0+2-1}(t)\right\}$$

$$\left[s + \lambda\left(\frac{1 + a(n_0 + 2)}{1 + \lambda at}\right)\right] L\left\{P_{n_0+2}(t)\right\} = P_{n_0+2}(0) + \lambda\left(\frac{1 + a(n_0 + 1)}{1 + \lambda at}\right) L\left\{P_{n_0+1}(t)\right\}$$

But

$$P_{n_0+2}(0) = 0$$

and

$$L\left\{P_{n_0+1}(t)\right\} = \frac{\lambda(1+an_0)}{(1+\lambda at)\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]}$$

Thus we have

$$\left[s+\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)\right]L\left\{P_{n_0+2}(t)\right\} = 0+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\left[\frac{\lambda(1+an_0)}{(1+\lambda at)\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]}\right]$$

$$\left[s+\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)\right]L\left\{P_{n_0+2}(t)\right\} = \frac{\lambda(1+an_0)\lambda\left[(1+a(n_0+1)\right]}{(1+\lambda at)^2\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]}$$

$$\Rightarrow L\left\{P_{n_0+2}(t)\right\} = \frac{\lambda(1+an_0)\lambda\left[(1+a(n_0+1)\right]}{(1+\lambda at)^2\left[s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)\right]}$$

$$= \frac{\prod_{i=0}^{1}\lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^2\prod_{i=0}^{2}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

Similarly for $n = n_0 + 3$ equation (5.79) becomes

$$\left[s+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\right]L\left\{P_{n_0+3}(t)\right\} = P_{n_0+3}(0)+\lambda\left(\frac{1+a(n_0+3-1)}{1+\lambda at}\right)L\left\{P_{n_0+3-1}(t)\right\}$$

$$\left[s+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\right]L\left\{P_{n_0+3}(t)\right\} = P_{n_0+3}(0)+\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)L\left\{P_{n_0+2}(t)\right\}$$

But $P_{n_0+3}(0)=0$

and

$$L\left\{P_{n_0+2}(t)\right\}=\frac{\prod_{i=0}^{1}\lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^2\prod_{i=0}^{2}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

Thus We have

$$\left[s+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\right]L\left\{P_{n_0+3}(t)\right\}=\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)\left[\frac{\prod_{i=0}^{1}\lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^2\prod_{i=0}^{2}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right]$$

$$\left[s+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\right]L\left\{P_{n_0+3}(t)\right\}=\frac{\prod_{i=0}^{2}\lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^3\prod_{i=0}^{2}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$\Rightarrow L\left\{P_{n_0+3}(t)\right\}=\frac{\prod_{i=0}^{2}\lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^3\prod_{i=0}^{3}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

Similarly for $n=n_0+4$, we have;

$$\left[s+\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)\right]L\left\{P_{n_0+4}(t)\right\}=P_{n_0+4}(0)+\lambda\left(\frac{1+a(n_0+4-1)}{1+\lambda at}\right)L\left\{P_{n_0+4-1}(t)\right\}$$

$$\left[s+\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)\right]L\left\{P_{n_0+4}(t)\right\}=P_{n_0+4}(0)+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)L\left\{P_{n_0+3}(t)\right\}$$

But

$$P_{n_0+4}(0)=0$$

and

$$L\left\{P_{n_0+3}(t)\right\} = \frac{\prod_{i=0}^{2} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^3 \prod_{i=0}^{3}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

Thus

$$\left[s+\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)\right] L\left\{P_{n_0+4}(t)\right\} = \lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\left[\frac{\prod_{i=0}^{2} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^3 \prod_{i=0}^{3}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right]$$

$$\left[s+\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)\right] L\left\{P_{n_0+4}(t)\right\} = \frac{\prod_{i=0}^{3} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^4 \prod_{i=0}^{3}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$\Rightarrow L\left\{P_{n_0+4}(t)\right\} = \frac{\prod_{i=0}^{3} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^4 \prod_{i=0}^{4}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

By mathematical induction, We assume that;

$$L\left\{P_{n_0+k-1}(t)\right\} = \frac{\prod_{i=0}^{k-1-1} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^{k-1} \prod_{i=0}^{k-1}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \frac{\prod_{i=0}^{k-2} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^{k-1} \prod_{i=0}^{k-1}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

Thus by equation (5.79) we had

$$\left[s+\lambda\left(\frac{1+an}{1+\lambda at}\right)\right] L\left\{P_n(t)\right\} = P_n(0) + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right) L\left\{P_{n-1}(t)\right\}$$

Setting $n = n_0 + k$ we obtain

$$\left[s + \lambda\left(\frac{1 + a(n_0 + k)}{1 + \lambda at}\right)\right] L\{P_{n_0+k}(t)\} = P_{n_0+k}(0) + \lambda\left(\frac{1 + a(n_0 + k - 1)}{1 + \lambda at}\right) L\{P_{n_0+k-1}(t)\}$$

But

$$P_{n_0+k}(0) = 0$$

and

$$L\{P_{n_0+k-1}(t)\} = \frac{\prod_{i=0}^{k-2} \lambda\left[(1 + a(n_0 + i)\right]}{(1 + \lambda at)^{k-1} \prod_{i=0}^{k-1}\left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

Thus we have;

$$\left[s + \lambda\left(\frac{1 + a(n_0 + k)}{1 + \lambda at}\right)\right] L\{P_{n_0+k}(t)\} = \left\{\begin{array}{c} \lambda\left(\frac{1+a(n_0+k-1)}{1+\lambda at}\right) \times \\ \left[\frac{\prod_{i=0}^{k-2} \lambda[(1+a(n_0+i)]}{(1+\lambda at)^{k-1} \prod_{i=0}^{k-1}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right] \end{array}\right\}$$

$$\left[s + \lambda\left(\frac{1 + a(n_0 + k)}{1 + \lambda at}\right)\right] L\{P_{n_0+k}(t)\} = \frac{\prod_{i=0}^{k-1} \lambda\left[(1 + a(n_0 + i)\right]}{(1 + \lambda at)^{k} \prod_{i=0}^{k-1}\left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \left[\frac{\prod_{i=0}^{k-1} \lambda\left[(1 + a(n_0 + i)\right]}{(1 + \lambda at)^{k} \prod_{i=0}^{k-1}\left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right] \left[s + \lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)\right]}\right]$$

$$\Rightarrow L\{P_n(t)\} = L\{P_{n_0+k}(t)\}$$
$$= \frac{\prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^k \prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]} \tag{5.80}$$

The next step is to determine the inverse Laplace transform of $L\{P_n(t)\}$ Applying the inverse Laplace transform to both sides of equation (5.80) , we have

$$L^{-1}\{L\{P_n(t)\}\} = L^{-1}\left\{\frac{\prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^k \prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\}$$

$$\Rightarrow P_n(t) = L^{-1}\left\{\frac{\prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right]}{(1+\lambda at)^k \prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\}$$

$$\therefore P_n(t) = \prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right] L^{-1}\left\{\frac{1}{(1+\lambda at)^k \prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\} \tag{5.81}$$

We now solve equation (5.81). The following methods have been considered

- Complex Inversion Formula
- Partial Fractions Method

Complex Inversion Formula

Using the complex inversion formula, If

$$L\{f(t)\} = \overline{f}(s)$$

Then

$$f(t) = L^{-1}\left(\overline{f}(s)\right)$$
$$= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\overline{f}(s)ds$$

Which is simply the sum of the residues of $e^{st}\overline{f}(s)$ at the poles of $\overline{f}(s)$ In our case;

$$\overline{f}(s) = \frac{1}{(1+\lambda at)^k \prod\limits_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$\Rightarrow e^{st}\overline{f}(s) = \frac{e^{st}}{(1+\lambda at)^k \prod\limits_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

It can be seen that $\overline{f}(s)$ is analytical is analytical at every other point except when

$$s = -\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right), \qquad i = 0,1,2,3,...k$$

Thus $\overline{f}(s)$ has $k+1$ simple poles as listed below;

$$s = -\lambda\left(\frac{1+an_0)}{1+\lambda at}\right)$$

$$s = -\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)$$

$$s = -\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)$$

$$s = -\lambda \left(\frac{1 + a(n_0 + 3)}{1 + \lambda at} \right)$$
$$\vdots$$
$$s = -\lambda \left(\frac{1 + a(n_0 + k)}{1 + \lambda at} \right)$$

We now determine the Residue of $e^{st}\overline{f}(s)$ at each pole. In general for simple poles say $s = a$, The residue of $e^{st}\overline{f}(s)$ is given by

$$Residue \left[e^{st}\overline{f}(s) \right] = \lim_{s \to a} (s - a) e^{st} \overline{f}(s)$$

Thus the Residues of $e^{st}\overline{f}(s)$ at poles of $\overline{f}(s)$ are;

At $s = -\lambda \left(\frac{1+an_0}{1+\lambda at} \right)$

$$Residue \left[e^{st}\overline{f}(s) \right] = \lim_{s \to -\lambda \left(\frac{1+an_0}{1+\lambda at} \right)} \left\{ \begin{array}{c} \left[s + \lambda \left(\frac{1+an_0}{1+\lambda at} \right) \right] \times \\ \\ \frac{e^{st}}{\left[s+\lambda \left(\frac{1+an_0)}{1+\lambda at} \right) \right] (1+\lambda at)^k \prod_{i=1}^{k} \left[s+\lambda \left(\frac{1+a(n_0+i)}{1+\lambda at} \right) \right]} \end{array} \right\}$$

$$= \lim_{s \to -\lambda \left(\frac{1+an_0}{1+\lambda at} \right)} \frac{e^{st}}{(1 + \lambda at)^k \prod_{i=1}^{k} \left[s + \lambda \left(\frac{1+a(n_0+i)}{1+\lambda at} \right) \right]}$$

$$= \frac{e^{-\lambda \left(\frac{1+an_0}{1+\lambda at} \right) t}}{(1 + \lambda at)^k \prod_{i=1}^{k} \left[-\lambda \left(\frac{1+an_0}{1+\lambda at} \right) + \lambda \left(\frac{1+a(n_0+i)}{1+\lambda at} \right) \right]}$$

$$= \frac{e^{-\lambda \left(\frac{1+an_0}{1+\lambda at} \right) t}}{(1 + \lambda at)^k \prod_{i=1}^{k} \left(\frac{-\lambda - \lambda an_0 + \lambda + \lambda an_0 + \lambda ai}{1+\lambda at} \right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t}}{(1+\lambda at)^{k-k}\prod_{i=1}^{k}\lambda ai}$$

$$= \frac{e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t}}{(\lambda a)^{k}\prod_{i=1}^{k} i}$$

$$= \frac{e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t}}{(\lambda a)^{k}k!}$$

At $s = -\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)$

$$Res\left[e^{st}\overline{f}(s)\right] = \lim_{s\to-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\left[\frac{\left[s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]e^{st}}{\begin{array}{c}\left[s+\lambda\left(\frac{1+an_0)}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)\right]\\ (1+\lambda at)^{k}\prod_{i=2}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\end{array}}\right]$$

$$= \lim_{s\to-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\frac{e^{st}}{(1+\lambda at)^{k}\left[s+\lambda\left(\frac{1+an_0)}{1+\lambda at}\right)\right]\prod_{i=2}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \left[\frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{\begin{array}{c}(1+\lambda at)^{k}\left[-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)+\lambda\left(\frac{1+an_0)}{1+\lambda at}\right)\right]\\ \prod_{i=2}^{k}\left[-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\end{array}}\right]$$

$$= \left[\frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \left(\frac{-\lambda-\lambda an_0-\lambda a+\lambda+\lambda an_0}{1+\lambda at}\right) \prod_{i=2}^{k} \left(\frac{-\lambda-\lambda an_0-\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right)} \right]$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \left(\frac{-\lambda a}{1+\lambda at}\right) \prod_{i=2}^{k} \left(\frac{\lambda ai-\lambda a}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \left(\frac{-\lambda a}{1+\lambda at}\right) \prod_{i=2}^{k} \left(\frac{\lambda a}{1+\lambda at}\right)(i-1)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(1+\lambda at)^k(-1) \left(\frac{\lambda a}{1+\lambda at}\right) \left(\frac{\lambda a}{1+\lambda at}\right)^{k-1} \prod_{i=2}^{k} (i-1)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(-1)(\lambda a)^k \prod_{i=2}^{k} (i-1)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(-1)(\lambda a)^k (k-1)!}$$

At $s=-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)$

Residue$\left[e^{st}\overline{f}(s)\right]$ is given by

$$Res = \lim_{s\to-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)}\left[\frac{\left[s+\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)\right]e^{st}}{(1+\lambda at)^k\prod_{i=0}^{1}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)\right]\prod_{i=3}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right]$$

$$= \lim_{s\to-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)}\frac{e^{st}}{(1+\lambda at)^k\prod_{i=0}^{1}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\prod_{i=3}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \left[\frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(1+\lambda at)^k\prod_{i=0}^{1}\left[-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\prod_{i=3}^{k}\left[-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right]$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(1+\lambda at)^k\prod_{i=0}^{1}\left(\frac{-\lambda-\lambda an_0-2\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right)\prod_{i=3}^{k}\left(\frac{-\lambda-\lambda an_0-2\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(1+\lambda at)^k\prod_{i=0}^{1}\left(\frac{\lambda ai-2\lambda a}{1+\lambda at}\right)\prod_{i=3}^{k}\left(\frac{\lambda ai-2\lambda a}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(1+\lambda at)^k\left(\frac{-2\lambda a}{1+\lambda at}\right)\left(\frac{-\lambda a}{1+\lambda at}\right)\prod_{i=2}^{k}\left(\frac{\lambda a}{1+\lambda at}\right)(i-2)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(1+\lambda at)^k(-1)^2\left(\frac{2\lambda a}{1+\lambda at}\right)\left(\frac{\lambda a}{1+\lambda at}\right)\left(\frac{\lambda a}{1+\lambda at}\right)^{k-2}\prod_{i=3}^{k}(i-2)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(-1)^2(\lambda a)^k(2)\prod_{i=3}^{k}(i-2)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(-1)^2(\lambda a)^k 2!(k-2)!}$$

At $s = -\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)$

Residue $\left[e^{st}\overline{f}(s)\right]$ is given by

$$\operatorname{Re} s = \lim_{s\to-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)} \frac{\left[s+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\right]e^{st}}{(1+\lambda at)^k\prod_{i=0}^{2}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\left[s+\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)\right]\prod_{i=4}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \lim_{s\to-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)} \frac{e^{st}}{(1+\lambda at)^k\prod_{i=0}^{2}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\prod_{i=4}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(1+\lambda at)^k\prod_{i=0}^{2}\left[-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]\prod_{i=4}^{k}\left[-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \prod_{i=0}^{2}\left(\frac{-\lambda-\lambda an_0-3\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right)\prod_{i=4}^{k}\left(\frac{-\lambda-\lambda an_0-3\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \prod_{i=0}^{2}\left(\frac{\lambda ai-3\lambda a}{1+\lambda at}\right)\prod_{i=4}^{k}\left(\frac{\lambda ai-3\lambda a}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \left(\frac{-3\lambda a}{1+\lambda at}\right)\left(\frac{-2\lambda a}{1+\lambda at}\right)\left(\frac{-\lambda a}{1+\lambda at}\right)\prod_{i=4}^{k}\left(\frac{\lambda a}{1+\lambda at}\right)(i-3)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(1+\lambda at)^k(-1)^3 \left(\frac{3\lambda a}{1+\lambda at}\right)\left(\frac{2\lambda a}{1+\lambda at}\right)\left(\frac{\lambda a}{1+\lambda at}\right)\left(\frac{\lambda a}{1+\lambda at}\right)^{k-3}\prod_{i=4}^{k}(i-3)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(-1)^3(\lambda a)^k(6)\prod_{i=4}^{k}(i-3)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(-1)^3(\lambda a)^k 3!(k-3)!}$$

At $s = -\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)$

Residue $\left[e^{st}\overline{f}(s)\right]$ is given by

$$\text{Res} = \lim_{s \to -\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)} \frac{\left[s + \lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)\right] e^{st}}{(1+\lambda at)^k \prod_{i=0}^{3} \left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right] \left[s + \lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)\right] \prod_{i=5}^{k} \left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \lim_{s \to -\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)} \frac{e^{st}}{(1+\lambda at)^k \prod_{i=0}^{3} \left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right] \prod_{i=5}^{k} \left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \prod_{i=0}^{3} \left[-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right) + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right] \prod_{i=5}^{k} \left[-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right) + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \prod_{i=0}^{3} \left(\frac{-\lambda-\lambda an_0-4\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right) \prod_{i=5}^{k} \left(\frac{-\lambda-\lambda an_0-4\lambda a+\lambda+\lambda an_0+\lambda ai}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \prod_{i=0}^{3} \left(\frac{\lambda ai-4\lambda a}{1+\lambda at}\right) \prod_{i=5}^{k} \left(\frac{\lambda ai-4\lambda a}{1+\lambda at}\right)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(1+\lambda at)^k \left(\frac{-4\lambda a}{1+\lambda at}\right) \left(\frac{-3\lambda a}{1+\lambda at}\right) \left(\frac{-2\lambda a}{1+\lambda at}\right) \left(\frac{-\lambda a}{1+\lambda at}\right) \prod_{i=5}^{k} \left(\frac{\lambda a}{1+\lambda at}\right)(i-4)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(1+\lambda at)^k (-1)^4 \left(\frac{4\lambda a}{1+\lambda at}\right) \left(\frac{3\lambda a}{1+\lambda at}\right) \left(\frac{2\lambda a}{1+\lambda at}\right) \left(\frac{\lambda a}{1+\lambda at}\right) \left(\frac{\lambda a}{1+\lambda at}\right)^{k-4} \prod_{i=5}^{k} (i-4)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(-1)^4(\lambda a)^k(24)\prod_{i=5}^{k}(i-4)}$$

$$= \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(-1)^4(\lambda a)^k 4!(k-4)!}$$

Generalizing the results above, the residue at the pole $s = -\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)$ we have;

$$\text{Residue}\left[e^{st}\overline{f}(s)\right] = \frac{e^{-[v+\lambda(n_0+k)]t}}{(-1)^k\, k!\lambda^k\,(k-k)!}$$

$$= \frac{e^{-[v+\lambda(n_0+k)]t}}{(-1)^k\, k!\lambda^k}$$

Therefore summing the residues at each pole yields,

$$L^{-1}\left\{\frac{1}{(1+\lambda at)^k\prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\} = \sum \text{Residue}\left(e^{st}\overline{f}(s)\right)$$

$$= \left\{\begin{array}{c} \frac{e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t}}{(\lambda a)^k k!} + \frac{e^{-\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)t}}{(-1)(\lambda a)^k 1!(k-1)!} + \\ \frac{e^{-\lambda\left(\frac{1+a(n_0+2)}{1+\lambda at}\right)t}}{(-1)^2(\lambda a)^k 2!(k-2)!} + \frac{e^{-\lambda\left(\frac{1+a(n_0+3)}{1+\lambda at}\right)t}}{(-1)^3(\lambda a)^k 3!(k-3)!} + \\ \frac{e^{-\lambda\left(\frac{1+a(n_0+4)}{1+\lambda at}\right)t}}{(-1)^4(\lambda a)^k 4!(k-4)!} + ... + \frac{e^{-\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)t}}{(-1)^k(\lambda a)^k k!} \end{array}\right\}$$

$$= \sum_{j=0}^{k} \left\{ \frac{e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{(-1)^j(\lambda a)^k j!(k-j)!} \right\}$$

$$= \frac{1}{(\lambda a)^k} \sum_{j=0}^{k} \left\{ \frac{e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{(-1)^j\, j!\, (k-j)!} \right\} \tag{5.82}$$

But from equation (5.81) we had

$$P_n(t) = \prod_{i=0}^{k-1} \lambda\, [(1 + a(n_0 + i)] L^{-1} \left\{ \frac{1}{(1+\lambda at)^k \prod_{i=0}^{k} \left[s + \lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]} \right\}$$

Thus using equation (5.82) in equation (5.81) we have

$$P_n(t) = \prod_{i=0}^{k-1} \lambda\, [(1 + a(n_0 + i)] \times \frac{1}{(\lambda a)^k} \sum_{j=0}^{k} \left\{ \frac{e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{(-1)^j\, j!\, (k-j)!} \right\}$$

But

$$\prod_{i=0}^{k-1} \lambda\, [(1 + a(n_0 + i)] = \prod_{i=0}^{k-1} \lambda a \left[\frac{1}{a} + (n_0 + i) \right]$$

$$= (\lambda a)^k \prod_{i=0}^{k-1} \left[\frac{1}{a} + (n_0 + i) \right]$$

$$= \left\{ \begin{array}{c} (\lambda a)^k \left(\frac{1}{a} + n_0\right) \left(\frac{1}{a} + n_0 + 1\right) \left(\frac{1}{a} + n_0 + 2\right) ... \\ \\ \left(\frac{1}{a} + n_0 + k - 2\right) \left(\frac{1}{a} + n_0 + k - 1\right) \end{array} \right\}$$

And

$$\left(\frac{1}{a}+n_0+k-1\right)! = \begin{Bmatrix} \left(\frac{1}{a}+n_0+k-1\right)\left(\frac{1}{a}+n_0+k-2\right)... \\ \\ \left(\frac{1}{a}+n_0+1\right)\left(\frac{1}{a}+n_0\right)\left(\frac{1}{a}+n_0-1\right)! \end{Bmatrix}$$

$$\Rightarrow \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!} = \left(\frac{1}{a}+n_0+k-1\right)\left(\frac{1}{a}+n_0+k-2\right)...\left(\frac{1}{a}+n_0+1\right)\left(\frac{1}{a}+n_0\right)$$

$$\therefore \prod_{i=0}^{k-1}\lambda\left[(1+a(n_0+i)\right] = (\lambda a)^k\,\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!} \tag{5.83}$$

Therefore

$$P_n(t) = \prod_{i=0}^{k-1}\lambda\left[(1+a(n_0+i)\right]\times\frac{1}{(\lambda a)^k}\sum_{j=0}^{k}\left\{\frac{e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{(-1)^j\,j!\,(k-j)!}\right\}$$

$$= (\lambda a)^k\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!}\times\frac{1}{(\lambda a)^k}\sum_{j=0}^{k}\left\{\frac{e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{(-1)^j\,j!\,(k-j)!}\right\}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!}\sum_{j=0}^{k}\left\{\frac{(-1)^j\,e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{j!\,(k-j)!}\right\}\times\frac{k!}{k!}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!}\sum_{j=0}^{k}\left\{\frac{k!\,(-1)^j\,e^{-\lambda\left(\frac{1+a(n_0+j)}{1+\lambda at}\right)t}}{j!\,(k-j)!}\right\}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} \sum_{j=0}^{k} \left\{ \frac{k!\,(-1)^j\, e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} e^{-\left(\frac{\lambda at}{1+\lambda at}\right)j}}{j!\,(k-j)!} \right\}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \sum_{j=0}^{k} \left\{ \frac{k!\left(-e^{-\left(\frac{\lambda at}{1+\lambda at}\right)}\right)^j}{j!\,(k-j)!} \right\}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \sum_{j=0}^{k} \binom{k}{j} \left(-e^{-\left(\frac{\lambda at}{1+\lambda at}\right)}\right)^j$$

$$= \binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \sum_{j=0}^{k} \binom{k}{j} \left(-e^{-\left(\frac{\lambda at}{1+\lambda at}\right)}\right)^j$$

$$= \binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \left(1-e^{-\left(\frac{\lambda at}{1+\lambda at}\right)}\right)^k \qquad k=0,1,2,...$$

$$= \binom{\left[n_0+\frac{1}{a}\right]+k-1}{k} \left(e^{-\frac{\lambda at}{1+\lambda at}}\right)^{n_0+\frac{1}{a}} \left(1-e^{-\left(\frac{\lambda at}{1+\lambda at}\right)}\right)^k \qquad k=0,1,2,...$$

Which is the pmf of a negative binomial distribution with parameters $r = n_0+\frac{1}{a}$ and $p = e^{-\left(\frac{\lambda at}{1+\lambda at}\right)}$

Partial Fractions Method

By equation (5.81) we had

$$P_n(t) = \prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right]L^{-1}\left\{ \frac{1}{(1+\lambda at)^k \prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]} \right\}$$

But from equation (5.83)

$$\prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right] = (\lambda a)^k \, \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!}$$

Therefore

$$P_n(t) = \prod_{i=0}^{k-1} \lambda\left[(1+a(n_0+i)\right] L^{-1}\left\{\frac{1}{(1+\lambda at)^k \prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\}$$

$$P_n(t) = (\lambda a)^k \, \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(1+\lambda at)^k} L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\} \tag{5.84}$$

We now determine

$$L^{-1}\left\{\frac{1}{\prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}\right\}$$

But

$$\frac{1}{\prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]} = \left\{\begin{array}{c} \left(\frac{1}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)\cdots \\ \\ \left(\frac{1}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right) \\ \\ \left(\frac{1}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)\cdots\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right) \end{array}\right\}$$

$$
\left\{
\begin{array}{c}
\left(\frac{1}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right) \\
\ldots\left(\frac{1}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right) \\
\left(\frac{1}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)\ldots\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)
\end{array}
\right\}
=
\left\{
\begin{array}{c}
\left(\frac{b_{n_0}}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)+ \\
\left(\frac{b_{n_0+1}}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)+\ldots \\
+\left(\frac{b_{j-1}}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)+ \\
\left(\frac{b_{j}}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right)+ \\
\left(\frac{b_{j+1}}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)+\ldots \\
+\left(\frac{b_{n_0+k}}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)
\end{array}
\right\}
$$

