School of Mathematics

University of Nairobi

HEAT EQUATIONS AND THEIR APPLICATIONS

(One and Two Dimension Heat Equations)

BY

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Project submitted to the School of Mathematics, University of Nairobi, in partial fulfillment of the requirement for the degree of Master of Science in Applied mathematics.

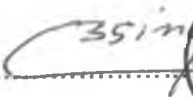August 2009

DECLARATION

By Candidate

This project is my original work and has not been presented for a degree in any other University except where due acknowledgement is made in the text.
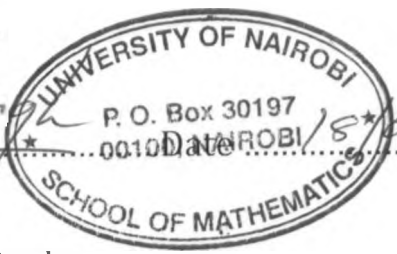
Signed : .... Date...... 18|8|09 ....

       Sammy Kihara Njogu

By Supervisor

This project has been submitted for examination with my approval as University Supervisor.

Signature: .... Date...... 18/ 08/09 ....

       Dr. C .B . Singh

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ACKNOWLEDGEMENTS

I would like to acknowledge my supervisor Dr. C. B. Singh for the unlimited support throughout this project. Special thanks to my colleagues Mpele ,Mary ,Fatma.Jane and Amos. I wish them well in all their endeavors.

Dedication

This project is dedicated to my beloved wife Roselucy, my brother Yusuf and family and my two children Derrick and Sharon

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Chapter One

1.0 Background of the Study

The heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time.

The heat equation has the general form

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

For a function $U(x,y,z,t)$ of three spatial variables x,y,z and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

or equivalently

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

where k is a constant.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In statistics, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

The heat equation predicts that if a hot body is placed in a box of cold water, the temperature of the body will decrease, and eventually (after infinite time, and subject to no external heat sources) the temperature in the box will equalize.

The heat equation is derived from Fourier's law and conservation of energy Cannon [1984]. By Fourier's law, the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface,

$$q = -k \cdot \nabla u$$

Where k is the thermal conductivity and u is the temperature. In one dimension, the gradient is an ordinary spatial derivative, and so Fourier's law is

$$q = -ku_x.$$

In the absence of work done, a change in internal energy per unit volume in the material, ΔQ is proportional to the change in temperature. That is,

$$\Delta Q = c_p \rho \Delta u$$

where c_p is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at temperature zero, this can be rewritten as

$$Q = c_p \rho u.$$

The increase in internal energy in a small spatial region of the material

$$x - \Delta x \leq \xi \leq x + \Delta x$$

over the time period

$$t - \Delta t \leq \tau \leq t + \Delta t$$

by the fundamental theorem of calculus. With no work done, and absent any heat sources or sinks, this change in internal energy in the interval $|x - \Delta x, x + \Delta x|$ is accounted for entirely by the flux of heat across the boundaries.

In the special case of heat propagation in an isotropic and homogeneous medium in the 3-dimensional space, this equation is

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$k(u_{xx} + u_{yy} + u_{zz})$$

Where:

$u = u(x, y, z, t)$ is temperature as a function of time and space;

$\frac{\partial u}{\partial t}$ is the rate of change of temperature at a point over time;

u_{xx}, u_{yy}, u_{zz} are the second spatial derivatives (*thermal conductions*) of temperature in the *x, y*, and *z* directions, respectively;

k is a material-specific quantity depending on the *thermal conductivity*, the *density* and the *heat capacity*. Specifically, $k = \kappa / c\rho$ where κ is the thermal conductivity, c is the capacity, and ρ the density.

The solutions of the unsteady heat conduction equations in cylindrical geometry in one and two dimensions are obtained using the Chebyshev polynomial expansions in the spatial domain.

Equations are discretized in the time domain using the trapezoidal rule. The resulting differential equations are reduced to backward recurrence relations for the coefficients occurring in the Chebyshev polynomial expansions, which are then solved using the Tau method. It is shown that the Chebyshev polynomial solutions produce results to the machine-precision accuracy in the spatial domain using only a modest number of terms, and are, therefore, excellent alternatives to the other techniques used.

Suppose one has a function u which describes the temperature at a given location (x, y, z). This function will change over time as heat spreads throughout space. The heat equation is used to determine the change in the function u over time. The image above is animated and has a description of the way heat changes in time along a metal bar. One of the interesting properties of the heat equation is the maximum principle which says that the maximum value of u is either earlier in time than the region of concern or on the edge of the region of concern.