Multiplying both sides by $s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)$ yields

$$\left\{\begin{array}{c} \left(\frac{1}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)\cdots \\ \left(\frac{1}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)\left(\frac{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right) \\ \left(\frac{1}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)\cdots\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right) \end{array}\right\} = \left\{\begin{array}{c} \left(\frac{b_{n_0}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right) \\ +\left(\frac{b_{n_0+1}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)+\ldots+ \\ \left(\frac{b_{j-1}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)+ \\ \left(\frac{b_{j}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right)+ \\ \left(\frac{b_{j+1}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)+\ldots \\ +\left(\frac{b_{n_0+k}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right) \end{array}\right\}$$

$$\left\{\begin{array}{c}\left(\frac{1}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)\dots \\ \left(\frac{1}{s+\lambda\left(\frac{1+a(j-2)}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right) \\ \left(\frac{1}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)\left(\frac{1}{s+\lambda\left(\frac{1+a(j+2)}{1+\lambda at}\right)}\right) \\ \dots\left(\frac{1}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)\end{array}\right\} = \left\{\begin{array}{c}\left(\frac{b_{n_0}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right) \\ +\left(\frac{b_{n_0+1}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)+\dots+ \\ \left(\frac{b_{j-1}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)+b_j+ \\ \left(\frac{b_{j+1}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)+\dots \\ +\left(\frac{b_{n_0+k}\left[s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{s+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)\end{array}\right\}$$

Setting $s=-\lambda\left(\frac{1+aj}{1+\lambda at}\right)$, implies

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)\times \\ \left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)\times ... \\ \times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j-2)}{1+\lambda at}\right)}\right)\times \\ \left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)\times \\ \left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)\times \\ \left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j+2)}{1+\lambda at}\right)}\right)\times ... \\ \times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)\end{array}\right\}=\left\{\begin{array}{c}\left(\frac{b_{n_0}\left[-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)+ \\ \left(\frac{b_{n_0+1}\left[-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)+...+ \\ \left(\frac{b_{j-1}\left[-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)+b_j+ \\ \left(\frac{b_{j+1}\left[-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)+... \\ +\left(\frac{b_{n_0+k}\left[-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+aj}{1+\lambda at}\right)\right]}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)\end{array}\right\}$$

$$\left\{\begin{array}{c}\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+an_0}{1+\lambda at}\right)}\right)\times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+1)}{1+\lambda at}\right)}\right)\times\ldots\\ \times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j-2)}{1+\lambda at}\right)}\right)\times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j-1)}{1+\lambda at}\right)}\right)\times\\ \left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j+1)}{1+\lambda at}\right)}\right)\times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(j+2)}{1+\lambda at}\right)}\right)\\ \times\ldots\times\left(\frac{1}{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)+\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)}\right)\end{array}\right\}=b_j$$

We now have;

$$\left\{\begin{array}{c}
\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda(1+an_0)}{1+\lambda at}}\right)\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(n_0+1)]}{1+\lambda at}}\right)\cdots \\
\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(n_0+2)]}{1+\lambda at}}\right)\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(j-2)]}{1+\lambda at}}\right) \\
\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(j-1)]}{1+\lambda at}}\right)\times\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(j+1)]}{1+\lambda at}}\right) \\
\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(j+2)]}{1+\lambda at}}\right)\cdots\times\left(\frac{1}{\frac{-\lambda(1+aj)+\lambda[1+a(n_0+k)]}{1+\lambda at}}\right)
\end{array}\right\} = b_j$$

$$\left\{\begin{array}{c}
\left(\frac{1}{\frac{-\lambda a}{1+\lambda at}(j-n_0)}\right)\left(\frac{1}{\frac{-\lambda a}{1+\lambda at}[j-(n_0+1)]}\right)\left(\frac{1}{\frac{-\lambda a}{1+\lambda at}[j-(n_0+2)]}\right)\cdots\left(\frac{1}{\frac{-\lambda a}{1+\lambda at}(2)}\right) \\
\left(\frac{1}{\frac{-\lambda a}{1+\lambda at}(1)}\right)\times\left(\frac{1}{\frac{\lambda a}{1+\lambda at}(1)}\right)\left(\frac{1}{\frac{\lambda a}{1+\lambda at}(2)}\right)\cdots\left(\frac{1}{\frac{\lambda a}{1+\lambda at}[(n_0+k)-j]}\right)
\end{array}\right\} = b_j$$

$$\left\{\begin{array}{c}\frac{1}{\left(\frac{-\lambda a}{1+\lambda at}\right)^{j-n_0}}\left(\frac{1}{j-n_0}\right)\left(\frac{1}{j-(n_0+1)}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{1}{1}\right)\\ \\ \times\frac{1}{\left(\frac{\lambda a}{1+\lambda at}\right)^{n_0+k-j}}\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{n_0+k-j}\right)\end{array}\right\}=b_j$$

$$\frac{1}{(-1)^{j-n_0}\left(\frac{\lambda a}{1+\lambda at}\right)^{j-n_0}(j-n_0)!}\times\frac{1}{\left(\frac{\lambda a}{1+\lambda at}\right)^{n_0+k-j}(n_0+k-j)!}=b_j$$

$$\frac{1}{(-1)^{j-n_0}\left(\frac{\lambda a}{1+\lambda at}\right)^{j-n_0+n_0+k-j}(j-n_0)!\,(n_0+k-j)!}=b_j$$

$$\Rightarrow b_j=\frac{1}{(-1)^{j-n_0}\left(\frac{\lambda a}{1+\lambda at}\right)^{k}(j-n_0)!\,(n_0+k-j)!}$$

$$=\frac{(-1)^{j-n_0}}{\left(\frac{\lambda a}{1+\lambda at}\right)^{k}(j-n_0)!\,(n_0+k-j)!}$$

With this it follows that

$$\begin{aligned}\frac{1}{\prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}&=\sum_{j=n_0}^{n_0+k}\left[\frac{b_j}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right]\\&=\sum_{j=n_0}^{n_0+k}\frac{(-1)^{j-n_0}}{\left(\frac{\lambda a}{1+\lambda at}\right)^{k}(j-n_0)!\,(n_0+k-j)!}\left(\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right)\end{aligned}$$

Thus

$$P_n(t) = (\lambda a)^k \frac{\left(\frac{1}{a} + n_0 + k - 1\right)!}{\left(\frac{1}{a} + n_0 - 1\right)!(1 + \lambda a t)^k} L^{-1} \left\{ \frac{1}{\prod_{i=0}^{k} \left[s + \lambda \left(\frac{1 + a(n_0 + i)}{1 + \lambda a t} \right) \right]} \right\}$$

$$= \left\{ \begin{array}{c} (\lambda a)^k \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(1+\lambda a t)^k} \times \\ \\ L^{-1} \left\{ \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{\left(\frac{\lambda a}{1+\lambda a t}\right)^k (j-n_0)!(n_0+k-j)!} \left(\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda a t}\right)} \right) \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (\lambda a)^k \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(1+\lambda a t)^k} \times \\ \\ \frac{1}{\left(\frac{\lambda a}{1+\lambda a t}\right)^k} L^{-1} \left\{ \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j-n_0)!(n_0+k-j)!} \left(\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda a t}\right)} \right) \right\} \end{array} \right\}$$

$$= \frac{\left(\frac{1}{a} + n_0 + k - 1\right)!}{\left(\frac{1}{a} + n_0 - 1\right)!} \times L^{-1} \left\{ \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j - n_0)!\,(n_0 + k - j)!} \left(\frac{1}{s + \lambda \left(\frac{1+aj}{1+\lambda a t}\right)} \right) \right\}$$

$$= \frac{\left(\frac{1}{a} + n_0 + k - 1\right)!}{\left(\frac{1}{a} + n_0 - 1\right)!} \times \sum_{j=n_0}^{n_0+k} \frac{(-1)^{j-n_0}}{(j - n_0)!\,(n_0 + k - j)!} L^{-1} \left\{ \frac{1}{s + \lambda \left(\frac{1+aj}{1+\lambda a t}\right)} \right\}$$

But

$$L\left\{e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t}\right\} = \int_0^\infty e^{-st}e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t}dt$$

$$= \int_0^\infty e^{-\left[\lambda\left(\frac{1+aj}{1+\lambda at}\right)+s\right]\,t}dt$$

$$= \left.\frac{e^{-\left[\lambda\left(\frac{1+aj}{1+\lambda at}\right)+s\right]\,t}}{-\left[\lambda\left(\frac{1+aj}{1+\lambda at}\right)+s\right]}\right|_0^\infty$$

$$= \frac{1}{-\left[\lambda\left(\frac{1+aj}{1+\lambda at}\right)+s\right]}\left[0-1\right]$$

$$= \frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}$$

$$\Rightarrow e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t} = L^{-1}\left\{\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right\}$$

Thus

$$P_n\left(t\right) = \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!} \times \sum_{j=n_0}^{n_0+k} \frac{\left(-1\right)^{j-n_0}}{\left(j-n_0\right)!\left(n_0+k-j\right)!}L^{-1}\left\{\frac{1}{s+\lambda\left(\frac{1+aj}{1+\lambda at}\right)}\right\}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!} \times \sum_{j=n_0}^{n_0+k} \left[\frac{\left(-1\right)^{j-n_0}}{\left(j-n_0\right)!\left(n_0+k-j\right)!}\right] e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t}$$

Multiplying the above equation by $\frac{k!}{k!}$ yields;

$$P_n(t) = \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!} \times \sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}}{(j-n_0)!\,(n_0+k-j)!}\right] e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t} \times \frac{k!}{k!}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} \sum_{j=n_0}^{n_0+k}\left[\frac{(-1)^{j-n_0}\,k!}{(j-n_0)!\,(n_0+k-j)!}\right] e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t}$$

$$= \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} \sum_{j=n_0}^{n_0+k}(-1)^{j-n_0}\binom{k}{j-n_0} e^{-\lambda\left(\frac{1+aj}{1+\lambda at}\right)t}$$

Letting $m = j - n_0 \Rightarrow j = m + n_0$ implies

$$P_n(t) = \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} \sum_{m=0}^{k}(-1)^m\binom{k}{m} e^{-\lambda\left[\frac{1+a(m+n_0)}{1+\lambda at}\right]t}$$

But

$$\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!} = \binom{\frac{1}{a}+n_0+k-1}{k}$$
$$\binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1}$$

Thus

$$\begin{aligned}
P_n(t) &= \binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \sum_{m=0}^{k}(-1)^m\binom{k}{m} e^{-\frac{\lambda amt}{1+\lambda at}} \\
&= \binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \sum_{m=0}^{k}(-1)^m\binom{k}{m}\left(e^{-\frac{\lambda at}{1+\lambda at}}\right)^m \\
&= \binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t} \sum_{m=0}^{k}\binom{k}{m}\left(-e^{-\frac{\lambda at}{1+\lambda at}}\right)^m \\
&= \binom{\frac{1}{a}+n_0+k-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t}\left(1-e^{-\frac{\lambda at}{1+\lambda at}}\right)^k \qquad k = 0, 1, 2, 3, ...
\end{aligned}$$

But $n = n_0 + k \Rightarrow k = n - n_0$ hence

$$P_n(t) = \binom{\frac{1}{a}+n-1}{\frac{1}{a}+n_0-1} e^{-\lambda\left(\frac{1+an_0}{1+\lambda at}\right)t}\left(1-e^{-\frac{\lambda at}{1+\lambda at}}\right)^{n-n_0} \qquad n = n_0, n_0+1, n_0+2, ...$$

$$= \binom{\left[n_0+\frac{1}{a}\right]+k-1}{k}\left(e^{-\frac{\lambda at}{1+\lambda at}}\right)^{n_0+\frac{1}{a}}\left(1-e^{-\frac{\lambda at}{1+\lambda at}}\right)^{k} \qquad k = 0,\ 1,\ 2, ...$$

Which is the negative binomial distribution with $r = n_0 + \frac{1}{a}$ and $p = e^{-\frac{\lambda at}{1+\lambda at}}$

Method 2: Using the pgf approach

The basic difference-differential equations for the polya process as defined by equations (5.78a) and (5.78b) are;

$$P_0'(t) = -\left(\frac{\lambda}{1+\lambda at}\right)P_0(t) \ n = 0$$
$$P_n'(t) = -\lambda\left(\frac{1+an}{1+\lambda at}\right)P_n(t) + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right)P_{n-1}(t) \qquad n = 1, 2, 3, ...$$

Multiplying both sides of equation (5.78b) by z^n and summing the result over n, we have

$$\sum_{n=1}^{\infty} P_n'(t)z^n = \lambda\sum_{n=1}^{\infty}\left[\frac{1+a(n-1)}{1+\lambda at}\right]P_{n-1}(t)z^n - \sum_{n=1}^{\infty}\left[\frac{1+an}{1+\lambda at}\right]P_n(t)z^n$$

$$= \frac{\lambda}{1+\lambda at}\left\{\sum_{n=1}^{\infty}\left[1+a(n-1)\right]P_{n-1}(t)z^n - \sum_{n=1}^{\infty}[1+an]P_n(t)\ z^n\right\}$$

$$\sum_{n=1}^{\infty} P_n'(t)z^n = \frac{\lambda}{1+\lambda at}\left\{\begin{array}{c} \underbrace{\sum_{n=1}^{\infty}P_{n-1}(t)z^{n-1}}_{I} + \underbrace{az\sum_{n=1}^{\infty}(n-1)P_{n-1}(t)z^n}_{II} \\ \\ -\underbrace{\sum_{n=1}^{\infty}P_n(t)z^n}_{III} - \underbrace{a\sum_{n=1}^{\infty}nP_n(t)z^n}_{IV} \end{array}\right\} \tag{5.85}$$

Let $G(z,t)$ be the probability generating function of X(t) defined as follows;

$$
\begin{aligned}
G(z,t) &= \sum_{n=0}^{\infty} P_n(t) z^n \\
&= P_0(t) + \sum_{n=1}^{\infty} P_n(t) z^n \\
\frac{\partial}{\partial t} G(z,t) &= \sum_{n=0}^{\infty} P'_n(t) z^n \\
&= P'_0(t) + \sum_{n=1}^{\infty} P'_n(t) z^n \\
\sum_{n=1}^{\infty} P'_n(t) z^n &= \frac{\partial}{\partial z} G(z,t) - P'_0(t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z} G(z,t) &= \sum_{n=0}^{\infty} n P_n(t) z^{n-1} \\
&= \frac{1}{z} \sum_{n=1}^{\infty} n P_n(t) z^n \\
&= \sum_{n=1}^{\infty} (n-1) P_n(t) z^{n-2}
\end{aligned}
$$

We now simplify the four parts of equation (5.85) separately

Part I

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_{n-1}(t) z^n &= z \sum_{n=1}^{\infty} P_{n-1}(t) z^{n-1} \\
&= zG(z,t)
\end{aligned}
$$

Part II

$$
\begin{aligned}
a \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) z^n &= az^2 \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) z^{n-2} \\
&= az^2 \frac{\partial}{\partial z} G(z,t)
\end{aligned}
$$

Part III

$$\sum_{n=1}^{\infty} P_n(t)z^n = G(z,t) - P_0(t)$$

Part IV

$$\begin{aligned} a\sum_{n=1}^{\infty} nP_n(t)z^n &= az\sum_{n=1}^{\infty} nP_n(t)z^{n-1} \\ &= az\frac{\partial}{\partial z}G(z,t) \end{aligned}$$

Substituting the above in equation (5.85) yields

$$\frac{\partial}{\partial t}G(z,t) - P_0'(t) = \frac{\lambda}{1+\lambda at}\left\{zG(z,t) + az^2\frac{\partial}{\partial z}G(z,t) - [G(z,t) - P_0(t)] - az\frac{\partial}{\partial z}G(z,t)\right\}$$
$$\frac{\partial}{\partial t}G(z,t) - P_0'(t) = \frac{\lambda}{1+\lambda at}\left\{zG(z,t) + az^2\frac{\partial}{\partial z}G(z,t) - G(z,t) + P_0(t) - az\frac{\partial}{\partial z}G(z,t)\right\}$$

But by equation (5.78a)

$$P_0'(t) = -\frac{\lambda}{1+\lambda at}P_0(t)$$

Hence we now have

$$\frac{\partial}{\partial t}G(z,t) + \frac{\lambda}{1+\lambda at}P_0(t) = \frac{\lambda}{1+\lambda at}\left\{zG(z,t) + az^2\frac{\partial}{\partial z}G(z,t) - G(z,t) + P_0(t) - az\frac{\partial}{\partial z}G(z,t)\right\}$$
$$\frac{\partial}{\partial t}G(z,t) = \frac{\lambda}{1+\lambda at}\left\{zG(z,t) + az^2\frac{\partial}{\partial z}G(z,t) - G(z,t) - az\frac{\partial}{\partial z}G(z,t)\right\}$$

Grouping the like terms together yields

$$\frac{\partial}{\partial t}G(z,t) = \frac{\lambda}{1+\lambda at}\left\{az(z-1)\frac{\partial}{\partial z}G(z,t) + (z-1)G(z,t)\right\}$$

$$\Rightarrow \frac{\partial}{\partial t}G(z,t) - \frac{\lambda az\,(z-1)}{1+\lambda at}\frac{\partial}{\partial z}G(z,t) - \frac{\lambda\,(z-1)}{1+\lambda at}G(z,t) = 0$$

$$\frac{\partial}{\partial t}G\left(z,t\right)-\frac{\lambda az(z-1)}{1+\lambda at}\frac{\partial}{\partial z}G\left(z,t\right)-\frac{\lambda(z-1)}{1+\lambda at}G(z,t)=0$$

Which is a partial differential equation. Multiplying both sides by $1+\lambda at$, we have

$$\left(1+\lambda at\right)\frac{\partial}{\partial t}G\left(z,t\right)-\lambda az(z-1)\frac{\partial}{\partial z}G\left(z,t\right)-\lambda(z-1)G(z,t)=0$$

Factoring λa out we have

$$\lambda a\left[\left(\frac{1}{\lambda a}+t\right)\frac{\partial}{\partial t}G\left(z,t\right)-z(z-1)\frac{\partial}{\partial z}G\left(z,t\right)-\frac{(z-1)}{a}G(z,t)\right]=0$$
$$\Rightarrow\left(\frac{1}{\lambda a}+t\right)\frac{\partial}{\partial t}G\left(z,t\right)-z(z-1)\frac{\partial}{\partial z}G\left(z,t\right)-\frac{(z-1)}{a}G(z,t)=0$$

Let $\alpha=\frac{1}{\lambda a}$

With this substitution, the above equation can be expressed as

$$\left(t+\alpha\right)\frac{\partial}{\partial t}G\left(z,t\right)-z(z-1)\frac{\partial}{\partial z}G\left(z,t\right)-\frac{(z-1)}{a}G(z,t)=0$$

To kill the persistent term $\left(t+\alpha\right)\frac{\partial}{\partial t}G\left(z,t\right)$ where $\lambda a=\frac{1}{\alpha}$. We need to change variable with $\tau=\ln\left|\frac{t+\alpha}{\alpha}\right|$

$$\frac{\partial G(z,\tau)}{\partial\tau}=\frac{\partial t}{\partial\tau}\frac{\partial G(z,\tau)}{\partial t}$$

But

$$\frac{\partial\tau}{\partial t}=\frac{\left|\frac{t+\alpha}{\alpha}\right|'}{\left|\frac{t+\alpha}{\alpha}\right|}$$

$$=\left|\frac{1}{\alpha}\right|\times\left|\frac{\alpha}{t+\alpha}\right|$$
$$=\left|\frac{1}{t+\alpha}\right|$$

$$\frac{\partial t}{\partial \tau} = \frac{1}{\frac{\partial \tau}{\partial t}} = \frac{1}{\frac{1}{t+\alpha}} = t + \alpha$$

$$\therefore \frac{\partial G(z,\tau)}{\partial \tau} = (t+\alpha)\,\frac{\partial G(z,\tau)}{\partial t}$$

Where $G(z,\tau) = G(z,t)$ just expressed in new variable. With this substitution the PDE is reduced to

$$\frac{\partial}{\partial \tau} G\left(z,\tau\right) - z(z-1)\frac{\partial}{\partial z} G\left(z,\tau\right) - \frac{(z-1)}{a} G(z,\tau) = 0$$
$$\frac{\partial}{\partial \tau} G\left(z,\tau\right) - (z-1)\left[z\frac{\partial}{\partial z} G\left(z,\tau\right) + \frac{G(z,\tau)}{a}\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau} G\left(z,\tau\right) = (z-1)\left[z\frac{\partial}{\partial z} G\left(z,\tau\right) + \frac{G(z,\tau)}{a}\right]$$

Which is an ODE. Taking Laplace transform of both sides yields

$$s\overline{G}(z,s) - G(z,0) = (z-1)\left[z\frac{d}{dz}\overline{G}\left(z,s\right) + \frac{\overline{G}(z,s)}{a}\right]$$
$$s\overline{G}(z,s) - z\left(z-1\right)\frac{d}{dz}\overline{G}\left(z,s\right) - \frac{(z-1)}{a}\overline{G}(z,s) = G(z,0)$$

Collecting like terms together yields

$$-z\left(z-1\right)\frac{d}{dz}\overline{G}\left(z,s\right) + \left[s - \frac{(z-1)}{a}\right]\overline{G}(z,s) = G(z,0)$$

$$\frac{d}{dz}\overline{G}\left(z,s\right) - \frac{1}{z\left(z-1\right)}\left[s - \frac{z-1}{a}\right]\overline{G}\left(z,s\right) = -\frac{G\left(z,0\right)}{z\left(z-1\right)}$$

$$\frac{d}{dz}\overline{G}\left(z,s\right) - \left[\frac{s}{z\left(z-1\right)} - \frac{1}{az}\right]\overline{G}\left(z,s\right) = -\frac{G\left(z,0\right)}{z\left(z-1\right)} \tag{5.86}$$

The remaining task is to find a solution to this differential equation. Two methods are considered.

- Dirac delta function approach
- Hyper geometric function approach

Dirac delta function approach

Equation (5.86) is an ODE of 1^{st} order. It is of the form $y' + Py = Q$ where

$$y = \overline{G}\left(z,s\right), P = -\left[\frac{s}{z\left(z-1\right)} - \frac{1}{az}\right] \text{ and } Q = -\frac{G(z,0)}{z\left(z-1\right)}$$

Now the solution formula for such an ODE is given by

$$ye^{\int Pdz} = f + \int Qe^{\int Pdz}$$

Where f does not depend on the variable z but can depend on the s parameter which implies that $f = f(s)$, in our case y is a function of z and s ,that is $y = y(z,s)$.
$e^{\int Pdz}$ is called the integrating factor. We first calculate the P integral.

$$\begin{aligned}
\int Pdz &= \int -\left[\frac{s}{z\left(z-1\right)} - \frac{1}{az}\right] \\
&= -\int \left[\frac{s}{z\left(z-1\right)} - \frac{1}{az}\right] dz \\
&= -\int \frac{s}{z\left(z-1\right)} dz + \int \frac{1}{az} dz \\
&= -s\int \frac{1}{z\left(z-1\right)} dz + \frac{1}{a}\int \frac{1}{z} dz \\
&= -s\int \frac{1}{z\left(z-1\right)} dz + \frac{1}{a}\int \frac{1}{z} dz \\
&= -s\int \frac{1}{z\left(z-1\right)} dz + \frac{1}{a}\ln\left|z\right|
\end{aligned}$$

But using partial fractions

$$\frac{1}{z\left(z-1\right)} = \frac{A}{z} + \frac{B}{z-1}$$

Multiplying both sides by $z(z-1)$ yields $1 = A(z-1) + Bz$ which holds for all values of z . Setting $z = 0$, we have

$$1 = A(0-1) + B(0)$$
$$\Rightarrow A = -1$$

Setting $z = 1$, we have

$$1 = A(1-1) + B(1)$$
$$\Rightarrow B = 1$$

Thus

$$= -\frac{1}{z} + \frac{1}{z-1}$$

$$\begin{aligned}
\int \frac{1}{z(z-1)} dz &= \int \left[-\frac{1}{z} + \frac{1}{z-1} \right] dz \\
&= \int -\frac{1}{z} dz + \int \frac{1}{z-1} dz \\
&= -\int \frac{1}{z} dz + \int \frac{1}{z-1} dz \\
&= -\ln|z| + \ln|z-1| \\
&= \ln\left| \frac{z-1}{z} \right|
\end{aligned}$$

With this P integral becomes

$$\begin{aligned}
\int P dz &= -s \ln\left| \frac{z-1}{z} \right| + \frac{1}{a} \ln|z| \\
&= -s \ln\left| \frac{z-1}{z} \right| + \ln|z|^{\frac{1}{a}}
\end{aligned}$$

Thus integrating factor becomes

$$\begin{aligned} I.F &= e^{\int Pdz} = e^{-s\ln\left|\frac{z-1}{z}\right| + \ln|z|^{\frac{1}{a}}} \\ &= |z|^{\frac{1}{a}} e^{-s\ln\left|\frac{z-1}{z}\right|} \end{aligned}$$

From above the general solution of the ODE is given by

$$ye^{\int Pdz} = f + \int Qe^{\int Pdz}$$

So substituting $y,\ P$ and Q we have

$$\overline{G}(z,s)\,|z|^{\frac{1}{a}}\,e^{-s\ln\left|\frac{z-1}{z}\right|} = f(s) + \int -\frac{G(z,0)}{z(z-1)}\,|z|^{\frac{1}{a}}\,e^{-s\ln\left|\frac{z-1}{z}\right|}dz$$

$$\Rightarrow \overline{G}(z,s)\,|z|^{\frac{1}{a}}\,e^{-s\ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{G(z,0)}{z(z-1)}\,|z|^{\frac{1}{a}}\,e^{-s\ln\left|\frac{z-1}{z}\right|}dz \tag{5.87}$$

We consider two cases

Case 1: When initial population $X(0) = 1$

Recall that

$$\begin{aligned} G(z,t) &= \sum_{n=0}^{\infty} P_n(t)z^n \\ \Rightarrow G(z,0) &= \sum_{n=0}^{\infty} P_n(0)z^n \\ &= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ... \end{aligned}$$

but for the initial condition $X(0) = 1$, we have

$$P_1(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq 1$$

$$\therefore G(z,0) = z$$

With this equation (5.87) becomes

$$\overline{G}\left(z,s\right)\left|z\right|^{\frac{1}{a}}e^{-s\ln\left|\frac{z-1}{z}\right|}=f(s)-\int\frac{z}{z\left(z-1\right)}\left|z\right|^{\frac{1}{a}}e^{-s\ln\left|\frac{z-1}{z}\right|}dz$$

$$\overline{G}\left(z,s\right)\left|z\right|^{\frac{1}{a}}e^{-s\ln\left|\frac{z-1}{z}\right|}=f(s)-\int\frac{1}{\left(z-1\right)}\left|z\right|^{\frac{1}{a}}e^{-s\ln\left|\frac{z-1}{z}\right|}dz \tag{5.88}$$

Now this looks like a complicated equation to solve, but at this juncture we can licitly apply the inverse Laplace transform to both sides.