This is essentially saying that temperature comes either from some source or from earlier in time because heat permeates but is not created from nothingness. This is a property of parabolic partial differential equations and is not difficult to prove mathematically

Another interesting property is that even if u has a discontinuity at an initial time t = t_0, the temperature becomes smooth as soon as $t > t_0$ For example, if a bar of metal has temperature 0 and another has temperature 100 and they are stuck together end to end, then very quickly the temperature at the point of connection is 50 and the graph of the temperature is smoothly running from 0 to 100. The heat equation is used in probability and describes random walks. It is also applied in financial mathematics for this reason. It is also important in Riemannian geometry and thus topology: it was adapted by Richard Hamilton when he defined the Ricci flow that was later used to solve the topological Poincaré conjecture.

1.1 Statement of the Problem

It is well known that heat equations take a central place among possible instruments for the modeling of different processes and phenomena.

A considerable work has been done on investigating operation equations associated with heat equations Grigull and Hauf, [1966]; Kuehn and Goldstein, [1974], Alshahrani and Zeitoun, [2005]. Farinas *et al.* [1997] investigated the effect of internal fins on flow pattern, temperature distribution and heat transfer between concentric horizontal cylinders for different fin orientations and fin tip geometry for Rayleigh numbers ranging from 10^3 to 10^6. They employed the two fin orientations used by Chai and Patankar [1993].

Alshahrani and Zeitoun [2006] investigated the natural convection heat transfer between two horizontal concentric cylinders with two fins, at various inclination angles, attached to inner cylinder numerically using finite element technique.

However, little has been done to investigate the application of heat equations. This study therefore seeks to give a deeper insight on heat equations and their applicability

Chapter two

2.0   LITERATURE REVIEW

2.1   Introduction

This chapter discusses the literature review in the conceptual form. Areas addressed by this chapter include the assumptions made in one and two dimension heat equations. The different approaches used in developing one or two dimensional heat equations as well as the applications of heat equations.

2.2   Theoretical Background

The heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function u(x, y, z, t) of three spatial variables (x, y, z) and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In statistics, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

Thermal stability of superconductors under the effect of a two-dimensional hyperbolic heat conduction model M. As explained by Al-Odat, M.A. Al-Nimr, M. Hamdan,[2003]. The thermal stability of superconductor is numerically investigated under the effect of a two-dimensional hyperbolic heat conduction model. Two types of superconductor wires are considered, Types II and I. The thermal stability of superconductor wires under the effect of different design, geometrical and operating conditions is studied.

The Effect of the time rate of change of the disturbance and the disturbance duration time is investigated. Generally, it is found that wave model predicts a wider stability region as compared to the predictions of the classical diffusion model.

2.3 Importance of Heat Equations

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In statistics, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes. The geothermal gases of constant wall temperature/concentration and constant heat/mass flux are electrically conducting and are affected by the presence of a magnetic field.

Chamkha and Khaled [2000] have studied the effect of magnetic field on the coupled heat and mass transfer by mixed convection in a linearly stratified stagnation flow in the presence of an internal heat generation or absorption. EL-Hakiem [2000] studied thermal radiation effects on hydromagnetic free convection and flow through a highly porous medium bounded by a vertical plane surface. Borjini *et al.* [1999] have considered the effect of radiation on unsteady natural convection in a two-dimensional participating medium between two horizontal concentric and vertically eccentric cylinders. Chamkha [2000] has analyzed hydromagnetic mixed convection from a permeable semi-infinite vertical plate embedded in porous medium in heat dimension, Duwairi [2005] investigated radiation and magnetic field effects on forced convection flow from isothermal porous surfaces considering Viscous and Joule heating. Hayat [2006] reported the modeling and exact analytic solutions for hydromagnetic oscillatory rotating flows of an incompressible Burgers fluid bounded by a plate.

The problem of magnetohydrodynamic (MHD) boundary layer flow of an upper-convected Maxwell fluid is investigated in a channel by Abbas *et al.* [2006]. Suppose one has a function u which describes the temperature at a given location (x, y, z). This function will change over time as heat spreads throughout space. The heat equation is used to determine the change in the function u over time. The image above is animated and has a description of the way heat changes in time along a metal bar. One of the interesting properties of the heat equation is the maximum principle which says that the maximum value of u is either earlier in time than the region of concern or on the edge of the region of concern.

This is essentially saying that temperature comes either from some source or from earlier in time because heat permeates but is not created from nothingness.

2.4 Application on Brownian motion

Brownian motion is a seemingly random movement of particles suspended in a fluid (i.e. a liquid or gas) or the mathematical model used to describe such random movements, often called a particle theory. The mathematical model of Brownian motion has several real-world applications. An often quoted example is stock market fluctuations.

Brownian motion is among the simplest of the continuous-time stochastic (or random) processes, and it is a limit of both simpler and more complicated stochastic processes. This universality is closely related to the universality of the normal distribution. In both cases, it is often mathematical convenience rather than the accuracy of the models that motivates their study. Although the mingling motion of dust particles is caused largely by air currents, the glittering, tumbling motion of small dust particles is, indeed, caused chiefly by true Brownian dynamics.

Jan Ingenhousz [1785] had described the irregular motion of coal dust particles on the surface of alcohol. It is believed that Brown was studying pollen particles floating in water under the microscope. He then observed minute particles within the vacuoles of the pollen grains executing a jittery motion. By repeating the experiment with particles of dust, he was able to rule out that the motion was due to pollen particles being 'alive', although the origin of the motion was yet to be explained. The first person to describe the mathematics behind Brownian motion was Thorvald N. Thiele [1880] in a paper on the method of least squares. This was followed independently by Louis Bachelier [1900] in his PhD thesis "The theory of speculation", in which he presented a stochastic analysis of the stock and option markets. However, it was Albert Einstein's [1905] and Marian Smoluchowski's [1906] independent research of the problem that brought the solution to the attention of physicists, and presented it as a way to indirectly confirm the existence of atoms and molecules. This is a property of parabolic partial differential equations and is

not difficult to prove mathematically Another interesting property is that even if u has a discontinuity at an initial time t = t0, the temperature becomes smooth as soon as t > t0. For example, if a bar of metal has temperature 0 and another has temperature 100 and they are stuck together end to end, then very quickly the temperature at the point of connection is 50 and the graph of the temperature is smoothly running from 0 to 100.

The heat equation is used in probability and describes random walks. It is also applied in financial mathematics for this reason.

2.5   Modalities of Heat Equations on Particle Diffusion

The heat equation is, technically, in violation of special relativity, because its solutions involve instantaneous propagation of a disturbance.

The part of the disturbance outside the forward light cone can usually be safely neglected, but if it is necessary to develop a reasonable speed for the transmission of heat, a hyperbolic problem

should be considered instead – like a partial differential equation involving a second-order time derivative. It is also important in Riemannian geometry and thus topology: it was adapted by Richard Hamilton when he defined the Ricci flow that was later used to solve the topological Poincaré conjecture.

On the other hand, classical Lie symmetry method can be used to find similarity solutions, invariants, integrals motion, etc. systematically, Ibragimov, [1999] and the usefulness of this approach has been widely illustrated by several authors in different contexts such as; Yurusoy and Pakdemirli [1997] found symmetry reductions of unsteady three-dimensional boundary layers of some non-Newtonian fluids. Also, Group classification of boundary layer equations of a non-Newtonial fluid flow problem has been performed by Yurusoy and Pakdemirli, [1999].

Recently, Soh [2005] has used Lie symmetry techniques to obtain all non-similar and similarity reductions of a non-linear diffusion equation arising in the study of the flow of a charged non-

Newtonian fluid over a flat plate. Soh et al. [2005] used symmetry methods to obtain non-equivalent similarity reductions of the steady two-dimensional thermal boundary layer equations of an incompressible laminar flow.

Hayat et al. [2005] examined the unsteady flow of a hydrodynamic fluid past a porous plate by implementation of the Lie group method. Sivasankaran et al. [2006] studied coupled heat and mass transfer fluid flow by natural convection past an inclined semi-infinite porous surface using Lie group analysis.

The flow of a third-grade fluid occupying the space over a wall is studied analytically using Lie group methods by Hayat et al. [2003] . Mohyuddin et al. [2004] applied Lie symmetry group method to obtain some steady as well as unsteady solutions of the equations of motion for incompressible Newtonian and non-Newtonian fluids.

Hayat and Mahomed [2007] obtained a new exact power law solution for the pipe flow of a third-grade fluid. Hayat and Kara

(n.d.) presented here deals with similarity solutions of the problem of the flow of a third grade fluid past an infinite plate which are in a state of rigid body rotation. Hayat et al. [2007] studied the flow generated in a semi-infinite expanse of an incompressible second-grade fluid bounded by a porous oscillating disk in the presence of a uniform transverse magnetic field.