We observe that $f(s)$ can be regarded as a Laplace transform of some unknown function$f(t)$

Applying inverse Laplace transform to both sides of equation (5.88) we have from the table of transform pairs in chapter 2 or examples 7 and 8 (see pages 38-39)

1. e^{-cs} is the Laplace transform of the Dirac delta function $\delta(t-c)$

2. $\overline{G}(z,s)e^{cs}$ is the Laplace transform of $G(t-c)\eta(t-c)$ where η is the Heaviside step function. In our case $c=-\ln\left|\frac{z-1}{z}\right|$

With this equation (5.88) can be rewritten as

$$\overline{G}\left(z,s\right)\left|z\right|^{\frac{1}{a}}e^{-cs}=f(s)-\int\frac{1}{\left(z-1\right)}\left|z\right|^{\frac{1}{a}}e^{-cs}dz$$

So all together inversely transforming both sides the above equation, we come to

$$G(z,\tau-c)\left|z\right|^{\frac{1}{a}}\eta(t-c)=f(\tau)-\int\frac{1}{\left(z-1\right)}\left|z\right|^{\frac{1}{a}}\delta(\tau-c)dz$$

Substituting the value of a we obtain

$$G\left(z,t-\ln\left|\frac{z-1}{z}\right|\right)\left|z\right|^{\frac{1}{a}}\eta\left(t-\ln\left|\frac{z-1}{z}\right|\right)=f(t)-\int\frac{1}{\left(z-1\right)}\left|z\right|^{\frac{1}{a}}\delta\left(t-\ln\left|\frac{z-1}{z}\right|\right)dz \tag{5.89}$$

Now this is a pretty large equation but it can be simplified, but to do that we need to make one big detour. We can free ourselves of this terrible integral by using the Dirac delta function, but what we have is delta of function of variable z , so we need to first simplify it to a common delta of variable. To do so, we shall use the following property of delta function

$$\delta\left(g\left(z\right)\right)=\sum_{z_i}\frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right) \tag{5.90}$$

Where $g'\left(z\right)$ is the first derivative of $g\left(z\right)$, z_i is a simple root of $g\left(z\right)$ such that $g'\left(z_i\right)$. In our case $g\left(z\right)=\tau-\ln\left|\frac{z-1}{z}\right|$. To obtain the roots of $g(z)$ we solve $g\left(z\right)=0$

$$\begin{aligned}
&\Rightarrow\tau-\ln\left|\frac{z-1}{z}\right|=0\\
&\tau=\ln\left|\frac{z-1}{z}\right|\\
&\tau=\ln\left|\frac{z-1}{z}\right|\\
&e^{\tau}=\left|\frac{z-1}{z}\right|\\
&e^{\tau}=\pm\frac{z-1}{z}
\end{aligned}$$

First root

$$\begin{aligned}
&e^{\tau}=\frac{z-1}{z}\\
&ze^{\tau}=z-1\\
&1=z-ze^{\tau}\\
&1=z\left(1-e^{\tau}\right)\\
&z_1=\frac{1}{1-e^{\tau}}
\end{aligned}$$

Second root

$$\begin{aligned} e^{\tau} &= -\frac{z-1}{z} \\ ze^{\tau} &= -(z-1) \\ 1 &= z + ze^{\tau} \\ 1 &= z(1+e^{\tau}) \\ z_2 &= \frac{1}{1+e^{\tau}} \end{aligned}$$

Therefore

$$z_i = \frac{1}{1 \pm e^{\tau}}$$

The next step is to determine

$$\begin{aligned} g'(z) &= \frac{d}{dz}\left[\tau - \ln\left|\frac{z-1}{z}\right|\right] \\ &= -\frac{d}{dz}\ln\left|\frac{z-1}{z}\right| \end{aligned}$$

Using the property $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

$$\begin{aligned} g'(z) &= -\left(\frac{\frac{d}{dz}\left|\frac{z-1}{z}\right|}{\left|\frac{z-1}{z}\right|}\right) \\ g'(z) &= -\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \end{aligned} \tag{5.91}$$

Now remembering that $|x| = x\,\mathrm{sgn}(x)$ and $\mathrm{sgn}(x) = 2\eta(x) - 1$, therefore

$$\left|\frac{z}{z-1}\right| = \frac{z}{z-1}\,\mathrm{sgn}\left(\frac{z}{z-1}\right) \tag{5.92}$$

Also

$$\left|\frac{z-1}{z}\right| = \frac{z-1}{z}\operatorname{sgn}\left(\frac{z-1}{z}\right)$$
$$= \frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right)-1\right]$$

Where sgn is the sign distribution and eta is the Heaviside distribution .Hence

$$\left|\frac{z-1}{z}\right|' = \frac{d}{dz}\left|\frac{z-1}{z}\right|$$

$$\left|\frac{z-1}{z}\right|' = \frac{d}{dz}\left\{\frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right)-1\right]\right\} \tag{5.93}$$

At this step we need to recall that

$$\eta'_{h(z)} = h'(z)\frac{\partial\eta(z)}{\partial h}$$

Using product rule of differentiation, we simplify equation (5.93) as follows Let

$$U = \frac{z-1}{z} = 1-\frac{1}{z} \Rightarrow U' = \frac{1}{z^2}$$
$$V = 2\eta\left(\frac{z-1}{z}\right)-1$$
$$= 2\eta\left(1-\frac{1}{z}\right)-1 \Rightarrow V' = \frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)$$

We now have

$$\left|\frac{z-1}{z}\right|' = U'V+V'U$$
$$= \frac{1}{z^2}\underbrace{\left[2\eta\left(\frac{z-1}{z}\right)-1\right]}_{\operatorname{sgn}\left(\frac{z-1}{z}\right)} + \frac{z-1}{z}\left[\frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)\right]$$

$$= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\eta'\left(1-\frac{1}{z}\right)$$

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\,\mathrm{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(1-\frac{1}{z}\right) \tag{5.94}$$

Since $\eta'(x) = \delta(x)$

But

$$\delta(g(z)) = \sum_{z_i} \frac{1}{|g'(z_i)|}\delta(z-z_i)$$

Let

$$g(z) = 1-\frac{1}{z}$$
$$g'(z) = \frac{1}{z^2}$$

Solving for roots of $g(z)$, we have

$$g(z) = 0$$
$$\Rightarrow 1-\frac{1}{z} = 0$$
$$1 = \frac{1}{z} \Rightarrow z = 1$$
$$g'(z) = \frac{1}{1^2}$$
$$\therefore \delta\left(1-\frac{1}{z}\right) = \delta(z-1)$$

Hence equation (5.94) becomes

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\,\mathrm{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta(z-1) \tag{5.95}$$

Using equations (5.92) and (5.95) in equation (5.91) we have

$$g'(z) = -\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right)$$

$$= -\left\{\frac{z}{z-1}\operatorname{sgn}\left(\frac{z}{z-1}\right)\left[\frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right)+2\left(\frac{z-1}{z^3}\right)\delta(z-1)\right]\right\}$$

$$= -\left[\frac{1}{z^2}\left(\frac{z}{z-1}\right)\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)+2\left(\frac{z}{z-1}\right)\left(\frac{z-1}{z^3}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta(z-1)\right]$$

$$g'(z) = -\left[\frac{1}{z(z-1)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)+\frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta(z-1)\right] \tag{5.96}$$

Recalling that we obtained the roots of $g(z)$ as

$$z_1 = \frac{1}{1+e^{\tau}}, z_2 = \frac{1}{1-e^{\tau}}$$

And the Dirac delta function is zero everywhere except when its argument is zero.

$$\arg\delta(z-1) = z-1$$

For both roots, the delta term in $g'(z)$ vanishes because for it to exist it requires that argument of delta function must be zero. That is

$$0 = z_i - 1 = \frac{1}{1\pm e^{\tau}} - 1$$

which is equivalent to $e^{\tau} = 0$ which can't be true since $\tau > 0$ so only one term remains after neglecting delta in equation (5.96) . Therefore

$$g'(z_1) = -\left[\frac{1}{z(z-1)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)+\frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta(z-1)\right]$$

But

$$\operatorname{sgn}(x)\operatorname{sgn}(y) = \operatorname{sgn}(xy)$$

$$\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) = \operatorname{sgn}\left[\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right)\right]$$
$$= \operatorname{sgn}(1)$$
$$= 1$$

This implies that

$$g'(z_1) = -\left[\frac{1}{z(z-1)}\right]$$

$$= -\left[\frac{1}{\frac{1}{1+e^{\tau}}\left(\frac{1}{1+e^{\tau}}-1\right)}\right]$$

$$= -\left[\frac{1+e^{\tau}}{\frac{1}{1+e^{\tau}}-1}\right]$$

$$= -\left[\frac{1+e^{\tau}}{\frac{1-(1+e^{\tau})}{1+e^{\tau}}}\right]$$

$$= -\left[\frac{(1+e^{\tau})^2}{1-(1+e^{\tau})}\right]$$

$$= -\left[\frac{(1+e^{\tau})^2}{-e^{\tau}}\right]$$

$$= \frac{(1+e^{\tau})^2}{e^{\tau}}$$

$$g'(z_2) = -\left[\frac{1}{\frac{1}{1-e^{\tau}}\left(\frac{1}{1-e^{\tau}}-1\right)}\right]$$

$$= -\left[\frac{1-e^{\tau}}{\frac{1}{1-e^{\tau}}-1}\right]$$

$$= -\left[\frac{1-e^{\tau}}{\frac{1-(1-e^{\tau})}{1-e^{\tau}}}\right]$$

$$= -\left[\frac{(1-e^{\tau})^2}{1-(1-e^{\tau})}\right]$$

$$= -\left[\frac{(1-e^{\tau})^2}{e^{\tau}}\right]$$

$$= -\frac{(1-e^{\tau})^2}{e^{\tau}}$$

Therefore

$$g'(z_i) = \pm (1 \pm e^{\tau})^2 \frac{1}{e^{\tau}}$$

But $g'(z_i)$ must not be zero, and for the initial condition that is at $t = 0$ the negative root makes $g'(z_i)$ to be singular $[g'(z_i) = 0]$ so that root needs to be discarded in the summation only one root of z_i is left and will be from now on $z_i = z_1$

It is now time to go back to equation (5.90)

$$\delta(g(z)) = \sum_{z_i} \frac{1}{|g'(z_i)|}\delta(z - z_i)$$

$$= \frac{1}{|g'(z_1)|}\delta(z - z_1)$$

$$= \frac{1}{\left|(1+e^{\tau})^2 \frac{1}{e^{\tau}}\right|} \delta\left(z - \frac{1}{1+e^{\tau}}\right)$$

$$= \frac{e^{\tau}}{(1+e^{\tau})^2} \delta\left[z - (1+e^{\tau})^{-1}\right]$$

And this is the formula we searched all this time. We can now use the delta function to eliminate the integral in equation (5.89) . In doing so we need to recall that

$$\int f(x)\delta(x-a)dx = f(a)$$

That is the integral of a function multiplied by a Dirac delta shifted by a units is the value of that function at a , With this we now have

$$\int \frac{1}{(z-1)} |z|^{\frac{1}{a}} \delta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) dz = \int \frac{1}{(z-1)} |z|^{\frac{1}{a}} \frac{e^{\tau}}{(1+e^{\tau})^2} \delta\left[z - (1+e^{\tau})^{-1}\right] dz$$

Integral disappears and instead we get value of integrand for $z = \frac{1}{1+e^{\tau}}$

$$= \frac{1}{(z-1)} |z|^{\frac{1}{a}} \frac{e^{\tau}}{(1+e^{\tau})^2} \Big|_{z=\frac{1}{1+e^{\tau}}}$$

$$= \frac{1}{\left(\frac{1}{1+e^{\tau}} - 1\right)} \frac{e^{\tau}}{(1+e^{\tau})^2} \left(\frac{1}{1+e^{\tau}}\right)^{\frac{1}{a}}$$

$$= \frac{1}{\left[\frac{1-(1+e^{\tau})}{1+e^{\tau}}\right]} \frac{e^{\tau}}{(1+e^{\tau})^2} \left(\frac{1}{1+e^{\tau}}\right)^{\frac{1}{a}}$$

$$= \left[\frac{e^{\tau}}{(1+e^{\tau})^2}\right] \left[\frac{(1+e^{\tau})}{-e^{\tau}}\right] \left(\frac{1}{1+e^{\tau}}\right)^{\frac{1}{a}}$$

$$= -\left(\frac{1}{1+e^{\tau}}\right)^{1+\frac{1}{a}}$$

$$= (1+e^{\tau})^{-1-\frac{1}{a}}$$

$$\therefore -\int \frac{1}{(z-1)} |z|^{-\frac{1}{a}} \delta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) dz = (1+e^{\tau})^{-1-\frac{1}{a}}$$

But this is not the end of our trouble, with the integral solved, our problem as in equation (5.89) is reduced to

$$G\left(z, \tau - \ln\left|\frac{z-1}{z}\right|\right) |z|^{\frac{1}{a}} \eta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) = f(\tau) + (1+e^{\tau})^{-1-\frac{1}{a}}$$

Multiplying both sides of the equation by $|z|^{-\frac{1}{a}}$ we have

$$G\left(z, \tau - \ln\left|\frac{z-1}{z}\right|\right) \eta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) = f(\tau)\,|z|^{-\frac{1}{a}} + |z|^{-\frac{1}{a}} (1+e^{\tau})^{-1-\frac{1}{a}}$$

There is no harm if we make another substitution. Thus the final step towards the solution is to change the variable from τ to $T = \tau - \ln\left|\frac{z-1}{z}\right|$.

$$\tau = T + \ln\left|\frac{z-1}{z}\right|$$

Exponentiating both sides we get

$$e^{\tau} = e^{T + \ln\left|\frac{z-1}{z}\right|}$$

$$= e^{T} e^{\ln\left|\frac{z-1}{z}\right|}$$

$$= e^{T}\left|\frac{z-1}{z}\right|$$

Remembering that for our root

$$\begin{aligned}
z &= \frac{1}{1+e^{\tau}} \\
\Rightarrow \frac{z-1}{z} &= 1 - \frac{1}{z} \\
&= 1 - \frac{1}{\left[\frac{1}{1+e^{\tau}}\right]} \\
&= 1 - \left[1+e^{\tau}\right] \\
&= -e^{\tau} \\
\therefore \left|-e^{\tau}\right| &= -\left(-e^{\tau}\right)
\end{aligned}$$

$$\Rightarrow \left|\frac{z-1}{z}\right| = -\frac{z-1}{z}$$

$$\begin{aligned}
\therefore e^{\tau} &= e^{T}\left|\frac{z-1}{z}\right| \\
&= e^{T}\left[-\frac{z-1}{z}\right] \\
\Rightarrow 1+e^{\tau} &= 1 - e^{T}\left[\frac{z-1}{z}\right]
\end{aligned}$$

Multiplying the right hand side by $1 = \frac{e^{-T}}{e^{-T}}$ yields

$$\Rightarrow 1+e^{\tau} = 1 - e^{T}\left[\frac{z-1}{z}\right]\left(\frac{e^{-T}}{e^{-T}}\right)$$

$$= 1 - \left[\frac{z-1}{ze^{-T}}\right]$$

$$= \frac{ze^{-T} - (z-1)}{ze^{-T}}$$

$$= \frac{ze^{-T} - z + 1}{ze^{-T}}$$

$$= \frac{1 - z\left[1 - e^{-T}\right]}{ze^{-T}}$$

$$\therefore 1 + e^{\tau} = \frac{1 - z\left[1 - e^{-T}\right]}{ze^{-T}}$$

Described by T variable, equation for G becomes

$$G(z, T)\,\eta(T) = f\left(T + \ln\left|\frac{z-1}{z}\right|\right)|z|^{-\frac{1}{a}} + |z|^{-\frac{1}{a}}\left(1 + e^{\tau}\right)^{-1-\frac{1}{a}}$$

$$= |z|^{-\frac{1}{a}} f\left(T + \ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\left[\frac{1 - z\left(1 - e^{-T}\right)}{ze^{-T}}\right]^{-1-\frac{1}{a}}$$

But we know that f does not depend on z

$$G(z, T)\,\eta(T) = |z|^{-\frac{1}{a}} f\left(T + \ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\left[\frac{1 - z\left(1 - e^{-T}\right)}{ze^{-T}}\right]^{-1-\frac{1}{a}}$$

When $T = 0$ it follows that

$$G(z, 0)\,\eta(0) = |z|^{-\frac{1}{a}} f\left(0 + \ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\left[\frac{1 - z\left(1 - e^{-(0)}\right)}{ze^{-(0)}}\right]^{-1-\frac{1}{a}}$$

But $\eta(0) = 1$ hence

$$G(z, 0) = |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\left[\frac{1 - z(1-1)}{z}\right]^{-1-\frac{1}{a}}$$

$$= |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\left[\frac{1}{z}\right]^{-1-\frac{1}{a}}$$

$$= |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}} z^{1+\frac{1}{a}}$$

But knowing that

$$z = G(z, 0) = |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + z$$

It follows that

$$|z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) = 0$$

So we have finally eliminated f. Thus the solution is

$$G(z,T)\,\eta(T) = z^{-\frac{1}{a}}\left[\frac{1-z\left(1-e^{-T}\right)}{ze^{-T}}\right]^{-1-\frac{1}{a}}$$

Since $T \geqslant 0$ it follows by definition of tau that $\eta(T) = 1$ hence

$$G(z,T)\,\eta(T) = z^{-\frac{1}{a}}\left[\frac{1-z\left(1-e^{-T}\right)}{ze^{-T}}\right]^{-1-\frac{1}{a}}$$

$$G(z,T) = z^{-\frac{1}{a}}\left[\frac{1-z\left(1-e^{-T}\right)}{ze^{-T}}\right]^{-1-\frac{1}{a}}$$

$$= z^{-\frac{1}{a}}\left[\frac{ze^{-T}}{1-z\left(1-e^{-T}\right)}\right]^{1+\frac{1}{a}}$$

$$= z^{-\frac{1}{a}+1+\frac{1}{a}}\left[\frac{e^{-T}}{1-z\left(1-e^{-T}\right)}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{e^{-T}}{1-z\left(1-e^{-T}\right)}\right]^{1+\frac{1}{a}}$$

We can rewrite it as

$$G(z,\tau) = z\left[\frac{e^{-\tau}}{1-z\left(1-e^{-\tau}\right)}\right]^{1+\frac{1}{a}}$$

We are now close to the final solution, the remaining task is to rewrite $G(z,\tau)$ in terms of the original variable t. Its now time to go back to our initial substitution

$$\tau = \ln\left|\frac{t+\alpha}{\alpha}\right|$$

Where

$$\alpha = \frac{1}{\lambda a}$$

Thus

$$\begin{aligned} -\tau &= -\ln\left|\frac{t+\alpha}{\alpha}\right| \\ &= \ln\left|\frac{t+\alpha}{\alpha}\right|^{-1} \end{aligned}$$

$$\begin{aligned} \Rightarrow e^{-\tau} &= e^{\ln\left|\frac{t+\alpha}{\alpha}\right|^{-1}} \\ &= \left|\frac{t+\alpha}{\alpha}\right|^{-1} \\ &= \left|\frac{t}{\alpha}+1\right|^{-1} \\ &= \left|\frac{t}{1/\lambda a}+1\right|^{-1} \\ &= (\lambda at+1)^{-1} \end{aligned}$$

Therefore

$$G(z,t) = z\left[\frac{(1+\lambda at)^{-1}}{1-z\left[1-(1+\lambda at)^{-1}\right]}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{1}{(1+\lambda at)\left(1-z\left[1-\frac{1}{1+\lambda at}\right]\right)}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{1}{(1+\lambda at)\left(1-z\left[\frac{1+\lambda at-1}{1+\lambda at}\right]\right)}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{1}{(1+\lambda at)\left(1-\frac{z\lambda at}{1+\lambda at}\right)}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{1}{1+\lambda at-\lambda atz}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{\frac{1}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{1+\frac{1}{a}}$$

We finally have

$$G\left(z,t\right) = z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{1+\frac{1}{a}}$$

Letting $p = \frac{1}{1+\lambda at}$ and $q = 1-\frac{1}{1+\lambda at} = \frac{\lambda at}{1+\lambda at}$

$G(z,t)$ can be expressed as

$$G\left(z,t\right) = z^{-\frac{1}{a}}\left[\frac{zp}{1-qz}\right]^{1+\frac{1}{a}}$$

$P_n\left(t\right)$ is the coefficient of z^n in the expansion of $G(z,t)$. By binomial expansion

$$G\left(z,t\right) = z^{-\frac{1}{a}}z^{1+\frac{1}{a}}p^{1+\frac{1}{a}}\left[1-qz\right]^{-\left(1+\frac{1}{a}\right)}$$

$$= zp^{1+\frac{1}{a}}\sum_{k=0}^{\infty}\binom{-\left[1+\frac{1}{a}\right]}{k}\left(-qz\right)^k$$

$$= zp^{1+\frac{1}{a}}\sum_{k=0}^{\infty}\binom{-\left[1+\frac{1}{a}\right]}{k}\left(-1\right)^k q^k z^k$$

Recall that

$$\binom{-r}{k}(-1)^k = \binom{r+k-1}{k}$$

$$\Rightarrow \binom{-\left[1+\frac{1}{a}\right]}{k}(-1)^k = \binom{\left[1+\frac{1}{a}\right]+k-1}{k}$$

Thus

$$G(z,t) = zp^{1+\frac{1}{a}} \sum_{k=0}^{\infty} \binom{\left[1+\frac{1}{a}\right]+k-1}{k} q^k z^k$$

$$= \sum_{k=0}^{\infty} \binom{\left[1+\frac{1}{a}\right]+k-1}{k} p^{1+\frac{1}{a}} q^k z^{1+k}$$

From this it follows that

$$P_{1+k}(t) = \binom{\left[n_0+\frac{1}{a}\right]+k-1}{k} p^{1+\frac{1}{a}} q^k \quad k = 0,\ 1,\ 2$$

but $n = 1+k$, Thus

$$P_n(t) = \binom{\left[1+\frac{1}{a}\right]+k-1}{k} p^{1+\frac{1}{a}} q^k \quad k = 0,\ 1,\ 2$$

$$= \binom{\left[1+\frac{1}{a}\right]+k-1}{k} \left(\frac{1}{1+\lambda at}\right)^{1+\frac{1}{a}} \left(1-\frac{1}{1+\lambda at}\right)^k \quad k = 0,\ 1,\ 2$$

which is the pmf of a negative binomial distribution with parameters $r = 1+\frac{1}{a}$ and $p = \frac{1}{1+\lambda at}$

Case 2: When initial population $X(0) = n_0$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$

$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$

$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = n_0$, we have

$$P_{n_0}(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq n_0$$

$$\therefore G(z, 0) = z^{n_0}$$

With this equation (5.87) becomes

$$\overline{G}(z, s)\, |z|^{\frac{1}{a}}\, e^{-s \ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{z^{n_0}}{z\,(z-1)}\, |z|^{\frac{1}{a}}\, e^{-s \ln\left|\frac{z-1}{z}\right|} dz$$

$$\overline{G}(z, s)\, |z|^{\frac{1}{a}}\, e^{-s \ln\left|\frac{z-1}{z}\right|} = f(s) - \int \frac{z^{n_0 - 1}}{(z-1)}\, |z|^{\frac{1}{a}}\, e^{-s \ln\left|\frac{z-1}{z}\right|} dz \tag{5.97}$$

Now this looks like a complicated equation to solve, but at this juncture we can licitly apply the inverse Laplace transform to both sides.

We observe that $f(s)$ can be regarded as a Laplace transform of some unknown function$f(t)$

Applying inverse Laplace transform to both sides of equation (5.97) we have from the table of transform pairs in chapter 2 (for details see pages 38-39)

1. e^{-cs} is the Laplace transform of the Dirac delta function $\delta(t - c)$

2. $\overline{G}(z, s)e^{cs}$ is the Laplace transform of $G(t - c)\eta(t - c)$ where η is the Heaviside step function. In our case $c = -\ln\left|\frac{z-1}{z}\right|$

With this equation 4 can be rewritten as

$$\Rightarrow \overline{G}(z, s)\, |z|^{\frac{1}{a}}\, e^{-cs} = f(s) - \int \frac{z^{n_0}}{z\,(z-1)}\, |z|^{\frac{1}{a}}\, e^{-cs} dz$$

So all together inversely transforming both sides the above equation, we come to

$$G(z, \tau - c)\, |z|^{\frac{1}{a}}\, \eta(t - c) = f(\tau) - \int \frac{z^{n_0}}{z\,(z-1)}\, |z|^{\frac{1}{a}} \delta(\tau - c) dz$$

Substituting the value of a we obtain

$$G\left(z, t-\ln\left|\frac{z-1}{z}\right|\right)\eta\left(t-\ln\left|\frac{z-1}{z}\right|\right)=f(t)-\int\frac{z^{n_0}}{\lambda z\left(z-1\right)}\delta\left(t-\ln\left|\frac{z-1}{z}\right|\right)dz \tag{5.98}$$

Now this is a pretty large equation but it can be simplified, but to do that we need to make one big detour. We can free ourselves of this terrible integral by using the Dirac delta function, but what we have is delta of function of variable z , so we need to first simplify it to a common delta of variable. To do so, we shall use the following property of delta function

$$\delta\left(g\left(z\right)\right)=\sum_{z_i}\frac{1}{\left|g'\left(z_i\right)\right|}\delta\left(z-z_i\right) \tag{5.99}$$

Where $g'\left(z\right)$ is the first derivative of $g\left(z\right)$, z_i is a simple root of $g\left(z\right)$ such that $g'\left(z_i\right)$. In our case $g\left(z\right)=\tau-\ln\left|\frac{z-1}{z}\right|$. To obtain the roots of $g(z)$, we solve $g\left(z\right)=0$

$$\Rightarrow\tau-\ln\left|\frac{z-1}{z}\right|=0$$

$$\tau=\ln\left|\frac{z-1}{z}\right|$$

$$\tau=\ln\left|\frac{z-1}{z}\right|$$

$$e^{\tau}=\left|\frac{z-1}{z}\right|$$

$$e^{\tau}=\pm\frac{z-1}{z}$$

First root

$$
\begin{aligned}
e^{\tau} &= \frac{z-1}{z} \\
ze^{\tau} &= z-1 \\
1 &= z - ze^{\tau} \\
1 &= z\left(1-e^{\tau}\right) \\
z_1 &= \frac{1}{1-e^{\tau}}
\end{aligned}
$$

Second root

$$
\begin{aligned}
e^{\tau} &= -\frac{z-1}{z} \\
ze^{\tau} &= -\left(z-1\right) \\
1 &= z + ze^{\tau} \\
1 &= z\left(1+e^{\tau}\right) \\
z_2 &= \frac{1}{1+e^{\tau}}
\end{aligned}
$$

Therefore

$$z_i = \frac{1}{1 \pm e^{\tau}}$$

The next step is to determine

$$
\begin{aligned}
g'\left(z\right) &= \frac{d}{dz}\left[\tau - \ln\left|\frac{z-1}{z}\right|\right] \\
&= -\frac{d}{dz}\ln\left|\frac{z-1}{z}\right|
\end{aligned}
$$

Using the property $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

$$g'(z) = -\left(\frac{\frac{d}{dz}\left|\frac{z-1}{z}\right|}{\left|\frac{z-1}{z}\right|}\right)$$

$$g'(z) = -\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right) \tag{5.100}$$

Now remembering that $|x| = x\,\text{sgn}\,(x)$ and $\text{sgn}\,(x) = 2\eta(x) - 1$, therefore

$$\left|\frac{z}{z-1}\right| = \frac{z}{z-1}\,\text{sgn}\left(\frac{z}{z-1}\right) \tag{5.101}$$

Also

$$\begin{aligned}\left|\frac{z-1}{z}\right| &= \frac{z-1}{z}\,\text{sgn}\left(\frac{z-1}{z}\right)\\ &= \frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\end{aligned}$$

Where sgn is the sign distribution and eta is the Heaviside distribution .Hence

$$\left|\frac{z-1}{z}\right|' = \frac{d}{dz}\left|\frac{z-1}{z}\right|$$

$$\left|\frac{z-1}{z}\right|' = \frac{d}{dz}\left\{\frac{z-1}{z}\left[2\eta\left(\frac{z-1}{z}\right) - 1\right]\right\} \tag{5.102}$$

At this step we need to recall that

$$\eta'_{h(z)} = h'(z)\frac{\partial\eta(z)}{\partial h}$$

Use product rule of differentiation, we simplify equation (5.102) as follows Let

$$\begin{aligned}U &= \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow U' = \frac{1}{z^2}\\ V &= 2\eta\left(\frac{z-1}{z}\right) - 1\\ &= 2\eta\left(1 - \frac{1}{z}\right) - 1 \Rightarrow V' = \frac{2}{z^2}\eta'\left(1 - \frac{1}{z}\right)\end{aligned}$$

We now have

$$\left|\frac{z-1}{z}\right|' = U'V + V'U$$

$$= \frac{1}{z^2}\underbrace{\left[2\eta\left(\frac{z-1}{z}\right)-1\right]}_{\operatorname{sgn}\left(\frac{z-1}{z}\right)} + \frac{z-1}{z}\left[\frac{2}{z^2}\eta'\left(1-\frac{1}{z}\right)\right]$$

$$= \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\eta'\left(1-\frac{1}{z}\right)$$

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(1-\frac{1}{z}\right) \tag{5.103}$$

Since $\eta'(x) = \delta(x)$

But

$$\delta(g(z)) = \sum_{z_i}\frac{1}{|g'(z_i)|}\delta(z-z_i)$$

Let

$$g(z) = 1 - \frac{1}{z}$$
$$g'(z) = \frac{1}{z^2}$$

Solving for roots of $g(z)$, we have

$$g(z) = 0$$
$$\Rightarrow 1 - \frac{1}{z} = 0$$
$$1 = \frac{1}{z} \Rightarrow z = 1$$
$$g'(z) = \frac{1}{1^2}$$
$$\therefore \delta\left(1-\frac{1}{z}\right) = \delta(z-1)$$

Hence equation (5.103) becomes

$$\left|\frac{z-1}{z}\right|' = \frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(z-1\right) \tag{5.104}$$

We now use equations (5.101) and (5.104) in equation (5.100). This yields

$$g'(z) = -\left(\left|\frac{z}{z-1}\right|\left|\frac{z-1}{z}\right|'\right)$$

$$= -\left\{\frac{z}{z-1}\operatorname{sgn}\left(\frac{z}{z-1}\right)\left[\frac{1}{z^2}\operatorname{sgn}\left(\frac{z-1}{z}\right) + 2\left(\frac{z-1}{z^3}\right)\delta\left(z-1\right)\right]\right\}$$

$$= -\left[\frac{1}{z^2}\left(\frac{z}{z-1}\right)\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + 2\left(\frac{z}{z-1}\right)\left(\frac{z-1}{z^3}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

$$= -\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

Recalling that we obtained the roots of $g(z)$ as

$$z_1 = \frac{1}{1+e^\tau},\ z_2 = \frac{1}{1-e^\tau}$$

And the Dirac delta function is zero everywhere except when its argument is zero.

$$\arg\delta\left(z-1\right) = z-1$$

For both roots, the delta term in $g'(z)$ vanish because for it to exist it requires that argument of delta function must be zero. That is

$$0 = z_i - 1 = \frac{1}{1\pm e^\tau} - 1$$

which is equivalent to $e^\tau = 0$ and that is never true in Mathematics so only one term remains after neglecting delta in Therefore

$$g'(z_1) = -\left[\frac{1}{z\left(z-1\right)}\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) + \frac{2}{z^2}\operatorname{sgn}\left(\frac{z}{z-1}\right)\delta\left(z-1\right)\right]$$

But

$$\operatorname{sgn}(x)\operatorname{sgn}(y) = \operatorname{sgn}(xy)$$

$$\begin{aligned}
\operatorname{sgn}\left(\frac{z-1}{z}\right)\operatorname{sgn}\left(\frac{z}{z-1}\right) &= \operatorname{sgn}\left[\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right)\right] \\
&= \operatorname{sgn}(1) \\
&= 1
\end{aligned}$$

This implies that

$$\begin{aligned}
g'(z_1) &= -\left[\frac{1}{z(z-1)}\right] \\
&= -\left[\frac{1}{\frac{1}{1+e^{\tau}}\left(\frac{1}{1+e^{\tau}}-1\right)}\right] \\
&= -\left[\frac{1+e^{\tau}}{\frac{1}{1+e^{\tau}}-1}\right] \\
&= -\left[\frac{1+e^{\tau}}{\frac{1-(1+e^{\tau})}{1+e^{\tau}}}\right] \\
&= -\left[\frac{(1+e^{\tau})^2}{1-(1+e^{\tau})}\right] \\
&= -\left[\frac{(1+e^{\tau})^2}{-e^{\tau}}\right] \\
&= \frac{(1+e^{\tau})^2}{e^{\tau}}
\end{aligned}$$

$$g'(z_2) = -\left[\frac{1}{\frac{1}{1-e^{\tau}}\left(\frac{1}{1-e^{\tau}} - 1\right)}\right]$$

$$= -\left[\frac{1-e^{\tau}}{\frac{1}{1-e^{\tau}} - 1}\right]$$

$$= -\left[\frac{1-e^{\tau}}{\frac{1-(1-e^{\tau})}{1-e^{\tau}}}\right]$$

$$= -\left[\frac{(1-e^{\tau})^2}{1-(1-e^{\tau})}\right]$$

$$= -\left[\frac{(1-e^{\tau})^2}{e^{\tau}}\right]$$

$$= -\frac{(1-e^{\tau})^2}{e^{\tau}}$$

Therefore

$$g'(z_i) = \pm(1 \pm e^{\tau})^2 \frac{1}{e^{\tau}}$$

But $g'(z_i)$ must not be zero, and for the initial condition that is at $t = 0$ the negative root makes $g'(z_i)$ to be singular $[g'(z_i) = 0]$ so that root needs to be discarded in the summation only one root of z_i is left and will be from now on $z_i = z_1$

It is now time to go back to equation (5.99)

$$\delta(g(z)) = \sum_{z_i} \frac{1}{|g'(z_i)|}\delta(z - z_i)$$

$$= \frac{1}{|g'(z_1)|}\delta(z - z_1)$$

$$= \frac{1}{\left|(1+e^{\tau})^2 \frac{1}{e^{\tau}}\right|}\delta\left(z - \frac{1}{1+e^{\tau}}\right)$$

$$= \frac{e^{\tau}}{(1+e^{\tau})^2}\delta\left[z - (1+e^{\tau})^{-1}\right]$$

And this is the formula we searched all this time. We can now use the delta function to eliminate the integral in equation (5.98). In doing so we need to recall that

$$\int f(x)\delta\,(x-a)dx = f(a)$$

That is the integral of a function multiplied by a Dirac delta shifted by a units is the value of that function at a . With this we now have

$$\int \frac{z^{n_0-1}}{(z-1)}\,|z|^{\frac{1}{a}}\,\delta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right)dz = \int \frac{z^{n_0}}{z\,(z-1)}\,|z|^{\frac{1}{a}}\,\frac{e^{\tau}}{(1+e^{\tau})^2}\delta\left[z-(1+e^{\tau})^{-1}\right]dz$$

Integral disappears and instead we get value of integrand for $z = \frac{1}{1+e^{\tau}}$

$$= \frac{z^{n_0}}{z\,(z-1)}\,|z|^{\frac{1}{a}}\,\frac{e^{\tau}}{(1+e^{\tau})^2}\Big|_{z=\frac{1}{1+e^{\tau}}}$$

$$= \frac{\left(\frac{1}{1+e^{\tau}}\right)^{n_0}}{\left(\frac{1}{1+e^{\tau}}\right)\left(\frac{1}{1+e^{\tau}}-1\right)}\frac{e^{\tau}}{(1+e^{\tau})^2}\left(\frac{1}{1+e^{\tau}}\right)^{\frac{1}{a}}$$

$$= \frac{\left(\frac{1}{1+e^{\tau}}\right)^{n_0}}{\left(\frac{1}{1+e^{\tau}}\right)\left[\frac{1-(1+e^{\tau})}{1+e^{\tau}}\right]}\frac{e^{\tau}}{(1+e^{\tau})^2}\left(\frac{1}{1+e^{\tau}}\right)^{\frac{1}{a}}$$

$$= \left(\frac{1}{1+e^{\tau}}\right)^{n_0}\left[\frac{e^{\tau}}{(1+e^{\tau})^2}\right]\left[\frac{(1+e^{\tau})^2}{e^{\tau}}\right]\left(\frac{1}{1+e^{\tau}}\right)^{\frac{1}{a}}$$

$$= \left(\frac{1}{1+e^{\tau}}\right)^{n_0+\frac{1}{a}}$$

$$= (1+e^{\tau})^{-n_0-\frac{1}{a}}$$

$$\therefore -\int \frac{z^{n_0-1}}{(z-1)} |z|^{-\frac{1}{a}} \delta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) dz = (1+e^{\tau})^{-n_0-\frac{1}{a}}$$

But this is not the end of our trouble, with the integral solved, our problem as in equation (5.98) is reduced to

$$G\left(z, \tau - \ln\left|\frac{z-1}{z}\right|\right) |z|^{\frac{1}{a}} \eta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) = f(\tau) + (1+e^{\tau})^{-n_0-\frac{1}{a}}$$

Multiplying both sides of the equation by $|z|^{-\frac{1}{a}}$ we have

$$G\left(z, \tau - \ln\left|\frac{z-1}{z}\right|\right) \eta\left(\tau - \ln\left|\frac{z-1}{z}\right|\right) = f(\tau)\,|z|^{-\frac{1}{a}} + |z|^{-\frac{1}{a}} (1+e^{\tau})^{-n_0-\frac{1}{a}}$$

There is no harm if we make another substitution. Thus the final step towards the solution is to change the variable from τ to $T = \tau - \ln\left|\frac{z-1}{z}\right|$.

$$\tau = T + \ln\left|\frac{z-1}{z}\right|$$

Exponentiating this we get

$$\begin{aligned} e^{\tau} &= e^{T+\ln\left|\frac{z-1}{z}\right|} \\ &= e^{T} e^{\ln\left|\frac{z-1}{z}\right|} \\ &= e^{T}\left|\frac{z-1}{z}\right| \end{aligned}$$

Remembering that for our root

$$\begin{aligned} z &= \frac{1}{1+e^{\tau}} \\ \Rightarrow \frac{z-1}{z} &= 1 - \frac{1}{z} \\ &= 1 - \frac{1}{\left[\frac{1}{1+e^{\tau}}\right]} \\ &= 1 - [1+e^{\tau}] \\ &= -e^{\tau} \\ \therefore |-e^{\tau}| &= -(-e^{\tau}) \\ \Rightarrow \left|\frac{z-1}{z}\right| &= -\frac{z-1}{z} \end{aligned}$$

$$\therefore e^{\tau} = e^{T}\left|\frac{z-1}{z}\right|$$
$$= e^{T}\left[-\frac{z-1}{z}\right]$$
$$\Rightarrow 1+e^{\tau} = 1-e^{T}\left[\frac{z-1}{z}\right]$$

Multiplying the right hand side by $1 = \frac{e^{-T}}{e^{-T}}$ yields

$$\Rightarrow 1+e^{\tau} = 1-e^{T}\left[\frac{z-1}{z}\right]\left(\frac{e^{-T}}{e^{-T}}\right)$$

$$= 1-\left[\frac{z-1}{ze^{-T}}\right]$$

$$= \frac{ze^{-T}-(z-1)}{ze^{-T}}$$

$$= \frac{ze^{-T}-z+1}{ze^{-T}}$$

$$= \frac{1-z\left[1-e^{-T}\right]}{ze^{-T}}$$

$$\therefore 1+e^{\tau} = \frac{1-z\left[1-e^{-T}\right]}{ze^{-T}}$$

Described by T variable, equation for G becomes

$$G\left(z,T\right)\eta\left(T\right) = f\left(T+\ln\left|\frac{z-1}{z}\right|\right)|z|^{-\frac{1}{a}} + |z|^{-\frac{1}{a}}\left(1+e^{\tau}\right)^{-n_0-\frac{1}{a}}$$

$$= |z|^{-\frac{1}{a}} f\left(T+\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\left[\frac{1-z\left(1-e^{-T}\right)}{ze^{-T}}\right]^{-n_0-\frac{1}{a}}$$

But we know that f does not depend on z , we know that

$$G(z,T)\,\eta(T) = |z|^{-\frac{1}{a}} f\left(T + \ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}} \left[\frac{1 - z\left(1 - e^{-T}\right)}{ze^{-T}}\right]^{-n_0 - \frac{1}{a}}$$

When $T = 0$ it follows that

$$G(z,0)\,\eta(0) = |z|^{-\frac{1}{a}} f\left(0 + \ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}} \left[\frac{1 - z\left(1 - e^{-(0)}\right)}{ze^{-(0)}}\right]^{-n_0 - \frac{1}{a}}$$

But $\eta(0) = 1$ hence

$$G(z,0) = |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}} \left[\frac{1 - z(1-1)}{z}\right]^{-n_0 - \frac{1}{a}}$$

$$= |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}} \left[\frac{1}{z}\right]^{-n_0 - \frac{1}{a}}$$

$$= |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + |z|^{-\frac{1}{a}}\, z^{n_0 + \frac{1}{a}}$$

But knowing that

$$z^{n_0} = G(z,0) = |z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) + z^{n_0}$$

It follows that

$$|z|^{-\frac{1}{a}} f\left(\ln\left|\frac{z-1}{z}\right|\right) = 0$$

So we have finally eliminated f. Thus the solution is

$$G(z,T)\,\eta(T) = z^{-\frac{1}{a}} \left[\frac{1 - z\left(1 - e^{-T}\right)}{ze^{-T}}\right]^{-n_0 - \frac{1}{a}}$$

Since $T \geqslant 0$ it follows from definition of eta that $\eta(T) = 1$ hence

$$G(z,T)\,\eta(T) = z^{-\frac{1}{a}} \left[\frac{1 - z\left(1 - e^{-T}\right)}{ze^{-T}}\right]^{-n_0 - \frac{1}{a}}$$

$$G(z,T) = z^{-\frac{v}{\lambda}} \left[\frac{1 - z\left(1 - e^{-T}\right)}{ze^{-T}}\right]^{-n_0 - \frac{v}{\lambda}}$$

$$= z^{-\frac{1}{a}} \left[\frac{ze^{-T}}{1 - z\left(1 - e^{-T}\right)}\right]^{n_0 + \frac{1}{a}}$$

$$= z^{-\frac{1}{a} + n_0 + \frac{1}{a}} \left[\frac{e^{-T}}{1 - z\left(1 - e^{-T}\right)}\right]^{n_0 + \frac{1}{a}}$$

$$= z^{n_0} \left[\frac{e^{-T}}{1 - z\left(1 - e^{-T}\right)}\right]^{n_0 + \frac{1}{a}}$$

We can rewrite it as

$$G(z,\tau) = z^{n_0} \left[\frac{e^{-\tau}}{1 - z\left(1 - e^{-\tau}\right)}\right]^{n_0 + \frac{1}{a}}$$

Which is the pmf of a negative binomial distribution
We now have something to smile about! The remaining task is to rewrite $G(z,\tau)$ in terms of the original variable t. Recall the initial substitution we made

$$\tau = \ln\left|\frac{t + \alpha}{\alpha}\right|$$

Where

$$\alpha = \frac{1}{\lambda a}$$

Thus

$$-\tau = -\ln\left|\frac{t + \alpha}{\alpha}\right|$$

$$= \ln \left| \frac{t+\alpha}{\alpha} \right|^{-1}$$

$$\Rightarrow e^{-\tau} = e^{\ln\left|\frac{t+\alpha}{\alpha}\right|^{-1}}$$

$$= \left| \frac{t+\alpha}{\alpha} \right|^{-1}$$

$$= \left| \frac{t}{\alpha} + 1 \right|^{-1}$$

$$= \left| \frac{t}{1/\lambda a} + 1 \right|^{-1}$$

$$= (\lambda at + 1)^{-1}$$

Therefore

$$G(z,t) = z^{n_0} \left[\frac{(1+\lambda at)^{-1}}{1 - z\left[1 - (1+\lambda at)^{-1}\right]} \right]^{n_0 + \frac{1}{a}}$$

$$= z^{n_0} \left[\frac{1}{(1+\lambda at)\left(1 - z\left[1 - \frac{1}{1+\lambda at}\right]\right)} \right]^{n_0 + \frac{1}{a}}$$

$$= z^{n_0} \left[\frac{1}{(1+\lambda at)\left(1 - z\left[\frac{1+\lambda at - 1}{1+\lambda at}\right]\right)} \right]^{n_0 + \frac{1}{a}}$$

$$= z^{n_0} \left[\frac{1}{(1+\lambda at)\left(1 - \frac{z\lambda at}{1+\lambda at}\right)} \right]^{n_0 + \frac{1}{a}}$$

$$= z^{n_0}\left[\frac{\frac{1}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{n_0+\frac{1}{a}}$$

We finally have

$$G\left(z,t\right) = z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{n_0+\frac{1}{a}}$$

Letting $p = \frac{1}{1+\lambda at}$ and $q = 1-\frac{1}{1+\lambda at} = \frac{\lambda at}{1+\lambda at}$

$G(z,t)$ can be expressed as

$$G\left(z,t\right) = z^{-\frac{1}{a}}\left[\frac{zp}{1-qz}\right]^{n_0+\frac{1}{a}}$$

$P_n\left(t\right)$ is the coefficient of z^n in the expansion of $G(z,t)$. By binomial expansion

$$G\left(z,t\right) = z^{-\frac{1}{a}}z^{n_0+\frac{1}{a}}p^{n_0+\frac{1}{a}}\left[1-qz\right]^{-\left(n_0+\frac{1}{a}\right)}$$

$$= z^{n_0}p^{n_0+\frac{1}{a}}\sum_{k=0}^{\infty}\binom{-\left[n_0+\frac{1}{a}\right]}{k}\left(-qz\right)^k$$

$$= z^{n_0}p^{n_0+\frac{1}{a}}\sum_{k=0}^{\infty}\binom{-\left[n_0+\frac{1}{a}\right]}{k}\left(-1\right)^k q^k z^k$$

Recall that

$$\binom{-r}{k}(-1)^k = \binom{r+k-1}{k}$$

$$\Rightarrow \binom{-\left[n_0+\frac{1}{a}\right]}{k}(-1)^k = \binom{\left[n_0+\frac{1}{a}\right]+k-1}{k}$$

Thus

$$G(z,t) = z^{n_0} p^{n_0+\frac{1}{a}} \sum_{k=0}^{\infty} \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} q^k z^k$$

$$= \sum_{k=0}^{\infty} \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} p^{n_0+\frac{1}{a}} q^k z^{n_0+k}$$

From this it follows that

$$P_{n_0+k}\left(t\right) = \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} p^{n_0+\frac{1}{a}} q^k \quad k = 0,\ 1,\ 2$$

but $n = n_0 + k$, Thus

$$P_n\left(t\right) = \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} p^{n_0+\frac{1}{a}} q^k \quad k = 0,\ 1,\ 2$$

$$= \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} \left(\frac{1}{1+\lambda at}\right)^{n_0+\frac{1}{a}} \left(1 - \frac{1}{1+\lambda at}\right)^k \quad k = 0,\ 1,\ 2$$

which is the pmf of a negative binomial distribution with parameters $r = n_0 + \frac{1}{a}$ and $p = \frac{1}{1+\lambda at}$

Hyper geometric function approach

From equation (5.86) we had

$$\frac{d}{dz}\overline{G}(z,s) - \left[\frac{s}{z(z-1)} - \frac{1}{az}\right]\overline{G}(z,s) = -\frac{G(z,0)}{z(z-1)}$$

but this can be rewritten as

$$\frac{d}{dz}\overline{G}(z,s) + \left[\frac{s}{z(1-z)} + \frac{1}{az}\right]\overline{G}(z,s) = \frac{G(z,0)}{z(1-z)} \tag{5.105}$$

This is an ODE of 1^{st} order. It is of the form $y' + Py = Q$ where

$$y = \overline{G}(z,s), P = \left[\frac{s}{z(1-z)} + \frac{1}{az}\right] \text{ and } Q = \frac{G(z,0)}{z(1-z)}$$

We use the integrating factor technique. The integrating factor is given by

$$I.F = e^{\int Pdz}$$

We first calculate the P integral

$$\begin{aligned}\int Pdz &= \int\left[\frac{s}{z(1-z)} + \frac{1}{az}\right]dz \\ &= s\int\frac{1}{z(1-z)}dz + \frac{1}{a}\int\frac{1}{z}dz \\ &= s\int\frac{1}{z(1-z)}dz + \frac{1}{a}\ln z\end{aligned}$$