El-Kabeir et al. [2007] have applied Group method to simulate problem of heat and mass transfer in boundary-layer flow of an electrically conducting fluid over a vertical permeable cone surface saturated porous medium in the presence of a uniform transverse magnetic field and thermal radiation effects. EL-Kabeir et al. [2008]

Chapter three

3.0 One and Two Dimension Heat Equation

3.1 Assumptions:

1) "Temperature and other scalar fields used in physics are assumed to be continuous, and this guarantees that if point x has temperature $\bar{A}(x)$ and point z has temperature $\bar{A}(z)$ and r is a real number between $\bar{A}(x)$ and $\bar{A}(z)$, then there will be a point y spatio-temporally between x and z such that $\bar{A}(y) = r$"

   Field [1980],

2) Not *all* mathematical properties transfer to temperatures.

3) There is no least real number but there is a lowest temperature.

4) Case study: For $u(x; t)$ representing the temperature of point x at time t, we can derive the partial differential equation ,Boyce and DiPrima [1986].

3.2 One Dimensional Heat Equations

The "one-dimensional" in the description of the differential equation refers to the fact that we are considering only one spatial dimension. There are two methods used to solve for the rate of heat flow through an object. The first method is derived from the properties of the object. The second method is derived by measuring the rate of heat flow through the boundaries of the object.

Imagine a thin rod that is given an initial temperature distribution, then insulated on the sides. The ends of the rod are kept at the same fixed temperature; e.g., suppose at the start of the experiment, both ends are immediately plunged into ice water. We are interested in how the temperatures along the rod vary with time. Suppose that the rod has a length L (in meters), and we establish a coordinate system along the rod as illustrated below.
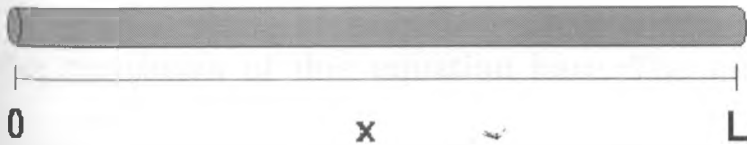
Let $u(x,t)$ represent the temperature at the point x meters along the rod

at time t (in seconds). We start with an initial temperature

distribution $u(x,0) = f(x)$ such as the one represented by the following graph

(with $L = 2$ meters).
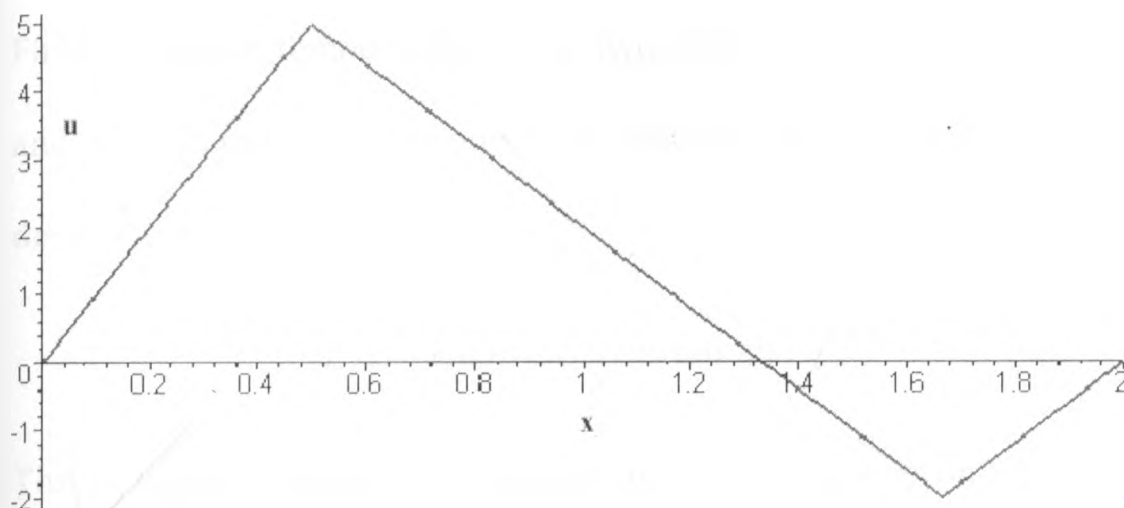
The partial differential equation

$$u_t = a^2 u_{xx}$$

is used to model one-dimensional temperature evolution. We will not discuss

the derivation of this equation here. The most important features of this

equation are the *second* spatial derivative u_{xx} and the *first* derivative with

respect to time, u_t.

The positive constant a^2 represents the *thermal diffusivity* of the rod. It depends on the thermal conductivity of the material composing the rod, the density of the rod, and the specific heat of the rod.