Using PF we have

$$\frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$

$$1 = A(1-z) + Bz$$

Which holds true for all values of z

Setting $z = 0$, we have

$$1 = A(1-0) + B(0)$$
$$\Rightarrow A = 1$$

Setting $z = 1$, we have

$$1 = A(1-1) + B(1)$$
$$\Rightarrow B = 1$$

Thus

$$\begin{aligned} \frac{1}{z(1-z)} &= \frac{1}{z} + \frac{1}{1-z} \\ \int \frac{1}{z(1-z)} &= \int \frac{1}{z}dz + \int \frac{1}{1-z}dz \\ &= \ln z - \ln(1-z) \\ &= \ln\left(\frac{z}{1-z}\right) \end{aligned}$$

$$\begin{aligned} \therefore \int P dz &= s\int \frac{1}{z(1-z)}dz + \frac{1}{a}\ln z \\ &= s\ln\left(\frac{z}{1-z}\right) + \frac{1}{a}\ln z \\ &= \ln\left(\frac{z}{1-z}\right)^s + \ln z^{\frac{1}{a}} \\ &= \ln\left[\left(\frac{z}{1-z}\right)^s z^{\frac{1}{a}}\right] \\ &= \ln\left[z^{s+\frac{1}{a}}(1-z)^{-s}\right] \end{aligned}$$

$$\begin{aligned} \therefore I.F &= e^{\int pdz} \\ &= e^{\ln\left[z^{s+\frac{1}{a}}(1-z)^{-s}\right]} \\ &= z^{s+\frac{1}{a}}(1-z)^{-s} \end{aligned}$$

Multiplying both sides of equation (5.105) by the integrating factor we get

$$z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\frac{d}{dz}\overline{G}\left(z,s\right)+z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\left[\frac{s}{z\left(1-z\right)}+\frac{1}{az}\right]\overline{G}\left(z,s\right)=z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\frac{G(z,0)}{z\left(1-z\right)}$$

$$\frac{d}{dz}\left[z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right)\right]=\frac{z^{s+\frac{1}{a}-1}G(z,0)}{\left(1-z\right)^{s+1}}$$

Integrating both sides with respect to z yields

$$\int\frac{d}{dz}\left[z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right)\right]dz=\int\frac{z^{s+\frac{1}{a}-1}G(z,0)}{\left(1-z\right)^{s+1}}dz$$

$$\int d\left[z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right)\right]=\int\frac{z^{s+\frac{1}{a}-1}G(z,0)}{\left(1-z\right)^{s+1}}dz$$

$$z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right)=\int\frac{z^{s+\frac{1}{a}-1}G(z,0)}{\left(1-z\right)^{s+1}}dz \tag{5.106}$$

We consider two cases

1. When $X(0)=1$
2. When $X(0)=n_0$

Case 1: When initial population $X(0)=1$

Recall that

$$\begin{aligned} G(z,t) &= \sum_{n=0}^{\infty}P_n(t)z^n \\ \Rightarrow G(z,0) &= \sum_{n=0}^{\infty}P_n(0)z^n \\ &= P_0(0)+P_1(0)z+P_2(0)z^2+...+P_{n_0}(0)z^{n_0}+... \end{aligned}$$

but for the initial condition $X(0) = 1$, we have

$$P_1(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq 1$$

$$\therefore G(z,0) = z$$

With this equation (5.106) becomes

$$z^{s+\frac{1}{a}} (1-z)^{-s} \overline{G}(z,s) = \int \frac{z^{s+\frac{1}{a}-1} z}{(1-z)^{s+1}} dz$$

$$z^{s+\frac{1}{a}} (1-z)^{-s} \overline{G}(z,s) = \int \frac{z^{s+\frac{1}{a}}}{(1-z)^{s+1}} dz \tag{5.107}$$

We now simplify the RHS of equation (5.107) Recall that

$$\int \frac{x^{a+b}}{(1-x)^{a+1}} dx = \frac{x^{a+b+1} \;\; {}_2F_1(a+1, a+b+1; a+b+2; x)}{a+b+1} + \text{constant}$$

Where ${}_2F_1(a+1, a+b+1; a+b+2; x)$ is the gauss Hyper geometric function. Thus

$$\int \frac{z^{s+\frac{1}{a}}}{(1-z)^{s+1}} dz = \frac{z^{s+\frac{1}{a}+1} \;\; {}_2F_1\left(s+1, s+\frac{1}{a}+1; s+\frac{1}{a}+2; z\right)}{s+\frac{1}{a}+1} + k_1 \tag{5.108}$$

Where k_1 is a constant of integration

But

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} \;\; {}_2F_1(c-a, c-b; c; z)$$

We now use this property to simplify ${}_2F_1\left(s+1, s+\frac{1}{a}+1; s+\frac{1}{a}+2; z\right)$ as follows.

Here

$$a = s+1, \quad b = s+\frac{1}{a}+1, \quad c = s+\frac{1}{a}+2$$

$$\begin{aligned} c-a &= s+\frac{1}{a}+2-(s+1) \\ &= s+\frac{1}{a}+2-s-1 \\ &= \frac{1}{a}+1 \end{aligned}$$

$$\begin{aligned} c-b &= s+\frac{1}{a}+2-\left(s+\frac{1}{a}+1\right) \\ &= s+\frac{1}{a}+2-s-\frac{1}{a}-1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} c-a-b &= s+\frac{1}{a}+2-(s+1)-\left(s+\frac{1}{a}+1\right) \\ &= s+\frac{1}{a}+2-s-1-s-\frac{1}{a}-1 \\ &= -s \end{aligned}$$

Thus

$${}_2F_1\left(s+1, s+\frac{1}{a}+1; s+\frac{1}{a}+2; z\right) = (1-z)^{-s} \quad {}_2F_1\left(\frac{1}{a}+1, 1; s+\frac{1}{a}+2; z\right)$$

With this equation (5.108) becomes

$$\int \frac{z^{s+\frac{1}{a}}}{(1-z)^{s+1}} dz = \frac{z^{s+\frac{1}{a}+1}(1-z)^{-s} \quad {}_2F_1\left(1+\frac{1}{a}, 1; s+\frac{1}{a}+2; z\right)}{s+\frac{1}{a}+1} + k_1$$

Remembering equation (5.107), we had

$$z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right)=\int\frac{z^{s+\frac{1}{a}}}{\left(1-z\right)^{s+1}}dz$$

$$\Rightarrow z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right)=\frac{z^{s+\frac{1}{a}+1}\left(1-z\right)^{-s}\ {}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)}{s+\frac{1}{a}+1}+k_1$$

$$\overline{G}\left(z,s\right)=\frac{1}{z^{s+\frac{1}{a}}\left(1-z\right)^{-s}}\frac{z^{s+\frac{1}{a}+1}\left(1-z\right)^{-s}\ {}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)}{s+\frac{1}{a}+1}+\frac{k_1}{z^{s+\frac{1}{a}}\left(1-z\right)^{-s}}$$

$$\therefore\overline{G}\left(z,s\right)=\frac{z\ {}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)}{s+\frac{1}{a}+1}+\frac{k_1\left(1-z\right)^{s}}{z^{s+\frac{1}{a}}}$$

But since for all $t,z\leqslant 1$ we have $G\left(z,t\right)\leqslant 1$, It follows that $k_1=0$

Thus

$$\overline{G}\left(z,s\right)=\frac{z\ {}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)}{s+\frac{1}{a}+1}\tag{5.109}$$

But according to Euler the Gauss hyper geometric series

$$\begin{aligned}{}_2F_1\left(a,b;c;x\right)&=\sum_{k=0}^{\infty}\frac{\left(a\right)_k\left(b\right)_k}{\left(c\right)_k}\frac{x^k}{k!}\\&=1+\frac{ab}{c}\frac{x}{1!}+\frac{a\left(a+1\right)b\left(b+1\right)}{c\left(c+1\right)}\frac{x^2}{2!}+\frac{a\left(a+1\right)\left(a+2\right)b\left(b+1\right)\left(b+2\right)}{c\left(c+1\right)\left(c+2\right)}\frac{x^3}{3!}+...\end{aligned}$$

Where a,b and c are complex numbers and

$$\left(a\right)_k=\prod_{i=0}^{k-i}\left(a+i\right)=a\left(a+1\right)\left(a+2\right)\left(a+3\right)...\left(a+k-1\right)$$

Thus Letting $a=1+\frac{1}{a}$,$b=1$ and $c=s+\frac{1}{a}+2$ we have

$$ {}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)=F\left(a,b;c;z\right) $$

$$ =\left\{\begin{matrix} 1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+ \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+... \end{matrix}\right\} $$

$$ =\left\{\begin{matrix} 1+\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+2\right)}+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)2!}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}\frac{z^2}{2!}+ \\ \frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)3!}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}\frac{z^3}{3!}+... \end{matrix}\right\} $$

$$ =\left\{\begin{matrix} 1+\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+2\right)}+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}+ \\ \frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}+... \end{matrix}\right\} $$

But by equation (5.109)

$$ \overline{G}\left(z,s\right)=\frac{z\ {}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)}{s+\frac{1}{a}+1} $$

$$ =\frac{z}{s+\frac{1}{a}+1}\left\{\begin{matrix} 1+\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+2\right)}+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}+ \\ \frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}+... \end{matrix}\right\} $$

$$= z\left\{\begin{array}{c}\frac{1}{s+\frac{1}{a}+1}+\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}+\\ \frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}+\ldots\end{array}\right\}$$

We can rewrite this as

$$\overline{G}\left(z,s\right)=zR \tag{5.110}$$

Where

$$R=\left\{\begin{array}{c}\frac{1}{s+\frac{1}{a}+1}+\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}+\\ \frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}+\ldots\end{array}\right\}$$

To obtain the explicit form of $G(z,t)$ we apply inverse Laplace transform to both sides of equation (5.110). This yields;

$$\begin{aligned}G(z,t)&=L^{-1}\left(zR\right)\\&=zL^{-1}\left(R\right)\end{aligned}$$

But

$$L^{-1}\left(R\right)=L^{-1}\left\{\begin{array}{c}\frac{1}{s+\frac{1}{a}+1}+\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}+\\ \frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}+\ldots\end{array}\right\}$$

$$= \begin{bmatrix} L^{-1}\left\{\frac{1}{s+\frac{1}{a}+1}\right\} + L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}\right\} + \\ L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}\right\} + \\ L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}\right\} + \ldots \end{bmatrix}$$

We shall solve the first four terms of the above equation separately

Remark

Remember that our variable is in terms of tau and not t

Part I

From the table of transform pairs in chapter 2.

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-a\tau}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s+\frac{1}{a}+1}\right\} = e^{-\left(\frac{1}{a}+1\right)\tau}$$

Part II

$$L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}\right\} = \left(\frac{1}{a}+1\right)zL^{-1}\left\{\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}\right\}$$

The function

$$\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}$$

has simple poles at $s = -\left(\frac{1}{a}+1\right)$ and at $s = -\left(\frac{1}{a}+2\right)$

Its residue at each pole is given by

At $s = -\left(\frac{1}{a}+1\right)$

$$a_{-i} = \lim_{s\to-\left(\frac{1}{a}+1\right)} \frac{\left(s+\frac{1}{a}+1\right)e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}$$

$$= \lim_{s\to-\left(\frac{1}{a}+1\right)} \frac{e^{s\tau}}{\left(s+\frac{1}{a}+2\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{\left[-\left(\frac{1}{a}+1\right)+\frac{1}{a}+2\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{\left[-\frac{1}{a}-1+\frac{1}{a}+2\right]}$$

$$= e^{-\left(\frac{1}{a}+1\right)\tau}$$

At $s = -\left(\frac{1}{a}+2\right)$

$$a_{-i} = \lim_{s\to-\left(\frac{1}{a}+2\right)} \frac{\left(s+\frac{1}{a}+2\right)e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}$$

$$= \lim_{s\to-\left(\frac{1}{a}+2\right)} \frac{e^{s\tau}}{\left(s+\frac{1}{a}+1\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{\left[-\left(\frac{1}{a}+2\right)+\frac{1}{a}+1\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{\left[-\frac{1}{a}-2+\frac{1}{a}+1\right]}$$

$$= -e^{-\left(\frac{1}{a}+2\right)\tau}$$

Therefore

$$L^{-1}\left\{\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}\right\} = \sum a_{-i}$$
$$= e^{-\left(\frac{1}{a}+1\right)\tau} - e^{-\left(\frac{1}{a}+2\right)\tau}$$
$$= e^{-\left(\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right)$$

Thus

$$L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)z}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)}\right\} = \left(\frac{1}{a}+1\right)ze^{-\left(\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right)$$
$$= \left(\frac{1}{a}+1\right)e^{-\left(\frac{1}{a}+1\right)\tau}z\left(1-e^{-\tau}\right)$$

Part III

$$L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}\right\}$$

$$= \left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2L^{-1}\left\{\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}\right\}$$

The function

$$\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}$$

has simple poles at $s=-\left(\frac{1}{a}+1\right),\quad s=-\left(\frac{1}{a}+2\right)$ and $s=-\left(\frac{1}{a}+3\right)$

It's residue at each pole is obtained as follows

At $s = -\left(\frac{1}{a} + 1\right)$

$$a_{-i} = \lim_{s \to -\left(\frac{1}{a}+1\right)} \frac{\left(s + \frac{1}{a} + 1\right) e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(\frac{1}{a}+1\right)} \frac{e^{s\tau}}{\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{\left[-\left(\frac{1}{a} + 1\right) + \frac{1}{a} + 2\right]\left[-\left(\frac{1}{a} + 1\right) + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{\left[-\frac{1}{a} - 1 + \frac{1}{a} + 2\right]\left[-\frac{1}{a} - 1 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{(1)(2)}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{2!}$$

At $s = -\left(\frac{1}{a} + 2\right)$

$$a_{-i} = \lim_{s \to -\left(\frac{1}{a}+2\right)} \frac{\left(s + \frac{1}{a} + 2\right) e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(\frac{1}{a}+2\right)} \frac{e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{\left[-\left(\frac{1}{a} + 2\right) + \frac{1}{a} + 1\right]\left[-\left(\frac{1}{a} + 2\right) + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{\left[-\frac{1}{a} - 2 + \frac{1}{a} + 1\right]\left[-\frac{1}{a} - 2 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{(-1)(1)}$$

$$= -e^{-\left(\frac{1}{a}+2\right)\tau}$$

At $s = -\left(\frac{1}{a} + 3\right)$

$$a_{-i} = \lim_{s \to -\left(\frac{1}{a}+3\right)} \frac{\left(s + \frac{1}{a} + 3\right) e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(\frac{1}{a}+3\right)} \frac{e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{\left[-\left(\frac{1}{a} + 3\right) + \frac{1}{a} + 1\right]\left[-\left(\frac{1}{a} + 3\right) + \frac{1}{a} + 2\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{\left[-\frac{1}{a} - 3 + \frac{1}{a} + 1\right]\left[-\frac{1}{a} - 3 + \frac{1}{a} + 2\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{(-2)(-1)}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{2!}$$

Therefore

$$L^{-1}\left\{\frac{1}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)}\right\} = \sum a_{-i}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{2!} - e^{-\left(\frac{1}{a}+2\right)\tau} + \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{2!}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{2!}\left(1 - 2e^{-\tau} + e^{-2\tau}\right)$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{2!}\left(1 - e^{-\tau}\right)^2$$

Thus

$$L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)}\right\}$$

$$= \left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)z^2\frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{2!}\left(1 - e^{-\tau}\right)^2$$

$$= \left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)e^{-\left(\frac{1}{a}+1\right)\tau}\frac{z^2}{2!}\left(1 - e^{-\tau}\right)^2$$

Part IV

$$L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}\right\}$$

$$= \left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3L^{-1}\left\{\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}\right\}$$

The function

$$\frac{1}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}$$

has simple poles at

$$s=-\left(\frac{1}{a}+1\right),\ s=-\left(\frac{1}{a}+2\right),\ s=-\left(\frac{1}{a}+3\right),\ s=-\left(\frac{1}{a}+4\right)$$

It's residue at each pole is obtained as follows

At $s=-\left(\frac{1}{a}+1\right)$

$$a_{-i}=\lim_{s\to-\left(\frac{1}{a}+1\right)}\frac{\left(s+\frac{1}{a}+1\right)e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}$$

$$=\lim_{s\to-\left(\frac{1}{a}+1\right)}\frac{e^{s\tau}}{\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}$$

$$=\frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{\left[-\left(\frac{1}{a}+1\right)+\frac{1}{a}+2\right]\left[-\left(\frac{1}{a}+1\right)+\frac{1}{a}+3\right]\left[-\left(\frac{1}{a}+1\right)+\frac{1}{a}+4\right]}$$

$$=\frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{\left[-\frac{1}{a}-1+\frac{1}{a}+2\right]\left[-\frac{1}{a}-1+\frac{1}{a}+3\right]\left[-\frac{1}{a}-1+\frac{1}{a}+4\right]}$$

$$=\frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{(1)\,(2)\,(3)}$$

$$=\frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{3!}$$

At $s=-\left(\frac{1}{a}+2\right)$

$$a_{-i}=\lim_{s\to-\left(\frac{1}{a}+2\right)}\frac{\left(s+\frac{1}{a}+2\right)e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}$$

$$=\lim_{s\to-\left(\frac{1}{a}+2\right)}\frac{e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{\left[-\left(\frac{1}{a}+2\right)+\frac{1}{a}+1\right]\left[-\left(\frac{1}{a}+2\right)+\frac{1}{a}+3\right]\left[-\left(\frac{1}{a}+2\right)+\frac{1}{a}+4\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{\left[-\frac{1}{a}-2+\frac{1}{a}+1\right]\left[-\frac{1}{a}-2+\frac{1}{a}+3\right]\left[-\frac{1}{a}-2+\frac{1}{a}+4\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{(-1)(1)(2)}$$

$$= \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{-2!}$$

At $s = -\left(\frac{1}{a}+3\right)$

$$a_{-i} = \lim_{s\to-\left(\frac{1}{a}+3\right)} \frac{\left(s+\frac{1}{a}+3\right)e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}$$

$$= \lim_{s\to-\left(\frac{1}{a}+3\right)} \frac{e^{s\tau}}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+4\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{\left[-\left(\frac{1}{a}+3\right)+\frac{1}{a}+1\right]\left[-\left(\frac{1}{a}+3\right)+\frac{1}{a}+2\right]\left[-\left(\frac{1}{a}+3\right)+\frac{1}{a}+4\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{\left[-\frac{1}{a}-3+\frac{1}{a}+1\right]\left[-\frac{1}{a}-3+\frac{1}{a}+2\right]\left[-\frac{1}{a}-3+\frac{1}{a}+4\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{(-2)(-1)(1)}$$

$$= \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{2!}$$

At $s = -\left(\frac{1}{a} + 4\right)$

$$a_{-i} = \lim_{s \to -\left(\frac{1}{a}+4\right)} \frac{\left(s + \frac{1}{a} + 4\right) e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)\left(s + \frac{1}{a} + 4\right)}$$

$$= \lim_{s \to -\left(\frac{1}{a}+4\right)} \frac{e^{s\tau}}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(\frac{1}{a}+4\right)\tau}}{\left[-\left(\frac{1}{a} + 4\right) + \frac{1}{a} + 1\right]\left[-\left(\frac{1}{a} + 4\right) + \frac{1}{a} + 2\right]\left[-\left(\frac{1}{a} + 4\right) + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+4\right)\tau}}{\left[-\frac{1}{a} - 4 + \frac{1}{a} + 1\right]\left[-\frac{1}{a} - 4 + \frac{1}{a} + 2\right]\left[-\frac{1}{a} - 4 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(\frac{1}{a}+4\right)\tau}}{(-3)(-2)(-1)}$$

$$= \frac{e^{-\left(\frac{1}{a}+4\right)\tau}}{-3!}$$

Therefore

$$L^{-1}\left\{\frac{1}{\left(s + \frac{1}{a} + 1\right)\left(s + \frac{1}{a} + 2\right)\left(s + \frac{1}{a} + 3\right)\left(s + \frac{1}{a} + 4\right)}\right\} = \sum a_{-i}$$

$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{3!} - \frac{e^{-\left(\frac{1}{a}+2\right)\tau}}{2!} + \frac{e^{-\left(\frac{1}{a}+3\right)\tau}}{2!} - \frac{e^{-\left(\frac{1}{a}+4\right)\tau}}{3!}$$
$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{3!}\left(1 - \frac{3!}{2!}e^{-\tau} + \frac{3!}{2!}e^{-2\tau} - e^{-3\tau}\right)$$
$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{3!}\left(1 - 3e^{-\tau} + 3e^{-2\tau} - e^{-3\tau}\right)$$
$$= \frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{3!}\left(1 - e^{-\tau}\right)^3$$

Thus

$$L^{-1}\left\{\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3}{\left(s+\frac{1}{a}+1\right)\left(s+\frac{1}{a}+2\right)\left(s+\frac{1}{a}+3\right)\left(s+\frac{1}{a}+4\right)}\right\}$$

$$=\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)z^3\frac{e^{-\left(\frac{1}{a}+1\right)\tau}}{3!}\left(1-e^{-\tau}\right)^3$$

$$=\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)e^{-\left(\frac{1}{a}+1\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3$$

Consolidating the above results we get

$$L^{-1}\{R\}=\left\{\begin{array}{c} e^{-\left(\frac{1}{a}+1\right)\tau}+\left(\frac{1}{a}+1\right)e^{-\left(\frac{1}{a}+1\right)\tau}z\left(1-e^{-\tau}\right)+ \\ \\ \left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)e^{-\left(\frac{1}{a}+1\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2+ \\ \\ \left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)e^{-\left(\frac{1}{a}+1\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3+... \end{array}\right\}$$

$$=e^{-\left(\frac{1}{a}+1\right)\tau}\left\{\begin{array}{c} 1+\left(\frac{1}{a}+1\right)z\left(1-e^{-\tau}\right)+\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2 \\ \\ +\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3+... \end{array}\right\}$$

$$=e^{-\left(\frac{1}{a}+1\right)\tau}\left\{\begin{array}{c} 1+\left(\frac{1}{a}+1\right)\left[z\left(1-e^{-\tau}\right)\right]+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)}{2!}\left[z\left(1-e^{-\tau}\right)\right]^2 \\ \\ +\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)}{3!}\left[z\left(1-e^{-\tau}\right)\right]^3+... \end{array}\right\}$$

Recall that

$$G(z,t) = zL^{-1}\{R\}$$

$$= ze^{-\left(\frac{1}{a}+1\right)\tau}\left\{\begin{array}{c} 1+\left(\frac{1}{a}+1\right)\left[z\left(1-e^{-\tau}\right)\right]+\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)}{2!}\left[z\left(1-e^{-\tau}\right)\right]^2 \\ +\frac{\left(\frac{1}{a}+1\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+3\right)}{3!}\left[z\left(1-e^{-\tau}\right)\right]^3+... \end{array}\right\}$$

But

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Implying that

$$\begin{aligned}\binom{\frac{1}{a}+1}{1} &= \frac{\left(\frac{1}{a}+1\right)!}{\left(\frac{1}{a}+1-1\right)!1!} \\ &= \frac{\left(\frac{1}{a}+1\right)\frac{1}{a}!}{\frac{1}{a}!1!} \\ &= \frac{1}{a}+1\end{aligned}$$

$$\begin{aligned}\binom{\frac{1}{a}+2}{2} &= \frac{\left(\frac{1}{a}+2\right)!}{\left(\frac{1}{a}+2-2\right)!2!} \\ &= \frac{\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+1\right)\frac{1}{a}!}{\frac{1}{a}!2!} \\ &= \frac{\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+1\right)}{2!}\end{aligned}$$

$$\begin{aligned}\binom{\frac{1}{a}+3}{3} &= \frac{\left(\frac{1}{a}+3\right)!}{\left(\frac{1}{a}+3-3\right)!3!} \\ &= \frac{\left(\frac{1}{a}+3\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+1\right)\frac{1}{a}!}{\frac{1}{a}!3!} \\ &= \frac{\left(\frac{1}{a}+3\right)\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+1\right)}{3!}\end{aligned}$$

Relating this to the above equation for $G(z,t)$ we get

$$G(z,t) = ze^{-\left(\frac{1}{a}+1\right)\tau}\left\{\begin{array}{c}\binom{\frac{1}{a}+0}{0} + \binom{\frac{1}{a}+1}{1}\left[z\left(1-e^{-\tau}\right)\right] + \\ \binom{\frac{1}{a}+2}{2}\left[z\left(1-e^{-\tau}\right)\right]^2 + \binom{\frac{1}{a}+3}{3}\left[z\left(1-e^{-\tau}\right)\right]^3 + ...\end{array}\right\}$$

$$= ze^{-\left(\frac{1}{a}+1\right)\tau}\sum_{j=0}^{\infty}\binom{\frac{1}{a}+j}{j}\left[z\left(1-e^{-\tau}\right)\right]^j$$

$$= ze^{-\left(\frac{1}{a}+1\right)\tau}\sum_{j=0}^{\infty}\binom{\left(\frac{1}{a}+1\right)+j-1}{j}\left[z\left(1-e^{-\tau}\right)\right]^j$$