The function $u(x,t)$ that models heat flow should satisfy the partial differential equation. However, in addition, we expect it to satisfy two other conditions. First, we fix the temperature at the two ends of the rod, i.e., we specify $u(0,t)$ and $u(L,t)$. In our sample problem, we will assume that both ends are kept at 0 degrees Celsius:

$$u(0,t) = u(L,t) = 0 \text{ For all } t > 0$$

This is called a boundary condition since it is imposed on the values of the desired function at the boundaries of the spatial domain.

The remaining condition represents the initial temperature distribution

$$u(x,o) = f(x)$$

Where f(x) is the temperature at position x at time t=0.

All together, the model function‴ $u(x,t)$ that we seek should satisfy

$$u_t = a^2 u_{xx}$$

$$u(0,t) = u(L,t) = 0 \text{ For all } t > 0$$

$$u(x,o) = f(x)$$

3.3 Derivation in one dimension

The heat equation is derived from Fourier's law and conservation of energy (Cannon 1984). By Fourier's law, the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface,

$$\mathbf{q} = -k\nabla u$$

where k is the thermal conductivity and u is the temperature. In one dimension, the gradient is an ordinary spatial derivative, and so Fourier's law is

$$\mathbf{q} = -ku_x.$$

In the absence of work done, a change in internal energy per unit volume in the material, ΔQ, is proportional to the change in temperature. That is,

$$\Delta Q = c_p \rho \Delta u$$

where cp is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at temperature zero, this can be rewritten as

$$Q = c_p \rho u.$$

The increase in internal energy in a small spatial region of the material

$$x - \Delta x \leq \xi \leq x + \Delta x$$

over the time period

$$t - \Delta t \leq \tau \leq t + \Delta t$$

is given by[1]

$$c_p \rho \int_{x-\Delta x}^{x+\Delta x} \left[u(\xi, t + \Delta t) - u(\xi, t - \Delta t) \right] d\xi = c_p \rho \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial u}{\partial \tau} d\xi d\tau$$

where the fundamental theorem of calculus was used. With no work done, and absent any heat sources or sinks, this change in internal energy in the interval

$[x - \Delta x, x + \Delta x]$ is accounted for entirely by the flux of heat across the boundaries. By Fourier's law, this is

$$k \int_{t-\Delta t}^{t+\Delta t} \left[\frac{\partial u}{\partial x}(x + \Delta x, \tau) - \frac{\partial u}{\partial x}(x - \Delta x, \tau) \right] d\tau = k \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial^2 u}{\partial \xi^2} d\xi d\tau$$

Again by the fundamental theorem of calculus. By conservation of energy,

$$\int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} [c_p \rho u_\tau - k u_{\xi\xi}] d\xi d\tau = 0.$$

This is true for any rectangle $[t - \Delta t, t + \Delta t] \times [x - \Delta x, x + \Delta x]$. Consequently, the integrand must vanish identically;

$$c_p \rho u_t - k u_{xx} = 0. \quad \text{Or,}$$

$$u_t = \frac{k}{c_p \rho} u_{xx},$$

This is the heat equation.

3.4 Internal heat generation

The function u above represents temperature of a body. Alternatively, it is sometimes convenient to change units and represent u as the heat density of a medium. Since heat density is proportional to temperature in a homogeneous medium, the heat equation is still obeyed in the new units.

Suppose that a body obeys the heat equation and, in addition, generates is own heat per unit volume (e.g., in watts/L) at a rate given by a known function q varying in space and time. Then the heat per unit volume u satisfies an equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + q.$$

3.5 One Dimensional Radial and Angular Neglects

3.5.1 Cylindrical Rod

(a) Two Dirichlet boundary conditions

(b)   One Dirichlet boundary conditions

One Neumann Two boundary conditions
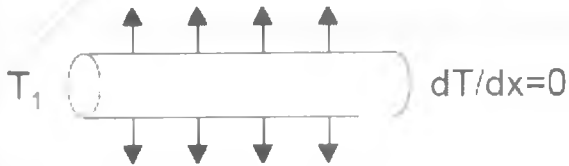
(c) One Dirichlet boundary conditions

One Neumann boundary conditions

3.5.2 Rectangular Slab

T_1 T_2
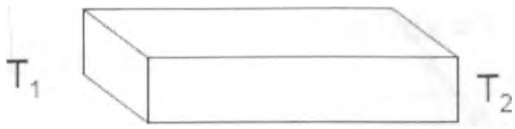
(d)   2 Dirichlet boundary conditions

T_1 $dT/dx=0$
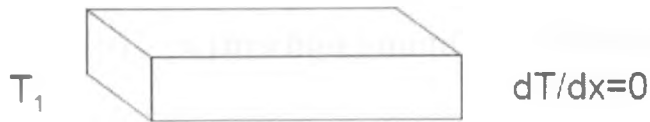
(e) 1 Dirichlet boundary conditions

1 Neumann boundary conditions

3.6   One Dimensional Radial and Angular Neglects

3.6.1 Rectangular Slab

T_4 at $y=L$

T_2 at $x=L$

T_1 at $x=0$

T_3 at $y=0$
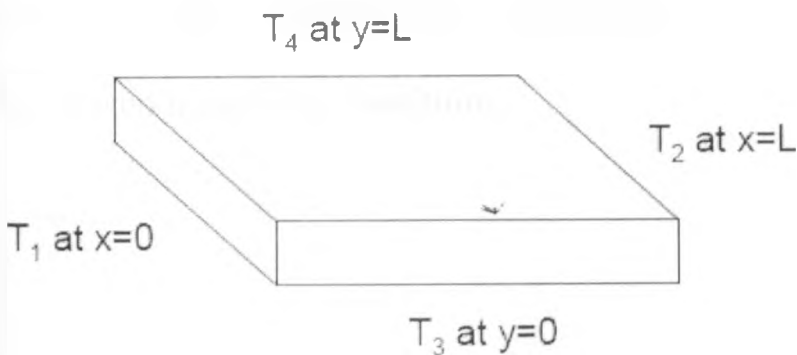
(a) 4 Dirichlet boundary conditions

3.6.2 Slab of Arbitrary Geometry
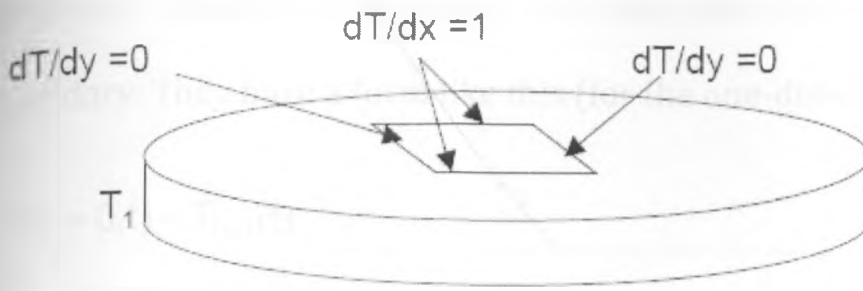
(b) 4 Dirichlet boundary condition on exterior boundary

Neumann boundary condition on interior boundary.

3.7 The Boundary Conditions

The heat equation is a second-order partial differential equation in the spatial coordinates. We need boundary conditions in order to specify how our system interacts with the outside surroundings.

There are three general types of boundary conditions: Dirichlet, Neumann, and Mixed boundary conditions.

3.71 Dirichlet Boundary Conditions

Dirichlet boundary conditions say that the temperature is set at the boundary. They have a form like this (for the one-dimensional case)

$$T(x = 0, t) = T_{bc1}(t) \qquad \text{... (10)}$$

This says that at the left-hand-side boundary of our one-dimensional system, the temperature is a specified function of time. If the temperature is constant, then we have

$$T(x = 0, t) = T_{bc1} \qquad \text{... (11)}$$

In this case, we have the physical situation where our system is touching an infinite heat reservoir that maintains a constant temperature.

In a one-dimensional system, we must have two boundary conditions, one at the left-hand-side boundary and the other at the right-hand-side boundary. If our one-dimensional system is of length **L** in the x-direction, then our second Dirichlet boundary condition would be of the form:

$$T(x = L, t) = T_{bc2}(t) \qquad \text{... (12)}$$

3.7.2 Neumann Boundary Conditions

Neumann boundary conditions say that the heat flux is set at the boundary. They have a form like this (for the one-dimensional case).

$$\frac{dT}{dx}(x = 0, t) = \frac{dT(t)}{dx}\bigg|_{bc1} \quad\text{.. (13)}$$

This says that at the left-hand-side boundary of our one-dimensional system, the heat flux is a specified function of time. If the heat flux is constant we have:

$$\frac{dT}{dx}(x = 0, t) = \frac{dT}{dx}\bigg|_{bc1} \quad\text{.. (14)}$$

In this case, we have the physical situation where our system is touching an infinite heat source that maintains a constant flux of heat into the system *regardless of the temperature*. One end of the rod is well insulated. No heat leaves it. The heat flux is zero. In this case, we would use a Neumann boundary condition.

In a one-dimensional system, we must have two boundary conditions, one at the left-hand-side boundary and the other at the right-hand-side boundary.