Recall that

$$\binom{-\tau}{j}(-1)^j = \binom{\tau+j-1}{j}$$

Thus

$$G(z,t) = ze^{-\left(\frac{1}{a}+1\right)\tau}\sum_{j=0}^{\infty}\binom{\left(\frac{1}{a}+1\right)+j-1}{j}\left[z\left(1-e^{-\tau}\right)\right]^j$$

$$= ze^{-\left(\frac{1}{a}+1\right)\tau}\left[1-z\left(1-e^{-\tau}\right)\right]^{-\left(\frac{1}{a}+1\right)}$$

$$= z\left[\frac{e^{-\tau}}{1-z\left(1-e^{-\tau}\right)}\right]^{\frac{1}{a}+1}$$

We are now close to the final solution, the remaining task is to rewrite $G(z,\tau)$ in terms of the original variable t. Its now time to go back to our initial substitution

$$\tau = \ln\left|\frac{t+\alpha}{\alpha}\right|$$

Where

$$\alpha = \frac{1}{\lambda a}$$

Thus

$$
\begin{aligned}
-\tau &= -\ln\left|\frac{t+\alpha}{\alpha}\right| \\
&= \ln\left|\frac{t+\alpha}{\alpha}\right|^{-1} \\
\Rightarrow e^{-\tau} &= e^{\ln\left|\frac{t+\alpha}{\alpha}\right|^{-1}} \\
&= \left|\frac{t+\alpha}{\alpha}\right|^{-1} \\
&= \left|\frac{t}{\alpha}+1\right|^{-1} \\
&= \left|\frac{t}{1/\lambda a}+1\right|^{-1} \\
&= (\lambda a t+1)^{-1}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G(z,t) &= z\left[\frac{(1+\lambda a t)^{-1}}{1-z\left[1-(1+\lambda a t)^{-1}\right]}\right]^{1+\frac{1}{a}} \\
&= z\left[\frac{1}{(1+\lambda a t)\left(1-z\left[1-\frac{1}{1+\lambda a t}\right]\right)}\right]^{1+\frac{1}{a}} \\
&= z\left[\frac{1}{(1+\lambda a t)\left(1-z\left[\frac{1+\lambda a t-1}{1+\lambda a t}\right]\right)}\right]^{1+\frac{1}{a}} \\
&= z\left[\frac{1}{(1+\lambda a t)\left(1-\frac{z\lambda a t}{1+\lambda a t}\right)}\right]^{1+\frac{1}{a}}
\end{aligned}
$$

$$= z\left[\frac{1}{1+\lambda at-\lambda atz}\right]^{1+\frac{1}{a}}$$

$$= z\left[\frac{\frac{1}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{1+\frac{1}{a}}$$

We finally have

$$G\left(z,t\right) = z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{1+\frac{1}{a}}$$

Letting $p = \frac{1}{1+\lambda at}$ and $q = 1 - \frac{1}{1+\lambda at} = \frac{\lambda at}{1+\lambda at}$

$G(z,t)$ can be expressed as

$$G\left(z,t\right) = z^{-\frac{1}{a}}\left[\frac{zp}{1-qz}\right]^{1+\frac{1}{a}}$$

$P_n\left(t\right)$ is the coefficient of z^n in the expansion of $G(z,t)$. By binomial expansion

$$G\left(z,t\right) = z^{-\frac{1}{a}}z^{1+\frac{1}{a}}p^{1+\frac{1}{a}}\left[1-qz\right]^{-\left(1+\frac{1}{a}\right)}$$

$$= zp^{1+\frac{1}{a}}\sum_{k=0}^{\infty}\binom{-\left[1+\frac{1}{a}\right]}{k}\left(-qz\right)^k$$

$$= zp^{1+\frac{1}{a}}\sum_{k=0}^{\infty}\binom{-\left[1+\frac{1}{a}\right]}{k}\left(-1\right)^k q^k z^k$$

Recall that

$$\binom{-r}{k}(-1)^k = \binom{r+k-1}{k}$$

$$\Rightarrow \binom{-\left[1+\frac{1}{a}\right]}{k}(-1)^k = \binom{\left[1+\frac{1}{a}\right]+k-1}{k}$$

Thus

$$G(z,t) = zp^{1+\frac{1}{a}} \sum_{k=0}^{\infty} \binom{\left[1+\frac{1}{a}\right]+k-1}{k} q^k z^k$$

$$= \sum_{k=0}^{\infty} \binom{\left[1+\frac{1}{a}\right]+k-1}{k} p^{1+\frac{1}{a}} q^k z^{1+k}$$

From this it follows that

$$P_{1+k}\left(t\right) = \binom{\left[1+\frac{1}{a}\right]+k-1}{k} p^{1+\frac{1}{a}} q^k \qquad k = 0,\ 1,\ 2$$

but $n = 1+k$, Thus

$$P_n\left(t\right) = \binom{\left[1+\frac{1}{a}\right]+k-1}{k} p^{1+\frac{1}{a}} q^k \qquad k = 0,\ 1,\ 2$$

$$= \binom{\left[1+\frac{1}{a}\right]+k-1}{k} \left(\frac{1}{1+\lambda at}\right)^{1+\frac{1}{a}} \left(1-\frac{1}{1+\lambda at}\right)^{k} \qquad k = 0,\ 1,\ 2$$

which is the pmf of a negative binomial distribution with parameters $r = 1+\frac{1}{a}$ and $p = \frac{1}{1+\lambda at}$

Case 2: When initial population $X(0) = n_0$

Recall that

$$G(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$

$$\Rightarrow G(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n$$

$$= P_0(0) + P_1(0)z + P_2(0)z^2 + ... + P_{n_0}(0)z^{n_0} + ...$$

but for the initial condition $X(0) = n_0$, we have

$$P_{n_0}(0) = 1, \quad P_n(0) = 0 \quad \forall n \neq n_0$$

$$\therefore G(z,0) = z^{n_0}$$

With this equation (5.106) becomes

$$z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right) = \int \frac{z^{s+\frac{1}{a}-1}z^{n_0}}{\left(1-z\right)^{s+1}}dz$$

$$z^{s+\frac{1}{a}}\left(1-z\right)^{-s}\overline{G}\left(z,s\right) = \int \frac{z^{s+n_0+\frac{1}{a}-1}}{\left(1-z\right)^{s+1}}dz \tag{5.111}$$

We now simplify the integral in the RHS of equation (5.111).

Recall that

$$\int \frac{x^{a+r-1}}{\left(1-x\right)^{a+1}}dx = x^{a+r}\left\{\begin{array}{c}\frac{{}_2F_1(a,a+r;a+r+1;x)}{a+r}+ \\ \\ x\frac{{}_2F_1(a+1,a+r+1;a+r+2;x)}{a+r+1}\end{array}\right\} + \text{constant}$$

$$\therefore \frac{1}{\lambda}\int \frac{z^{s+\left(n_0+\frac{1}{a}\right)-1}}{\left(1-z\right)^{s+1}}dz = z^{s+n_0+\frac{1}{a}}\left\{\begin{array}{c}\frac{{}_2F_1\left(s,s+n_0+\frac{1}{a};s+n_0+\frac{1}{a}+1;z\right)}{s+n_0+\frac{1}{a}}+ \\ \\ z\frac{{}_2F_1\left(s+1,s+n_0+\frac{1}{a}+1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\} + d_2$$

where d_2 is a constant of integration

But

$${}_2F_1\left(a,b;c;z\right) = \left(1-z\right)^{c-a-b}\ {}_2F_1\left(c-a,c-b;c;z\right)$$

We now use this property to simplify the terms in the RHS, We start with

$$ {}_2F_1\left(s, s+n_0+\frac{1}{a}; s+n_0+\frac{1}{a}+1; z\right) $$

Here

$$ a = s, \quad b = s+n_0+\frac{1}{a}, \quad c = s+n_0+\frac{1}{a}+1 $$

$$ \begin{aligned} c-a &= s+n_0+\frac{1}{a}+1-s \\ &= n_0+\frac{1}{a}+1 \end{aligned} $$

$$ \begin{aligned} c-b &= s+n_0+\frac{1}{a}+1-\left(s+n_0+\frac{1}{a}\right) \\ &= s+n_0+\frac{1}{a}+1-s-n_0-\frac{1}{a} \\ &= 1 \end{aligned} $$

$$ \begin{aligned} c-a-b &= s+n_0+\frac{1}{a}+1-s-\left(s+n_0+\frac{1}{a}\right) \\ &= s+n_0+\frac{1}{a}+1-s-s-n_0-\frac{1}{a} \\ &= 1-s \end{aligned} $$

Thus

$$ {}_2F_1\left(s, s+n_0+\frac{1}{a}; s+n_0+\frac{1}{a}+1; z\right) = (1-z)_2^{1-s}F_1\left(n_0+\frac{1}{a}+1, 1; s+n_0+\frac{1}{\lambda}+1; z\right) $$

Similarly for

$$ {}_2F_1\left(s+1, s+n_0+\frac{1}{a}+1; s+n_0+\frac{1}{a}+2; z\right) $$

We have

$$
{}_2F_1\left(s+1, s+n_0+\frac{1}{a}+1; s+n_0+\frac{1}{a}+2; z\right)
$$

$$
a = s+1, \quad b = s+n_0+\frac{1}{a}+1, \quad c = s+n_0+\frac{1}{a}+2
$$

$$
\begin{aligned}
c-a &= s+n_0+\frac{1}{a}+2-(s+1) \\
&= s+n_0+\frac{1}{a}+2-s-1 \\
&= n_0+\frac{1}{a}+1
\end{aligned}
$$

$$
\begin{aligned}
c-b &= s+n_0+\frac{1}{a}+2-\left(s+n_0+\frac{1}{a}+1\right) \\
&= s+n_0+\frac{1}{a}+2-s-n_0-\frac{1}{a}-1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
c-a-b &= s+n_0+\frac{1}{a}+2-(s+1)-\left(s+n_0+\frac{1}{a}+1\right) \\
&= s+n_0+\frac{1}{a}+2-s-1-s-n_0-\frac{1}{a}-1 \\
&= -s
\end{aligned}
$$

Thus

$$
{}_2F_1\left(s+1, s+n_0+\frac{1}{a}+1; s+n_0+\frac{1}{a}+2; z\right) = (1-z)_2^{-s} F_1\left(n_0+\frac{1}{a}+1, 1; s+n_0+\frac{1}{a}+2; z\right)
$$

We now have

$$\therefore \frac{1}{\lambda}\int \frac{z^{s+\left(n_0+\frac{1}{a}\right)-1}}{(1-z)^{s+1}}dz == z^{s+n_0+\frac{1}{a}}\left\{\begin{array}{l}\frac{(1-z)_2^{1-s}F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{\lambda}+1;z\right)}{s+n_0+\frac{1}{a}}+\\ \\ z\frac{(1-z)^{-s}\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\}+k_1$$

$$=z^{s+n_0+\frac{1}{a}}(1-z)^{-s}\left\{\begin{array}{l}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{\lambda}+1;z\right)}{s+n_0+\frac{1}{a}}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\}+k_1$$

where k_1 is a constant of integration

With this equation (5.111) becomes

$$z^{s+\frac{1}{a}}(1-z)^{-s}\,\overline{G}(z,s)=\int \frac{z^{s+n_0+\frac{1}{a}-1}}{(1-z)^{s+1}}dz$$

$$=z^{s+n_0+\frac{1}{a}}(1-z)^{-s}\left\{\begin{array}{l}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{\lambda}+1;z\right)}{s+n_0+\frac{1}{a}}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\}+k_1$$

$$\therefore G(z,s)=\frac{1}{z^{s+\frac{1}{a}}(1-z)^{-s}}z^{s+n_0+\frac{1}{a}}(1-z)^{-s}\left\{\begin{array}{l}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{\lambda}+1;z\right)}{s+n_0+\frac{1}{a}}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\}$$

$$+\frac{k_1}{z^{s+\frac{1}{a}}(1-z)^{-s}}$$

$$= z^{n_0}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{\lambda}+1;z\right)}{s+n_0+\frac{1}{a}}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\}+\frac{k_1}{z^{s+\frac{1}{a}}\left(1-z\right)^{-s}}$$

But since for all $t, z \leqslant 1$,$G\left(z,t\right) \leqslant 1$, It therefore follows that $k_1 = 0$
Thus

$$\overline{G}\left(z,s\right) = z^{n_0}\left\{\begin{array}{c}\frac{(1-z)\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+1;z\right)}{s+n_0+\frac{1}{a}}+\\ \\ z\frac{{}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\end{array}\right\}$$

But according to Euler the Gauss hyper geometric series

$$\begin{aligned}{}_2F_1\left(a,b;c;x\right) &= \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k}\frac{x^k}{k!}\\ &= 1+\frac{ab}{c}\frac{x}{1!}+\frac{a\left(a+1\right)b\left(b+1\right)}{c\left(c+1\right)}\frac{x^2}{2!}+\frac{a\left(a+1\right)\left(a+2\right)b\left(b+1\right)\left(b+2\right)}{c\left(c+1\right)\left(c+2\right)}\frac{x^3}{3!}+...\end{aligned}$$

Where a, b and c are complex numbers and

$$(a)_k = \prod_{i=0}^{k-i}\left(a+i\right) = a\left(a+1\right)\left(a+2\right)\left(a+3\right)...\left(a+k-1\right)$$

With this we simplify the RHS as follows, For

$${}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+1;z\right)$$

Letting $a = n_0 + \frac{1}{a} + 1$,$b = 1$ and $c = s + n_0 + \frac{1}{a} + 1$ we have

$$F\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+1;z\right) = F\left(a,b;c;z\right)$$

$$= \left\{ \begin{array}{c} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} \\ \\ + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)2!}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\frac{z^2}{2!} + \\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)3!}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)} + \\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \ldots \end{array} \right\}$$

Similarly for

$${}_2F_1\left(n_0 + \frac{1}{a} + 1, 1; s + n_0 + \frac{1}{a} + 2; z\right)$$

We let $a = n_0 + \frac{1}{a} + 1$,$b = 1$ and $c = s + n_0 + \frac{1}{a} + 2$ implying that

$$F\left(n_0 + \frac{1}{a} + 1, 1; s + n_0 + \frac{1}{a} + 2; z\right) = F\left(a, b; c; z\right)$$

$$= \left\{ \begin{array}{c} 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \\ \\ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+2\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)2!}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\frac{z^2}{2!} + \\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)3!}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}\frac{z^3}{3!} + ... \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+2\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)} + ... \end{array} \right\}$$

Consolidating the above results we get

$$\overline{G}\left(z,s\right) = z^{n_0} \left\{ \begin{array}{l} \frac{(1-z)}{s+n_0+\frac{1}{a}} \; {}_2F_1\left(n_0+\frac{1}{a}+1, 1; s+n_0+\frac{1}{a}+1; z\right) + \\ \\ \frac{z}{s+n_0+\frac{1}{a}+1} \; {}_2F_1\left(n_0+\frac{1}{a}+1, 1; s+n_0+\frac{1}{a}+2; z\right) \end{array} \right\}$$

$$= z^{n_0} \left\{ \begin{array}{l} \frac{(1-z)}{s+n_0+\frac{1}{a}} \left[\begin{array}{l} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)} + \\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + ... \end{array} \right] + \\ \\ \frac{z}{s+n_0+\frac{1}{a}+1} \left[\begin{array}{l} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+2\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)} + ... \end{array} \right] \end{array} \right\}$$

$$\overline{G}(z,s) = z^{n_0}\left\{\begin{array}{l}\frac{(1-z)}{s+n_0+\frac{1}{a}}\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+1;z\right)+\\ \\ \frac{z}{s+n_0+\frac{1}{a}+1}\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)\end{array}\right\}$$

$$= z^{n_0}\left\{\begin{array}{l}\frac{(1-z)}{s+n_0+\frac{1}{a}}\left[\begin{array}{l}1+\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)}+\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}+\\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}+\ldots\end{array}\right]+\\ \\ \frac{z}{s+n_0+\frac{1}{a}+1}\left[\begin{array}{l}1+\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+2\right)}+\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}+\\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}+\ldots\end{array}\right]\end{array}\right\}$$

$$= z^{n_0} \left\{ \begin{array}{c} (1-z) \left[\begin{array}{c} \frac{1}{\left(s+n_0+\frac{1}{a}\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \ldots \end{array} \right] + \\ z \left[\begin{array}{c} \frac{1}{\left(s+n_0+\frac{1}{a}+1\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)} + \ldots \end{array} \right] \end{array} \right\}$$

Therefore

$$\overline{G}(z,s) = z^{n_0}\left[(1-z)M + zN\right] \tag{5.112}$$

Where

$$M = \left\{ \begin{array}{c} \frac{1}{\left(s+n_0+\frac{1}{a}\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \ldots \end{array} \right\}$$

And

$$N=\left\{\begin{array}{c}\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)}+\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}+\\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}+\\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}+...\end{array}\right\}$$

Applying inverse Laplace transform to both sides of equation (5.112) we get

$$\begin{aligned}L^{-1}\left\{\overline{G}(z,s)\right\}&=L^{-1}\left\{z^{n_0}\left[(1-z)M+zN\right]\right\}\\&=z^{n_0}\left[(1-z)L^{-1}\{M\}+zL^{-1}\{N\}\right]\\ \therefore G(z,t)&=z^{n_0}\left[(1-z)L^{-1}\{M\}+zL^{-1}\{N\}\right]\end{aligned}$$

But

$$L^{-1}\{M\}=L^{-1}\left\{\begin{array}{c}\frac{1}{s+n_0+\frac{1}{a}}+\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}+\\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}+\\ \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}+..\end{array}\right\}$$

$$
= \begin{bmatrix}
L^{-1}\left\{\frac{1}{s+n_0+\frac{1}{a}}\right\}+ \\
L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}\right\}+ \\
L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\}+ \\
L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}
\end{bmatrix}
$$

We again simplify the above first four terms of separately

First term
From the table of transform pairs

$$
L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}
$$
$$
\Rightarrow L^{-1}\left\{\frac{1}{s+n_0+\frac{1}{a}}\right\} = e^{-\left(n_0+\frac{1}{a}\right)\tau}
$$

Second term

$$
L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}\right\} = \left(n_0+\frac{1}{a}+1\right)zL^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}\right\}
$$

But

$$
\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}
$$

has simple poles at $s = -\left(n_0+\frac{1}{a}\right)$ and $s = -\left(n_0+\frac{1}{a}+1\right)$

Its residue at each pole is obtained as follows

At $s = -\left(n_0 + \frac{1}{a}\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a}\right)} \frac{\left(s + n_0 + \frac{1}{a}\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a}\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{\left[-\left(n_0 + \frac{1}{a}\right) + n_0 + \frac{1}{a} + 1\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{\left[-n_0 - \frac{1}{a} + n_0 + \frac{1}{a} + 1\right]}$$

$$= e^{-\left(n_0 + \frac{1}{a}\right)\tau}$$

At $s = -\left(n_0 + \frac{1}{a} + 1\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{\left(s + n_0 + \frac{1}{a} + 1\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 1\right) + n_0 + \frac{1}{a}\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-n_0 - \frac{1}{a} - 1 + n_0 + \frac{1}{a}\right]}$$

$$= -e^{-\left(n_0+\frac{1}{a}+1\right)\tau}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}\right\} = \sum a_{-i} = e^{-\left(n_0+\frac{1}{a}\right)\tau} - e^{-\left(n_0+\frac{1}{a}+1\right)\tau}$$
$$= e^{-\left(n_0+\frac{1}{a}\right)\tau}\left(1-e^{-\tau}\right)$$

Therefore

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)}\right\} = \left(n_0+\frac{1}{a}+1\right)ze^{-\left(n_0+\frac{1}{a}\right)\tau}\left(1-e^{-\tau}\right)$$

Third term

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\}$$

is simply

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\}$$

But

$$\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}$$

has simple poles at $s = -\left(n_0+\frac{1}{a}\right)$,$s = -\left(n_0+\frac{1}{a}+1\right)$ and $s = -\left(n_0+\frac{1}{a}+2\right)$

Its residue at each pole is obtained as follows

At $s = -\left(n_0+\frac{1}{a}\right)$

$$a_{-i} = \lim_{s\to-\left(n_0+\frac{1}{a}\right)}\frac{\left(s+n_0+\frac{1}{a}\right)e^{s\tau}}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a}\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{\left[-\left(n_0 + \frac{1}{a}\right) + n_0 + \frac{1}{a} + 1\right]\left[-\left(n_0 + \frac{1}{a}\right) + n_0 + \frac{1}{a} + 1\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{\left[-n_0 - \frac{1}{a} + n_0 + \frac{1}{a} + 1\right]\left[-n_0 - \frac{1}{a} + n_0 + \frac{1}{a} + 2\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{1\,(2)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{2!}$$

At $s = -\left(n_0 + \frac{1}{a} + 1\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{\left(s + n_0 + \frac{1}{a} + 1\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 1\right) + n_0 + \frac{1}{a}\right]\left[-\left(n_0 + \frac{1}{a} + 1\right) + n_0 + \frac{1}{a} + 2\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-n_0 - \frac{1}{a} - 1 + n_0 + \frac{1}{a}\right]\left[-n_0 - \frac{1}{a} - 1 + n_0 + \frac{1}{a} + 2\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{-1\,(1)}$$

$$= -e^{-\left(n_0+\frac{1}{a}+1\right)\tau}$$

At $s = -\left(n_0 + \frac{1}{a} + 2\right)$

$$a_{-i} = \lim_{s \to -\left(n_0+\frac{1}{a}+2\right)} \frac{\left(s + n_0 + \frac{1}{a} + 2\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \lim_{s \to -\left(n_0+\frac{1}{a}+2\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a}\right]\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 1\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{\left[-n_0 - \frac{1}{a} - 2 + n_0 + \frac{1}{a}\right]\left[-n_0 - \frac{1}{a} - 2 + n_0 + \frac{1}{a} + 1\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{-2\,(-1)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{2!}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)}\right\} = \sum a_{-i}$$

But

$$\sum a_{-i} = \frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{2!} - e^{-\left(n_0+\frac{1}{a}+1\right)\tau} + \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{2!}$$
$$= \frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{2!}\left(1 - 2e^{-\tau} + e^{-2\tau}\right)$$
$$= \frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{2!}\left(1 - e^{-\tau}\right)^2$$

Therefore

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\}$$

will be

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2\frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{2!}\left(1-e^{-\tau}\right)^2$$
$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2$$

Fourth term

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}$$

which is equal to

$$\left\{\begin{array}{c}\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3 \quad \times \\ L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}\end{array}\right\}$$

The function

$$\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

has simple poles at $s = -\left(n_0 + \frac{1}{a}\right)$,$s = -\left(n_0 + \frac{1}{a} + 1\right)$,$s = -\left(n_0 + \frac{1}{a} + 2\right)$ and $s = -\left(n_0 + \frac{1}{a} + 3\right)$

Thus its residue at each pole is obtained as follows

At $s = -\left(n_0 + \frac{1}{a}\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a}\right)} \frac{\left(s + n_0 + \frac{1}{a}\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a}\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{\left[-\left(n_0 + \frac{1}{a}\right) + n_0 + \frac{1}{a} + 1\right]\left[-\left(n_0 + \frac{1}{a}\right) + n_0 + \frac{1}{a} + 2\right]\left[-\left(n_0 + \frac{1}{a}\right) + n_0 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{\left[-n_0 - \frac{1}{a} + n_0 + \frac{1}{a} + 1\right]\left[-n_0 - \frac{1}{a} + n_0 + \frac{1}{a} + 2\right]\left[-n_0 - \frac{1}{a} + n_0 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{1\,(2)\,(3)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a}\right)\tau}}{3!}$$

At $s = -\left(n_0 + \frac{1}{a} + 1\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{\left(s + n_0 + \frac{1}{a} + 1\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a}\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{\left[-\left(n_0+\frac{1}{a}+1\right)+n_0+\frac{1}{a}\right]\left[-\left(n_0+\frac{1}{a}+1\right)+n_0+\frac{1}{a}+2\right]\left[-\left(n_0+\frac{1}{a}+1\right)+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{\left[-n_0-\frac{1}{a}-1+n_0+\frac{1}{a}\right]\left[-n_0-\frac{1}{a}-1+n_0+\frac{1}{a}+2\right]\left[-n_0-\frac{1}{a}-1+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{-1\,(1)\,(2)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{-2!}$$

At $s = -\left(n_0+\frac{1}{a}+2\right)$

$$a_{-i} = \lim_{s\to-\left(n_0+\frac{1}{a}+2\right)} \frac{\left(s+n_0+\frac{1}{a}+2\right)e^{s\tau}}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

$$= \lim_{s\to-\left(n_0+\frac{1}{a}+2\right)} \frac{e^{s\tau}}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{\left[-\left(n_0+\frac{1}{a}+2\right)+n_0+\frac{1}{a}\right]\left[-\left(n_0+\frac{1}{a}+2\right)+n_0+\frac{1}{a}+1\right]\left[-\left(n_0+\frac{1}{a}+2\right)+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{\left[-n_0-\frac{1}{a}-2+n_0+\frac{1}{a}\right]\left[-n_0-\frac{1}{a}-2+n_0+\frac{1}{a}+1\right]\left[-n_0-\frac{1}{a}-2+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{-2\,(-1)\,(1)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{2!}$$

At $s=-\left(n_0+\frac{1}{a}+3\right)$

$$a_{-i}=\lim_{s\to-\left(n_0+\frac{1}{a}+3\right)}\frac{\left(s+n_0+\frac{1}{a}+3\right)e^{s\tau}}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

$$=\lim_{s\to-\left(n_0+\frac{1}{a}+3\right)}\frac{e^{s\tau}}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{\left[-\left(n_0+\frac{1}{a}+3\right)+n_0+\frac{1}{a}\right]\left[-\left(n_0+\frac{1}{a}+3\right)+n_0+\frac{1}{a}+1\right]\left[-\left(n_0+\frac{1}{a}+3\right)+n_0+\frac{1}{a}+2\right]}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}\right]\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+1\right]\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+2\right]}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{-3\left(-2\right)\left(-1\right)}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{-3!}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}=\sum a_{-i}$$

But

$$\sum a_{-i}=\frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{3!}-\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{2!}+\frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{2!}-\frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{3!}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{3!}\left(1-\frac{3!}{2!}e^{-\tau}+\frac{3!}{2!}e^{-2\tau}-e^{-3\tau}\right)$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{3!}\left(1-3e^{-\tau}+3e^{-2\tau}-e^{-3\tau}\right)$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{3!}\left(1-e^{-\tau}\right)^3$$

Therefore

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}\right)\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}$$

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3\frac{e^{-\left(n_0+\frac{1}{a}\right)\tau}}{3!}\left(1-e^{-\tau}\right)^3$$

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3$$

Consolidating the above results we get

$$L^{-1}\{M\} = \left\{\begin{array}{c} e^{-\left(n_0+\frac{1}{a}\right)\tau} + \left(n_0+\frac{1}{a}+1\right)ze^{-\left(n_0+\frac{1}{a}\right)\tau}\left(1-e^{-\tau}\right) + \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2 + \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3 + ... \end{array}\right\}$$

In a similar fashion

$$L^{-1}\{N\} = L^{-1}\left\{\begin{array}{c} \frac{1}{\left(s+n_0+\frac{1}{a}+1\right)} + \frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)} + \\ \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)} + ... \end{array}\right\}$$

$$
= \left[\begin{array}{c} L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)}\right\} + L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\} + \\ L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\} + \\ L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}\right\} + \ldots \end{array} \right]
$$

We now simplify the first four terms of the above equation separately

First term
From the table of transform pairs

$$
\begin{aligned}
&L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \\
&\Rightarrow L^{-1}\left\{\frac{1}{s+n_0+\frac{1}{a}+1}\right\} = e^{-\left(n_0+\frac{1}{a}+1\right)\tau}
\end{aligned}
$$

Second term

$$
L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\}
$$

$$
= \left(n_0+\frac{1}{a}+1\right) z L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\}
$$

The function

$$
\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}
$$

has simple poles at $s = -\left(n_0+\frac{1}{a}+1\right)$ and $s = -\left(n_0+\frac{1}{a}+2\right)$

Its residue at each pole is obtained as follows

At $s = -\left(n_0 + \frac{1}{a} + 1\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{\left(s + n_0 + \frac{1}{a} + 1\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 1\right) + n_0 + \frac{1}{a} + 2\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-n_0 - \frac{1}{a} - 1 + n_0 + \frac{1}{a} + 2\right]}$$

$$= e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}$$

At $s = -\left(n_0 + \frac{1}{a} + 2\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 2\right)} \frac{\left(s + n_0 + \frac{1}{a} + 2\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 2\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 2\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 1\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{\left[-n_0-\frac{1}{a}-2+n_0+\frac{1}{a}+1\right]}$$

$$= -e^{-\left(n_0+\frac{1}{a}+2\right)\tau}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\} = \sum a_{-i}$$
$$= e^{-\left(n_0+\frac{1}{a}+1\right)\tau} - e^{-\left(n_0+\frac{1}{a}+2\right)\tau}$$
$$= e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right)$$

Therefore

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)z}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}\right\} = \left(n_0+\frac{1}{a}+1\right)ze^{-\left(n_0+\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right)$$

Third term

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}$$

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}$$

The function

$$\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

has simple poles at $s=-\left(n_0+\frac{1}{a}+1\right)$, $s=-\left(n_0+\frac{1}{a}+2\right)$ and $s=-\left(n_0+\frac{1}{a}+3\right)$

Its residue at each pole is obtained as follows

At $s = -\left(n_0 + \frac{1}{a} + 1\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{\left(s + n_0 + \frac{1}{a} + 1\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 1\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 1\right) + n_0 + \frac{1}{a} + 2\right]\left[-\left(n_0 + \frac{1}{a} + 1\right) + n_0 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{\left[-n_0 - \frac{1}{a} - 1 + n_0 + \frac{1}{a} + 2\right]\left[-n_0 - \frac{1}{a} - 1 + n_0 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{1\,(2)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 1\right)\tau}}{2!}$$

At $s = -\left(n_0 + \frac{1}{a} + 2\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 2\right)} \frac{\left(s + n_0 + \frac{1}{a} + 2\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 2\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 3\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 2\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 1\right]\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{\left[-n_0-\frac{1}{a}-2+n_0+\frac{1}{a}+1\right]\left[-n_0-\frac{1}{a}-2+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{-1\,(1)}$$

$$= -e^{-\left(n_0+\frac{1}{a}+2\right)\tau}$$

At $s = -\left(n_0 + \frac{1}{a} + 3\right)$

$$a_{-i} = \lim_{s\to-\left(n_0+\frac{1}{a}+3\right)} \frac{\left(s+n_0+\frac{1}{a}+3\right)e^{s\tau}}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

$$= \lim_{s\to-\left(n_0+\frac{1}{a}+3\right)} \frac{e^{s\tau}}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{\left[-\left(n_0+\frac{1}{a}+3\right)+n_0+\frac{1}{a}+1\right]\left[-\left(n_0+\frac{1}{a}+3\right)+n_0+\frac{1}{a}+2\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+1\right]\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+2\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{-2\,(-1)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{2!}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\} = \sum a_{-i}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{2!} - e^{-\left(n_0+\frac{1}{a}+2\right)\tau} + \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{2!}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{2!} \left(1 - 2e^{-\tau} + e^{-2\tau}\right)$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{2!} \left(1 - e^{-\tau}\right)^2$$

Therefore

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}\right\}$$

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)z^2\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{2!}\left(1-e^{-\tau}\right)^2$$

$$= \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2$$

Fourth term

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}\right\}$$

$$= \left\{\begin{array}{c}\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3 * \\ \\ L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}\right\}\end{array}\right\}$$

The function

$$\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}$$

has simple poles at

$$s=-\left(n_0+\frac{1}{a}+1\right),\ \ s=-\left(n_0+\frac{1}{a}+2\right),\ \ s=-\left(n_0+\frac{1}{a}+3\right) \text{ and } s=-\left(n_0+\frac{1}{a}+4\right)$$

Thus its residue at each pole is obtained as follows

At $s=-\left(n_0+\frac{1}{a}+1\right)$

$$a_{-i}=\lim_{s\to-\left(n_0+\frac{1}{a}+1\right)}\frac{\left(s+n_0+\frac{1}{a}+1\right)e^{s\tau}}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}$$

$$=\lim_{s\to-\left(n_0+\frac{1}{a}+1\right)}\frac{e^{s\tau}}{\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{\left[-\left(n_0+\frac{1}{a}+1\right)+n_0+\frac{1}{a}+2\right]\left[-\left(n_0+\frac{1}{a}+1\right)+n_0+\frac{1}{a}+3\right]\left[-\left(n_0+\frac{1}{a}+1\right)+n_0+\frac{1}{a}+4\right]}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{\left[-n_0-\frac{1}{a}-1+n_0+\frac{1}{a}+2\right]\left[-n_0-\frac{1}{a}-1+n_0+\frac{1}{a}+3\right]\left[-n_0-\frac{1}{a}-1+n_0+\frac{1}{a}+4\right]}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{1\,(2)\,(3)}$$

$$=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{3!}$$

At $s = -\left(n_0 + \frac{1}{a} + 2\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 2\right)} \frac{\left(s + n_0 + \frac{1}{a} + 2\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)\left(s + n_0 + \frac{1}{a} + 4\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 2\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 3\right)\left(s + n_0 + \frac{1}{a} + 4\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 2\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 1\right]\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 3\right]\left[-\left(n_0 + \frac{1}{a} + 2\right) + n_0 + \frac{1}{a} + 4\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 2\right)\tau}}{\left[-n_0 - \frac{1}{a} - 2 + n_0 + \frac{1}{a} + 1\right]\left[-n_0 - \frac{1}{a} - 2 + n_0 + \frac{1}{a} + 3\right]\left[-n_0 - \frac{1}{a} - 2 + n_0 + \frac{1}{a} + 4\right]}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 2\right)\tau}}{-1\,(1)\,(2)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 2\right)\tau}}{-2!}$$

At $s = -\left(n_0 + \frac{1}{a} + 3\right)$

$$a_{-i} = \lim_{s \to -\left(n_0 + \frac{1}{a} + 3\right)} \frac{\left(s + n_0 + \frac{1}{a} + 3\right) e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 3\right)\left(s + n_0 + \frac{1}{a} + 4\right)}$$

$$= \lim_{s \to -\left(n_0 + \frac{1}{a} + 3\right)} \frac{e^{s\tau}}{\left(s + n_0 + \frac{1}{a} + 1\right)\left(s + n_0 + \frac{1}{a} + 2\right)\left(s + n_0 + \frac{1}{a} + 4\right)}$$

$$= \frac{e^{-\left(n_0 + \frac{1}{a} + 3\right)\tau}}{\left[-\left(n_0 + \frac{1}{a} + 3\right) + n_0 + \frac{1}{a} + 1\right]\left[-\left(n_0 + \frac{1}{a} + 3\right) + n_0 + \frac{1}{a} + 2\right]\left[-\left(n_0 + \frac{1}{a} + 3\right) + n_0 + \frac{1}{a} + 4\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+1\right]\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+2\right]\left[-n_0-\frac{1}{a}-3+n_0+\frac{1}{a}+4\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{-2\,(-1)\,(1)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{2!}$$

At $s = -\left(n_0 + \frac{1}{a} + 4\right)$

$$a_{-i} = \lim_{s\to-\left(n_0+\frac{1}{a}+4\right)} \frac{\left(s+n_0+\frac{1}{a}+4\right)e^{s\tau}}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}$$

$$= \lim_{s\to-\left(n_0+\frac{1}{a}+4\right)} \frac{e^{s\tau}}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+4\right)\tau}}{\left[-\left(n_0+\frac{1}{a}+4\right)+n_0+\frac{1}{a}+1\right]\left[-\left(n_0+\frac{1}{a}+4\right)+n_0+\frac{1}{a}+2\right]\left[-\left(n_0+\frac{1}{a}+4\right)+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+4\right)\tau}}{\left[-n_0-\frac{1}{a}-4+n_0+\frac{1}{a}+1\right]\left[-n_0-\frac{1}{a}-4+n_0+\frac{1}{a}+2\right]\left[-n_0-\frac{1}{a}-4+n_0+\frac{1}{a}+3\right]}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+4\right)\tau}}{-3\,(-2)\,(-1)}$$

$$= \frac{e^{-\left(n_0+\frac{1}{a}+4\right)\tau}}{-3!}$$

Thus

$$L^{-1}\left\{\frac{1}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}\right\}=\sum a_{-i}$$

$$\begin{aligned}
&=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{3!}-\frac{e^{-\left(n_0+\frac{1}{a}+2\right)\tau}}{2!}+\frac{e^{-\left(n_0+\frac{1}{a}+3\right)\tau}}{2!}-\frac{e^{-\left(n_0+\frac{1}{a}+4\right)\tau}}{3!}\\
&=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{3!}\left(1-\frac{3!}{2!}e^{-\tau}+\frac{3!}{2!}e^{-2\tau}-e^{-3\tau}\right)\\
&=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{3!}\left(1-3e^{-\tau}+3e^{-2\tau}-e^{-3\tau}\right)\\
&=\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{3!}\left(1-e^{-\tau}\right)^3
\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3}{\left(s+n_0+\frac{1}{a}+1\right)\left(s+n_0+\frac{1}{a}+2\right)\left(s+n_0+\frac{1}{a}+3\right)\left(s+n_0+\frac{1}{a}+4\right)}\right\}$$

$$\begin{aligned}
&=\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)z^3\frac{e^{-\left(n_0+\frac{1}{a}+1\right)\tau}}{3!}\left(1-e^{-\tau}\right)^3\\
&=\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3
\end{aligned}$$

Consolidating the above results We get

$$L^{-1}\left\{N\right\}=\left\{\begin{array}{c}
e^{-\left(n_0+\frac{1}{a}+1\right)\tau}+\left(n_0+\frac{1}{a}+1\right)ze^{-\left(n_0+\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right)+\\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2+\\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3+...
\end{array}\right\}$$

Its now time to go back to our expression/equation for $G(z,t)$ .Recall that we had

$$G(z,t) = z^{n_0}\left[(1-z)L^{-1}\{M\} + zL^{-1}\{N\}\right]$$

Thus

$$G(z,t) = z^{n_0}\left\{\begin{array}{c}(1-z)\left[\begin{array}{c} e^{-\left(n_0+\frac{1}{a}\right)\tau} + \left(n_0+\frac{1}{a}+1\right) z e^{-\left(n_0+\frac{1}{a}\right)\tau}\left(1-e^{-\tau}\right) + \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2 + \\ \left[\begin{array}{c}\left(n_0+\frac{1}{a}+1\right)* \\ \left(n_0+\frac{1}{a}+2\right)* \\ \left(n_0+\frac{1}{a}+3\right)e^{-\left(n_0+\frac{1}{a}\right)\tau}\end{array}\right]\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3 + ... \end{array}\right] \\ +z\left[\begin{array}{c} e^{-\left(n_0+\frac{1}{a}+1\right)\tau} + \left(n_0+\frac{1}{a}+1\right) z e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right) + \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2 + \\ \left[\begin{array}{c}\left(n_0+\frac{1}{a}+1\right)* \\ \left(n_0+\frac{1}{a}+2\right)* \\ \left(n_0+\frac{1}{a}+3\right)e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\end{array}\right]\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3 + ... \end{array}\right]\end{array}\right\}$$

Simplifying the terms in the brackets we obtain

$$
G\left(z,t\right) = z^{n_0} \left\{ \begin{array}{c}
e^{-\left(n_0+\frac{1}{a}\right)\tau} + \left(n_0+\frac{1}{a}+1\right) z e^{-\left(n_0+\frac{1}{a}\right)\tau}\left(1-e^{-\tau}\right) + \\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2 + \\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right) e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3 - \\
z e^{-\left(n_0+\frac{1}{a}\right)\tau} - \left(n_0+\frac{1}{a}+1\right) z^2 e^{-\left(n_0+\frac{1}{a}\right)\tau}\left(1-e^{-\tau}\right) - \\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^3}{2!}\left(1-e^{-\tau}\right)^2 - \\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right) e^{-\left(n_0+\frac{1}{a}\right)\tau}\frac{z^4}{3!}\left(1-e^{-\tau}\right)^3 + \\
z e^{-\left(n_0+\frac{1}{a}+1\right)\tau} + \left(n_0+\frac{1}{a}+1\right) z^2 e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\left(1-e^{-\tau}\right) + \\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^3}{2!}\left(1-e^{-\tau}\right)^2 + \\
\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right) e^{-\left(n_0+\frac{1}{a}+1\right)\tau}\frac{z^4}{3!}\left(1-e^{-\tau}\right)^3 + ...
\end{array} \right\}
$$

Factoring $e^{-\left(n_0+\frac{1}{a}\right)\tau}$ out we get

$$G\left(z,t\right)=z^{n_0}e^{-\left(n_0+\frac{1}{a}\right)\tau}\left\{\begin{array}{c} 1+\left(n_0+\frac{1}{a}+1\right)z\left(1-e^{-\tau}\right)+ \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\frac{z^2}{2!}\left(1-e^{-\tau}\right)^2+ \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)\frac{z^3}{3!}\left(1-e^{-\tau}\right)^3- \\ z+\left(n_0+\frac{1}{a}+1\right)z^2\left(1-e^{-\tau}\right)- \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\frac{z^3}{2!}\left(1-e^{-\tau}\right)^2- \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)\frac{z^4}{3!}\left(1-e^{-\tau}\right)^3+ \\ ze^{-\tau}+\left(n_0+\frac{1}{a}+1\right)z^2e^{-\tau}\left(1-e^{-\tau}\right)+ \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)e^{-\tau}\frac{z^3}{2!}\left(1-e^{-\tau}\right)^2+ \\ \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right)\left(n_0+\frac{1}{a}+3\right)e^{-\tau}\frac{z^4}{3!}\left(1-e^{-\tau}\right)^3+... \end{array}\right\}$$

At this juncture, we can decide to rewrite $G(z,t)$ as

$$G\left(z,t\right)=z^{n_0}e^{-\left(n_0+\frac{1}{a}\right)\tau}Q$$

Where

$$Q = \left\{ \begin{array}{c} 1 + \left(n_0 + \frac{1}{a} + 1\right) z\left(1 - e^{-\tau}\right) + \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\frac{z^2}{2!}\left(1 - e^{-\tau}\right)^2 + \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\left(n_0 + \frac{1}{a} + 3\right)\frac{z^3}{3!}\left(1 - e^{-\tau}\right)^3 - \\ z + \left(n_0 + \frac{1}{a} + 1\right) z^2\left(1 - e^{-\tau}\right) - \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\frac{z^3}{2!}\left(1 - e^{-\tau}\right)^2 - \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\left(n_0 + \frac{1}{a} + 3\right)\frac{z^4}{3!}\left(1 - e^{-\tau}\right)^3 + \\ ze^{-\tau} + \left(n_0 + \frac{1}{a} + 1\right) z^2 e^{-\tau}\left(1 - e^{-\tau}\right) + \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right) e^{-\tau}\frac{z^3}{2!}\left(1 - e^{-\tau}\right)^2 + \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\left(n_0 + \frac{1}{a} + 3\right) e^{-\tau}\frac{z^4}{3!}\left(1 - e^{-\tau}\right)^3 + ... \end{array} \right\}$$

In order to simplify Q we put together /factor the coefficients of corresponding powers of $\left(1 - e^{-\tau}\right)$. To avoid confusion we shall make use of the following table.

With this we now have

$$Q = \left\{ \begin{array}{c} 1 - z + ze^{-\tau} + \\ \left[\left(n_0 + \frac{1}{a} + 1\right) z - \left(n_0 + \frac{1}{a} + 1\right) z^2 + \left(n_0 + \frac{1}{a} + 1\right) z^2 e^{-\tau} \right] (1 - e^{-\tau}) + \\ \left[\begin{array}{c} \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right) \frac{z^2}{2!} - \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right) \frac{z^3}{2!} + \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right) \frac{z^3}{2!} e^{-\tau} \end{array} \right] (1 - e^{-\tau})^2 + \\ \left[\begin{array}{c} \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\left(n_0 + \frac{1}{a} + 3\right) \frac{z^3}{3!} - \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\left(n_0 + \frac{1}{a} + 3\right) \frac{z^4}{3!} + \\ \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right)\left(n_0 + \frac{1}{a} + 3\right) \frac{z^4}{3!} e^{-\lambda t} \end{array} \right] (1 - e^{-\tau})^3 + ... \end{array} \right\}$$

Factoring common terms, we get

$$Q = \left\{ \begin{array}{c} 1 - z + ze^{-\tau} + \left(n_0 + \frac{1}{a} + 1\right) z \left[1 - z + ze^{-\tau}\right] \left(1 - e^{-\tau}\right) + \\ \\ \left(n_0 + \frac{1}{a} + 1\right) \left(n_0 + \frac{1}{a} + 2\right) \frac{z^2}{2!} \left[1 - z + ze^{-\tau}\right] \left(1 - e^{-\tau}\right)^2 + \\ \\ \left(n_0 + \frac{1}{a} + 1\right) \left(n_0 + \frac{1}{a} + 2\right) \left(n_0 + \frac{1}{a} + 3\right) \frac{z^3}{3!} \left[1 - z + ze^{-\lambda t}\right] \left(1 - e^{-\tau}\right)^3 + ... \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left[1 - z\left(1 - e^{-\tau}\right)\right] + \left(n_0 + \frac{1}{a} + 1\right) z \left[1 - z\left(1 - e^{-\tau}\right)\right] \left(1 - e^{-\tau}\right) + \\ \\ \left(n_0 + \frac{1}{a} + 1\right) \left(n_0 + \frac{1}{a} + 2\right) \frac{z^2}{2!} \left[1 - z\left(1 - e^{-\tau}\right)\right] \left(1 - e^{-\tau}\right)^2 + \\ \\ \left(n_0 + \frac{1}{a} + 1\right) \left(n_0 + \frac{1}{a} + 2\right) \left(n_0 + \frac{1}{a} + 3\right) \frac{z^3}{3!} \left[1 - z\left(1 - e^{-\tau}\right)\right] \left(1 - e^{-\tau}\right)^3 + ... \end{array} \right\}$$

Factoring $1 - z\left(1 - e^{-\tau}\right)$ out yields

$$Q = \left[1 - z\left(1 - e^{-\tau}\right)\right] \left\{ \begin{array}{c} 1 + \left(n_0 + \frac{1}{a} + 1\right) z \left(1 - e^{-\tau}\right) + \\ \\ \left(n_0 + \frac{1}{a} + 1\right) \left(n_0 + \frac{1}{a} + 2\right) \frac{z^2}{2!} \left(1 - e^{-\tau}\right)^2 + \\ \\ \left(n_0 + \frac{1}{a} + 1\right) \left(n_0 + \frac{1}{a} + 2\right) \left(n_0 + \frac{1}{a} + 3\right) \frac{z^3}{3!} \left(1 - e^{-\tau}\right)^3 + ... \end{array} \right\}$$

Multiplying the RHS by $\frac{\left(n_0 + \frac{1}{a}\right)!}{\left(n_0 + \frac{1}{a}\right)!}$ yields

$$Q = \left[1 - z\left(1 - e^{-\tau}\right)\right] \frac{\left(n_0 + \frac{1}{a}\right)!}{\left(n_0 + \frac{1}{a}\right)!} \left\{ \begin{array}{c} 1 + \left(n_0 + \frac{1}{a} + 1\right) z\left(1 - e^{-\tau}\right) + \\ \\ \left[\begin{array}{c} \left(n_0 + \frac{1}{a} + 1\right) * \\ \\ \left(n_0 + \frac{1}{a} + 2\right) \end{array} \right] \frac{z^2}{2!}\left(1 - e^{-\tau}\right)^2 + \\ \\ \left[\begin{array}{c} \left(n_0 + \frac{1}{a} + 1\right) * \\ \\ \left(n_0 + \frac{1}{a} + 2\right) * \\ \\ \left(n_0 + \frac{1}{a} + 3\right) \end{array} \right] \frac{z^3}{3!}\left(1 - e^{-\tau}\right)^3 + ... \end{array} \right\}$$

$$= \left[1 - z\left(1 - e^{-\tau}\right)\right] \left\{ \begin{array}{c} \frac{\left(n_0 + \frac{1}{a}\right)!}{\left(n_0 + \frac{1}{a}\right)!} + \frac{\left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a}\right)!}{\left(n_0 + \frac{1}{a}\right)!} z\left(1 - e^{-\tau}\right) + \\ \\ \left[\begin{array}{c} \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right) * \\ \\ \frac{\left(n_0 + \frac{1}{a}\right)!}{\left(n_0 + \frac{1}{a}\right)!} \end{array} \right] \frac{z^2}{2!}\left(1 - e^{-\tau}\right)^2 + \\ \\ \left[\begin{array}{c} \left(n_0 + \frac{1}{a} + 1\right)\left(n_0 + \frac{1}{a} + 2\right) * \\ \\ \frac{\left(n_0 + \frac{1}{a} + 3\right)\left(n_0 + \frac{1}{a}\right)!}{\left(n_0 + \frac{1}{a}\right)!} \end{array} \right] \frac{z^3}{3!}\left(1 - e^{-\tau}\right)^3 + ... \end{array} \right\}$$

We thus have

$$Q = \left[1 - z\left(1-e^{-\tau}\right)\right] \left\{ \begin{array}{c} 1 + \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}\right)!}{\left(n_0+\frac{1}{a}\right)!} z\left(1-e^{-\tau}\right) + \\ \\ \left[\begin{array}{c} \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) * \\ \\ \frac{\left(n_0+\frac{1}{a}\right)!}{\left(n_0+\frac{1}{a}\right)!} \end{array} \right] \frac{z^2}{2!}\left(1-e^{-\tau}\right)^2 + \\ \\ \left[\begin{array}{c} \left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}+2\right) * \\ \\ \frac{\left(n_0+\frac{1}{a}+3\right)\left(n_0+\frac{1}{a}\right)!}{\left(n_0+\frac{1}{a}\right)!} \end{array} \right] \frac{z^3}{3!}\left(1-e^{-\tau}\right)^3 + ... \end{array} \right\}$$

Since

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We have

$$\binom{n_0+\frac{1}{a}+0}{0} = 1$$

$$\begin{aligned} \binom{n_0+\frac{1}{a}+1}{1} &= \frac{\left(n_0+\frac{1}{a}+1\right)!}{\left(n_0+\frac{1}{a}\right)!1!} \\ &= \frac{\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}\right)!}{\left(n_0+\frac{1}{a}\right)!1!} \end{aligned}$$

$$\begin{aligned} \binom{n_0+\frac{1}{a}+2}{2} &= \frac{\left(n_0+\frac{1}{a}+2\right)!}{\left(n_0+\frac{1}{a}\right)!2!} \\ &= \frac{\left(n_0+\frac{1}{a}+2\right)!\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}\right)!}{\left(n_0+\frac{1}{a}\right)!2!} \end{aligned}$$

$$\binom{n_0+\frac{1}{a}+3}{3} = \frac{\left(n_0+\frac{1}{a}+3\right)!}{\left(n_0+\frac{1}{a}\right)!3!}$$

$$= \frac{\left(n_0+\frac{1}{a}+3\right)\left(n_0+\frac{1}{a}+2\right)!\left(n_0+\frac{1}{a}+1\right)\left(n_0+\frac{1}{a}\right)!}{\left(n_0+\frac{1}{a}\right)!3!}$$