If our one-dimensional system is of length **L** in the x-direction, then our second Neumann boundary condition would be of the form:

$$\frac{dT}{dx}(x = L, t) = \frac{dT(t)}{dx}\bigg|_{bc2}$$

.... (15)

3.7.3 Mixed Boundary Conditions

Mixed Boundary Conditions, as the name implies, is a mixture of the Dirichlet and Neumann boundary conditions. They have a form like this (for the one-dimensional case).

$$\frac{dT}{dx}(x = 0, t) - T(x = 0, t) = T_{bc1}(t) + \frac{dT(t)}{dx}\bigg|_{bc1} = f(t)$$

.... (16)

There are very relevant physical systems which require these elaborate boundary conditions.

3.8 Initial Conditions

3.81 Generalized initial condition

The heat equation is first order in time. We need to know the temperature at every point in our system at time equals zero. In general, this initial condition can be written as

$$T(x,y,t=0) = T_{ic}(x,y)$$

.... (21)

3.82 Constant temperature initial condition

If the temperature is constant then the initial condition becomes:

$$T(x,y,t=0) = T_{ic}$$

.... (22)

3.83 Steady-state initial condition

If the temperature profile is initially a steady-state (linear) profile between two boundary condition temperatures T_{bc1} and T_{bc2}, then we would have the formula for the linear interpolation between them, which in one-dimension looks like:

$$T(x,t=0) = T_{bc1} - \frac{x}{L}(T_{bc2} - T_{bc1})$$

.... (23)

Chapter four

4.0 Solution using Fourier series

The following solution technique for the heat equation was proposed by Joseph Fourier in his treatise Théorie analytique de la chaleur, [1822].

Let us consider the heat equation for one space variable. This could be used to model heat conduction in a rod. The equation is

$$(1)\ u_t = ku_{xx}$$

where u = u(t, x) is a function of two variables t and x. Here x is the space variable, so x ∈ [0 ,L], where L is the length of the rod ,t is the time variable, so t ≥ 0. We assume the initial condition

$$(2)\ u(0,x) = f(x) \quad \forall x \in [0, L]$$

Where the function f is given and the boundary conditions

$$(3)\ u(t,0) = 0 = u(t, L) \quad \forall t > 0.$$

Let us attempt to find a solution of (1) which is not identically zero satisfying the boundary conditions (3) but with the following property: u is a product in which the dependence of u on x, t is separated, that is:

$$(4)\ u(t,x) = X(x)T(t).$$

This solution technique is called separation of variables. Substituting u back into equation (1),

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

Since the right hand side depends only on x and the left hand side only on t, both sides are equal to some constant value $-\lambda$. Thus:

$$(5)\ T'(t) = -\lambda kT(t)$$

and

$$(6)\ X''(x) = -\lambda X(x).$$

We will now show that solutions for (6) for values of $\lambda \le 0$ cannot occur:

Suppose that $\lambda < 0$. Then there exist real numbers B, C such that

$$X(x) = Be^{\sqrt{-\lambda}\,x} + Ce^{-\sqrt{-\lambda}\,x}.$$

From (3) we get

$$X(0) = 0 = X(L).$$

And therefore B = 0 = C which implies u is identically 0.

Suppose that λ = 0. Then there exist real numbers B, C such that

$$X(x) = Bx + C.$$

From equation (3) we conclude in the same manner as in 1 that u is identically 0.

Therefore, it must be the case that λ > 0. Then there exist real numbers A, B, C such that

$$T(t) = Ae^{-\lambda k t}$$

and

$$X(x) = B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x).$$

From (3) we get C = 0 and that for some positive integer n,

$$\sqrt{\lambda} = n\frac{\pi}{L}.$$

This solves the heat equation in the case that the dependence of u has the special form (4).

In general, the sum of solutions to (1) which satisfies the boundary conditions (3) also satisfies (1) and (3). We can show that the solution to (1), (2) and (3) is given by

$$u(t,x) = \sum_{n=1}^{+\infty} D_n \left(\sin \frac{n\pi x}{L} \right) e^{-\frac{n^2\pi^2 kt}{L^2}}$$

Where

$$D_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

4.01 Generalizing the solution technique

The solution technique used above can be greatly extended to many other types of equations. The idea is that the operator U_{xx} with the zero boundary conditions can be represented in terms of its eigenvectors. This leads naturally to one of the basic ideas of the spectral theory of linear self-adjoint operators.

Consider the linear operator $\Delta U = U_{xx}$ The infinite sequence of functions

$$e_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$ for n ≥ 1 are eigenvectors of Δ.