And so on

With this it follows that;

$$Q = \left[1 - z\left(1-e^{-\tau}\right)\right]\left\{\begin{matrix} 1 + \binom{n_0+\frac{1}{a}+1}{1} z\left(1-e^{-\tau}\right) + \binom{n_0+\frac{1}{a}+2}{2}\left[z\left(1-e^{-\tau}\right)\right]^2 + \\ \binom{n_0+\frac{1}{a}+3}{3}\left[z\left(1-e^{-\tau}\right)\right]^3 + \ldots \end{matrix}\right\}$$

$$= \left[1 - z\left(1-e^{-\tau}\right)\right]\sum_{j=0}^{\infty}\binom{n_0+\frac{1}{a}+j}{j}\left[z\left(1-e^{-\tau}\right)\right]^j$$

But

$$\binom{n_0+\frac{1}{a}+j}{j} = \binom{\left[n_0+\frac{1}{a}+1\right]+j-1}{j}$$

Thus

$$Q = \left[1 - z\left(1-e^{-\tau}\right)\right]\sum_{j=0}^{\infty}\binom{\left[n_0+\frac{1}{a}+1\right]+j-1}{j}\left[z\left(1-e^{-\tau}\right)\right]^j$$

Also

$$\binom{-r}{j}(-1)^j = \binom{r+j-1}{j}$$

$$\Rightarrow \binom{-\left[n_0+\frac{1}{a}+1\right]}{j}(-1)^j = \binom{\left[n_0+\frac{1}{a}+1\right]+j-1}{j}$$

Therefore

$$Q = \left[1 - z\left(1 - e^{-\tau}\right)\right] \sum_{j=0}^{\infty} \binom{-\left[n_0 + \frac{1}{a} + 1\right]}{j} (-1)^j \left[z\left(1 - e^{-\tau}\right)\right]^j$$

$$= \left[1 - z\left(1 - e^{-\tau}\right)\right] \sum_{j=0}^{\infty} \binom{-\left[n_0 + \frac{1}{a} + 1\right]}{j} \left[-z\left(1 - e^{-\tau}\right)\right]^j$$

$$= \left[1 - z\left(1 - e^{-\tau}\right)\right] \left[1 - z\left(1 - e^{-\tau}\right)\right]^{-\left(n_0 + \frac{1}{a} + 1\right)}$$
$$= \left[1 - z\left(1 - e^{-\tau}\right)\right]^{-\left(n_0 + \frac{1}{a}\right)}$$

Recollecting our memory, we had obtained $G(z, \tau)$ as

$$G\left(z, \tau\right) = z^{n_0} e^{-\left(n_0 + \frac{1}{a}\right)\tau} Q$$

Thus

$$G\left(z, \tau\right) = z^{n_0} e^{-\left(n_0 + \frac{1}{a}\right)\tau} \left[1 - z\left(1 - e^{-\tau}\right)\right]^{-\left(n_0 + \frac{1}{a}\right)} \qquad = z^{n_0} \left[\frac{e^{-\tau}}{1 - z\left(1 - e^{-\tau}\right)}\right]^{n_0 + \frac{1}{a}}$$

We now have something to smile about! The remaining task is to rewrite $G(z, \tau)$ in terms of the original variable t. Recall the initial substitution we made

$$\tau = \ln\left|\frac{t + \alpha}{\alpha}\right|$$

Where

$$\alpha = \frac{1}{\lambda a}$$

Thus

$$
\begin{aligned}
-\tau &= -\ln\left|\frac{t+\alpha}{\alpha}\right| \\
&= \ln\left|\frac{t+\alpha}{\alpha}\right|^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow e^{-\tau} &= e^{\ln\left|\frac{t+\alpha}{\alpha}\right|^{-1}} \\
&= \left|\frac{t+\alpha}{\alpha}\right|^{-1} \\
&= \left|\frac{t}{\alpha}+1\right|^{-1} \\
&= \left|\frac{t}{1/\lambda a}+1\right|^{-1} \\
&= (\lambda a t+1)^{-1}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G(z,t) &= z^{n_0}\left[\frac{(1+\lambda a t)^{-1}}{1-z\left[1-(1+\lambda a t)^{-1}\right]}\right]^{n_0+\frac{1}{a}} \\
&= z^{n_0}\left[\frac{1}{(1+\lambda a t)\left(1-z\left[1-\frac{1}{1+\lambda a t}\right]\right)}\right]^{n_0+\frac{1}{a}} \\
&= z^{n_0}\left[\frac{1}{(1+\lambda a t)\left(1-z\left[\frac{1+\lambda a t-1}{1+\lambda a t}\right]\right)}\right]^{n_0+\frac{1}{a}} \\
&= z^{n_0}\left[\frac{1}{(1+\lambda a t)\left(1-\frac{z\lambda a t}{1+\lambda a t}\right)}\right]^{n_0+\frac{1}{a}} \\
&= z^{n_0}\left[\frac{1}{1+\lambda a t-\lambda a t z}\right]^{n_0+\frac{1}{a}}
\end{aligned}
$$

We finally have

$$G(z,t) = z^{n_0}\left[\frac{\frac{1}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{n_0+\frac{1}{a}}$$

$$= z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{n_0+\frac{1}{a}}$$

Letting $p = \frac{1}{1+\lambda at}$ and $q = 1 - \frac{1}{1+\lambda at} = \frac{\lambda at}{1+\lambda at}$

$G(z,t)$ can be expressed as

$$G(z,t) = z^{-\frac{1}{a}}\left[\frac{zp}{1-qz}\right]^{n_0+\frac{1}{a}}$$

$P_n(t)$ is the coefficient of z^n in the expansion of $G(z,t)$. By binomial expansion

$$G(z,t) = z^{-\frac{1}{a}} z^{n_0+\frac{1}{a}} p^{n_0+\frac{1}{a}} \left[1-qz\right]^{-\left(n_0+\frac{1}{a}\right)}$$

$$= z^{n_0} p^{n_0+\frac{1}{a}} \sum_{k=0}^{\infty} \binom{-\left[n_0+\frac{1}{a}\right]}{k} (-qz)^k$$

$$= z^{n_0} p^{n_0+\frac{1}{a}} \sum_{k=0}^{\infty} \binom{-\left[n_0+\frac{1}{a}\right]}{k} (-1)^k q^k z^k$$

Recall that

$$\binom{-r}{k}(-1)^k = \binom{r+k-1}{k}$$

$$\Rightarrow \binom{-\left[n_0+\frac{1}{a}\right]}{k}(-1)^k = \binom{\left[n_0+\frac{1}{a}\right]+k-1}{k}$$

Thus

$$G(z,t) = z^{n_0} p^{n_0+\frac{1}{a}} \sum_{k=0}^{\infty} \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} q^k z^k$$

$$= \sum_{k=0}^{\infty} \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} p^{n_0+\frac{1}{a}} q^k z^{n_0+k}$$

From this it follows that

$$P_{n_0+k}\left(t\right) = \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} p^{n_0+\frac{1}{a}} q^k \qquad k = 0,\ 1,\ 2$$

but $n = n_0 + k$, Thus

$$P_n\left(t\right) = \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} p^{n_0+\frac{1}{a}} q^k \qquad k = 0,\ 1,\ 2$$

$$= \binom{\left[n_0 + \frac{1}{a}\right] + k - 1}{k} \left(\frac{1}{1+\lambda at}\right)^{n_0+\frac{1}{a}} \left(1 - \frac{1}{1+\lambda at}\right)^{k} \qquad k = 0,\ 1,\ 2$$

which is the pmf of a negative binomial distribution with parameters $r = n_0 + \frac{1}{a}$ and $p = \frac{1}{1+\lambda at}$

Chapter 6

Conclusion and Recommendation

6.1 Conclusion

As seen in all the special cases of pure birth the Laplace transform approach resulted in the same distribution as documented in cases where other methods were used. In particular the inversion methods applied in the first case yielded the same distributions. This was no different from the case where the Dirac delta function and the Gauss hyper geometric function were used to solve the DEs obtained while using the pgf technique. The only exception is in the case of the Polya where the parameter p differed. In a nut shell the distributions emerging from difference differential equations of a pure birth process are power series distributions. They are the Geometric distribution, the negative Binomial Distribution and the Poisson distribution

6.2 Recommendation

In this research, we confined ourselves to the pure birth process only. One possible recommendation is to extend it to other cases of the birth-death process.

6.3 Summary

6.3 Summary

1. **Poisson Process**

Parameter	$\lambda_n = \lambda$
Basic difference differential equations	$P_0'(t) = -\lambda P_0(t)$ $P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$
Method 1 :By Iteration	
$L\{P_n(t)\}$	$\dfrac{\lambda^n}{(s+\lambda)^{n+1}}$
$P_n(t)$	$\dfrac{e^{-\lambda t}\lambda t^n}{n!}$
Method 2 : pgf technique	
Partial Differential equation	$\dfrac{\partial}{\partial t}G(z,t) + \lambda(1-z)G(z,t) = 0$
$\bar{G}(z,s)$	$\dfrac{1}{s+\lambda(1-z)}$
$G(z,t)$	$e^{-\lambda(1-z)t}$
$P_n(t)$	$\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$

2. Simple Birth Process

Parameter	$\lambda_n = \lambda$	
Basic difference differential equations	$P_0'(t) = -\lambda P_0(t)$ $P_n'(t) = -n\lambda P_n(t) + \lambda(n-1)P_{n-1}(t)$	
Initial Condition	$X(0) = 1$	$X(0) = n_0$
	Method 1:By Iteration	
$L\{P_n(t)\}$	Not done explicitly	$\dfrac{\lambda^k \prod_{i=0}^{k-1}(n_0+i)}{\prod_{i=0}^{k}[s+(n_0+i)\lambda]}$
$P_n(t)$	$e^{-\lambda t}(1-e^{-\lambda t})^{n-1} \quad n = 1,2,3\ldots$	$\binom{n_0+k-1}{k}(e^{-\lambda t})^{n_0}(1-e^{-\lambda t})^k \quad k = 0,1,2\ldots$
	Method 2 : pgf technique	
Partial Differential equation	$\dfrac{\partial}{\partial t}G(z,t) - \lambda z(z-1)\dfrac{\partial}{\partial z}G(z,t) = 0$	

a) Dirac Approach		
Ordinary Differential equation	$\frac{d}{dz}\bar{G}(z,s)-\frac{s}{\lambda z(z-1)}\bar{G}(z,s)=-\frac{1}{\lambda(z-1)}$	$\frac{d}{dz}\bar{G}(z,s)-\frac{s}{\lambda z(z-1)}\bar{G}(z,s)=-\frac{z^{n_0}}{\lambda z(z-1)}$
$\bar{G}(z,s)e^{-\frac{s}{\lambda}\ln\left\|\frac{z-1}{z}\right\|}$	$=f(s)-\int\frac{1}{\lambda(z-1)}e^{-\frac{s}{\lambda}\ln\left\|\frac{z-1}{z}\right\|}dz$	$=f(s)-\int\frac{z^{n_0}}{\lambda z(z-1)}e^{-\frac{s}{\lambda}\ln\left\|\frac{z-1}{z}\right\|}dz$
$G(z,t)$	$\frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}$	$\left[\frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0}$
$P_n(t)$	$e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}\quad n=1,2,3\ldots$	$\binom{n_0+k-1}{k}\left(e^{-\lambda t}\right)^{n_0}\left(1-e^{-\lambda t}\right)^{k}\quad k=0,1,2\ldots$
a) Hypergeometric function approach		
Ordinary Differential equation	$\frac{d}{dz}\bar{G}(z,s)+\frac{s}{\lambda z(1-z)}\bar{G}(z,s)=\frac{1}{\lambda(1-z)}$	$\frac{d}{dz}\bar{G}(z,s)+\frac{s}{\lambda z(1-z)}\bar{G}(z,s)=\frac{z^{n_0}}{\lambda z(1-z)}$
$\bar{G}(z,s)$	$\frac{z}{\lambda\left(\frac{s}{\lambda}+1\right)}\,{}_2F_1\left(1,1;\frac{s}{\lambda}+2;z\right)$	$z^{n_0}\left\{\frac{(1-z)}{(s+\lambda n_0)}\,{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+1;z\right)+\frac{z}{\left[s+\lambda(n_0+1)\right]}\,{}_2F_1\left(n_0+1,1;\frac{s}{\lambda}+n_0+2;z\right)\right\}$
$G(z,t)$	$\frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}$	$z^{n_0}\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0+\frac{v}{\lambda}}$
$P_n(t)$	$e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}\quad n=1,2,3\ldots$	$\binom{n_0+k-1}{k}\left(e^{-\lambda t}\right)^{n_0}\left(1-e^{-\lambda t}\right)^{k}\quad k=0,1,2\ldots$

3. Simple birth with immigration

Parameter	$\lambda_n = n\lambda + v$	
Basic difference differential equations	$P_0'(t) = -vP_0(t)$ $P_n'(t) = [(n-1)\lambda + v]P_{n-1}(t) - (n\lambda + v)P_n(t)$	
Initial Population	$X(0)=1$	$X(0)=n_0$
Method 1:By Iteration		
$L\{P_n(t)\}$	Not done explicitly	$\dfrac{\prod_{i=0}^{k-1}\left[(n_0+i)\lambda+v\right]}{\prod_{i=0}^{k}\left[s+(n_0+i)\lambda+v\right]}$
$P_n(t)$	$\binom{\left[1+\frac{v}{\lambda}\right]+k-1}{k}\left(e^{-\lambda t}\right)^{1+\frac{v}{\lambda}}\left(1-e^{-\lambda t}\right)^k \quad k=0,1,\ldots$	$\binom{n_0+\frac{v}{\lambda}+k-1}{k}\left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}}\left(1-e^{-\lambda t}\right)^k \quad k=0,1,2,\ldots$
Method 2 : pgf technique		
Partial Differential equation	$\frac{\partial}{\partial t}G(z,t) - \lambda z(z-1)\frac{\partial}{\partial z}G(z,t) - v(z-1)G(z,t) = 0$	
a) Dirac Approach		
Ordinary Differential equation	$\frac{d}{dz}\bar{G}(z,s) - \frac{\left[s-v(z-1)\right]}{\lambda z(z-1)}\bar{G}(z,s) = -\frac{1}{\lambda(z-1)}$	$\frac{d}{dz}\bar{G}(z,s) - \frac{\left[s-v(z-1)\right]}{\lambda z(z-1)}\bar{G}(z,s) = -\frac{z^{n_0}}{\lambda(z-1)}$
$\bar{G}(z,s)\lvert z\rvert^{\frac{v}{\lambda}} e^{-\frac{s}{\lambda}\ln\left\lvert\frac{z-1}{z}\right\rvert}$	$= f(s) - \int \frac{1}{\lambda(z-1)}\lvert z\rvert^{\frac{v}{\lambda}} e^{-\frac{s}{\lambda}\ln\left\lvert\frac{z-1}{z}\right\rvert}dz$	$= f(s) - \int \frac{z^{n_0}}{\lambda z(z-1)}\lvert z\rvert^{\frac{v}{\lambda}} e^{-\frac{s}{\lambda}\ln\left\lvert\frac{z-1}{z}\right\rvert}dz$
$G(z,t)$	$z\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{1+\frac{v}{\lambda}}$	$z^{n_0}\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0+\frac{v}{\lambda}}$
$P_n(t)$	$\binom{\frac{v}{\lambda}+1+k-1}{k}\left(e^{-\lambda t}\right)^{\frac{v}{\lambda}+1}\left(1-e^{-\lambda t}\right)^k$	$\binom{\left[n_0+\frac{v}{\lambda}\right]+k-1}{k}\left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}}\left(1-e^{-\lambda t}\right)^k$

b) Hyper geometric function approach		
Ordinary Differential equation	$\frac{d}{dz}\bar{G}(z,s)+\frac{\left[s+v(1-z)\right]}{\lambda z(1-z)}\bar{G}(z,s)=\frac{1}{\lambda(1-z)}$	$\frac{d}{dz}\bar{G}(z,s)+\frac{\left[s+v(1-z)\right]}{\lambda z(1-z)}\bar{G}(z,s)=\frac{z^{n_0}}{\lambda z(1-z)}$
$\bar{G}(z,s)$	$\frac{z}{s+v+\lambda}\ {}_2F_1\left(\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+\frac{v}{\lambda}+2;z\right)$	$z^{n_0}\left\{\frac{(1-z)\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+1;z\right)}{(s+v+\lambda n_0)}+\frac{z\ {}_2F_1\left(n_0+\frac{v}{\lambda}+1,1;\frac{s}{\lambda}+n_0+\frac{v}{\lambda}+2;z\right)}{\left[s+v+\lambda(n_0+1)\right]}\right\}$
$G(z,t)$	$z^{-\frac{v}{\lambda}}\left[\frac{ze^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{\frac{v}{\lambda}+1}$	$z^{n_0}\left[\frac{e^{-\lambda t}}{1-z\left(1-e^{-\lambda t}\right)}\right]^{n_0+\frac{v}{\lambda}}$
$P_n(t)$	$\binom{\left[\frac{v}{\lambda}+1\right]+k-1}{k}(e^{-\lambda t})^{\frac{v}{\lambda}+1}(1-e^{-\lambda t})^{k}$ $k=0,1,2\ldots$	$\binom{\left[n_0+\frac{v}{\lambda}\right]+k-1}{k}\left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}}\left(1-e^{-\lambda t}\right)^{k}\quad k=0,1,2,\ldots$

4. Polya process

Parameter	$\lambda_n = \lambda\left(\frac{1+an}{1+\lambda at}\right)$	
Basic difference differential equations	$P_0'(t) = -\lambda\left(\frac{1}{1+\lambda at}\right)P_0(t)$ $P_n'(t) = -\lambda\left(\frac{1+an}{1+\lambda at}\right)P_n(t) + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right)P_{n-1}(t)$	
Initial Population	When $X(0)=1$	When $X(0)=n_0$
Method 1: By Iteration		
$L\{P_n(t)\}$	Not done explicitly	$\frac{\prod_{i=0}^{k-1}\lambda[(1+a(n_0+i)]}{(1+\lambda at)^k\prod_{i=0}^{k}\left[s+\lambda\left(\frac{1+a(n_0+i)}{1+\lambda at}\right)\right]}$
$P_n(t)$	$\binom{\left[1+\frac{1}{a}\right]+k-1}{k}\left(e^{-\frac{\lambda at}{1+\lambda at}}\right)^{1+\frac{1}{a}}\left(1-e^{-\frac{\lambda at}{1+\lambda at}}\right)^k$ $k=0,1,2,\ldots$	$\binom{\left[n_0+\frac{1}{a}\right]+k-1}{k}\left(e^{-\frac{\lambda at}{1+\lambda at}}\right)^{n_0+\frac{1}{a}}\left(1-e^{-\frac{\lambda at}{1+\lambda at}}\right)^k$ $k=0,1,2\ldots$
Method 2 : pgf technique		
Partial Differential equation	$\frac{\partial}{\partial t}G(z,t) - \frac{\lambda az(z-1)}{1+\lambda at}\frac{\partial}{\partial z}G(z,t) - \frac{\lambda(z-1)}{1+\lambda at}G(z,t) = 0$	
a) Dirac Approach		
Ordinary Differential equation	$\frac{d}{dz}\bar{G}(z,s) - \left[\frac{s}{z(z-1)} - \frac{1}{az}\right]\bar{G}(z,s) = -\frac{1}{(z-1)}$	$\frac{d}{dz}\bar{G}(z,s) - \left[\frac{s}{z(z-1)} - \frac{1}{az}\right]\bar{G}(z,s) = \frac{z^{n_0}}{-z(z-1)}$
$\bar{G}(z,s)\lvert z\rvert^{\frac{1}{a}}e^{-s\ln\left\lvert\frac{z-1}{z}\right\rvert}$	$= f(s) - \int\frac{1}{(z-1)}\lvert z\rvert^{\frac{1}{a}}e^{-s\ln\left\lvert\frac{z-1}{z}\right\rvert}dz$	$= f(s) - \int\frac{z^{n_0-1}}{(z-1)}\lvert z\rvert^{\frac{1}{a}}e^{-s\ln\left\lvert\frac{z-1}{z}\right\rvert}dz$
$G(z,t)$	$z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{1+\frac{1}{a}}$	$z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{n_0+\frac{1}{a}}$
$P_n(t)$	$\binom{\left[1+\frac{1}{a}\right]+k-1}{k}\left(\frac{1}{1+\lambda at}\right)^{1+\frac{1}{a}}\left(1-\frac{1}{1+\lambda at}\right)^k$ $k=0,1,2$	$\binom{\left[n_0+\frac{1}{a}\right]+k-1}{k}\left(\frac{1}{1+\lambda at}\right)^{n_0+\frac{1}{a}}\left(1-\frac{1}{1+\lambda at}\right)^k$ $k=0,1,2$

a) **Hypergeometric function approach**		
Ordinary Differential equation	$\frac{d}{dz}\bar{G}(z,s)+\left[\frac{s}{z(1-z)}+\frac{1}{az}\right]\bar{G}(z,s)=\frac{1}{(1-z)}$	$\frac{d}{dz}\bar{G}(z,s)+\left[\frac{s}{z(1-z)}+\frac{1}{az}\right]\bar{G}(z,s)=\frac{z^{n_0}}{z(1-z)}$
$\bar{G}(z,s)$	$z\frac{{}_2F_1\left(1+\frac{1}{a},1;s+\frac{1}{a}+2;z\right)}{s+\frac{1}{a}+1}$	$z^{n_0}\left\{\frac{(1-z)\ {}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+1;z\right)}{s+n_0+\frac{1}{a}}+z\frac{{}_2F_1\left(n_0+\frac{1}{a}+1,1;s+n_0+\frac{1}{a}+2;z\right)}{s+n_0+\frac{1}{a}+1}\right\}$
$G(z,t)$	$z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{1+\frac{1}{a}}$	$z^{-\frac{1}{a}}\left[\frac{\frac{z}{1+\lambda at}}{1-\frac{\lambda atz}{1+\lambda at}}\right]^{n_0+\frac{1}{a}}$
$P_n(t)$	$\binom{\left[1+\frac{1}{a}\right]+k-1}{k}\left(\frac{1}{1+\lambda at}\right)^{1+\frac{1}{a}}\left(1-\frac{1}{1+\lambda at}\right)^{k}$ $k=0,1,2$	$\binom{\left[n_0+\frac{1}{a}\right]+k-1}{k}\left(\frac{1}{1+\lambda at}\right)^{n_0+\frac{1}{a}}\left(1-\frac{1}{1+\lambda at}\right)^{k}$ $k=0,1,2$

Laplace transform of Some Probability Distributions

Distribution $f(x)$	Laplace Transform $L_X(s)$	Mean $E(X)$	Variance $Var(X)$	Moments $E\left[X^r\right]$
Uniform $(a,\ b)$	$\frac{1}{b-a}\left[\frac{e^{-sa}-e^{-sb}}{s}\right]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$(-1)^r\frac{1}{r+1}\sum_{i=0}^{r}b^{r-i}a^i$
Uniform $(0,\ 1)$	$\frac{1-e^{-s}}{s}$	$\frac{1}{2}$	$\frac{1}{12}$	
Gamma (α)	$\frac{1}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(s+1)x}dx$	α	α	$\frac{(\alpha+r-1)!}{(\alpha-1)!}$
Gamma $(\alpha,\ \beta)$	$\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(s+\beta)x}dx$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\frac{(\alpha+r-1)!}{(\alpha-1)!\,\beta^r}$
Exponential (λ)	$\frac{\lambda}{\lambda+s}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{r!}{\lambda^r}$
Inverse Gamma $(\alpha,\ \beta)$	$\frac{2}{\Gamma\alpha}\left(\sqrt{\beta s}\right)^\alpha K_\alpha\left(2\sqrt{\beta s}\right)$			
Inverse Gaussian $(\mu,\ \phi)$	$\exp\left\{-\frac{\mu}{\beta}\left(\sqrt{1+2\beta s}-1\right)\right\}$	μ	$\frac{\mu^3}{\phi}$	
Generalized Inverse Gaussian $(\lambda,\ \chi,\psi)$	$\left(\frac{\psi}{2s+\psi}\right)^{\frac{\lambda}{2}}\frac{K_\lambda\left(\sqrt{\chi(2s+\psi)}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)}$			
Reciprocal of Inverse Gaussian $(\mu,\ \phi)$	$\left(1+\frac{2}{\phi}s\right)^{-1/2}\exp\left\{-\frac{\phi}{\mu}\left(\sqrt{1+\frac{2}{\phi}s}-1\right)\right\}$	$\frac{1}{\phi}+\frac{1}{\mu}$	$\frac{2}{\phi^2}+\frac{1}{\mu\phi}$	
Rayleigh Distribution	$1-\sqrt{\frac{\pi}{2}}\sigma s e^{\frac{\sigma^2 s^2}{2}} erfc\left(\frac{\sigma s}{\sqrt{2}}\right)$	$\sigma\sqrt{\frac{\pi}{2}}$	$\sigma^2\left(\frac{4-\pi}{2}\right)$	

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