Indeed

$$\Delta e_n = -\frac{n^2\pi^2}{L^2} e_n.$$

Moreover, any eigenvector f of Δ with the boundary conditions f(0)=f(L)=0 is of the form e_n for some n ≥ 1. The functions e_n for n ≥ 1 form an orthonormal sequence with respect to a certain inner product on the space of real-valued functions on [0, L].

This means

$$\langle e_n, e_m \rangle = \int_0^L e_n(x)e_m(x)dx = \begin{cases} 0 & n \neq m \\ 1 & m = n \end{cases}.$$

Finally, the sequence $\{e_n\}, n \in \mathbb{N}$ spans a dense linear subspace of $L_2(0, L)$. This shows that in effect we have diagonalized the operator Δ.

4.02 Fundamental problem solution

A fundamental solution, also called a heat kernel, is a solution of the heat equation corresponding to the initial condition of an initial point source of heat at a known position. These can be used to find a general solution of the heat equation over certain domains; In one variable, the Green's function is a solution of the initial value problem

$$\begin{cases} u_t(x,t) - ku_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$

where δ is the Dirac delta function. The solution to this problem is the fundamental solution

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right).$$

One can obtain the general solution of the one variable heat equation with initial condition

$U(x, 0) = g(x)$ for $-\infty < x < \infty$ and $0 < t < \infty$ by applying a convolution:

$$u(x,t) = \int \Phi(x - y, t)g(y)dy.$$

In several spatial variables, the fundamental solution solves the analogous problem

$$\begin{cases} u_t(\mathbf{x}, t) - k\sum_{i=1}^{n} u_{x_i x_i}(\mathbf{x}, t) = 0 \\ u(\mathbf{x}, t = 0) = \delta(\mathbf{x}) \end{cases}$$

in $-\infty < x_i < \infty$, $i = 1,...,n$, and $0 < t < \infty$. The n-variable fundamental solution is the product of the fundamental solutions in each variable; i.e.,

$$\Phi(\mathbf{x}, t) = \Phi(x_1, t)\Phi(x_2, t)\ldots\Phi(x_n, t) = \frac{1}{(4\pi kt)^{n/2}}e^{-\mathbf{x}\cdot\mathbf{x}/4kt}.$$

The general solution of the heat equation on $\Re^n$ is then obtained by a convolution, so that to solve the initial value problem with $u(x,0) = g(x)$, one has

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t)g(\mathbf{y})d^n\mathbf{y}.$$

The general problem on a domain Ω in $\mathfrak{R}^n$ is

$$\begin{cases} u_t(\mathbf{x},t) - k \sum_{i=1}^{n} u_{x_i x_i}(\mathbf{x},t) = 0 & \mathbf{x} \in \Omega \quad 0 < t < \infty \\ u(\mathbf{x}, t = 0) = g(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases}$$

With either Dirichlet or Neumann boundary data. A Green's function always

exists, but unless the domain Ω can be readily decomposed into one-variable

problems , it may not be possible to write it down explicitly. The method of

images provides one additional technique for obtaining Green's functions for

non-trivial domains. Some Green's function solutions in 1D

A variety of elementary Green's function solutions in one-dimension are

recorded here. In some of these, the spatial domain is the entire real line (-

∞,∞). In others, it is the semi-infinite interval (0,∞) with either Neumann or

Dirichlet boundary conditions. One further variation is that some of these

solve the inhomogeneous equation.

$$U_t = kU_{xx} + f$$

Where f is some given function of x and t.

Initial value problem on (-∞, ∞)

$$\begin{cases} u_t = k u_{xx} & -\infty < x < \infty, \, 0 < t < \infty \\ u(x,0) = g(x) & IC \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) g(y)\, dy$$

Initial value problem on (0,∞) with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t = k u_{xx} & 0 \le x < \infty, \, 0 < t < \infty \\ u(x,0) = g(x) & IC \\ u(0,t) = 0 & BC \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(\exp\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right) \right) g(y)\, dy$$

Initial value problem on (0, ∞) with homogeneous Neumann boundary conditions

$$\begin{cases} u_t = k u_{xx} & 0 \le x < \infty, \, 0 < t < \infty \\ u(x,0) = g(x) & IC \\ u_x(0,t) = 0 & BC \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(\exp\left(-\frac{(x-y)^2}{4kt}\right) + \exp\left(-\frac{(x+y)^2}{4kt}\right) \right) g(y)\, dy$$

Problem on $(0, \infty)$ with homogeneous initial conditions and non-homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t = ku_{xx} & 0 \le x < \infty, 0 < t < \infty \\ u(x,0) = 0 & IC \\ u(0,t) = h(t) & BC \end{cases}$$

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi k(t-s)^3}} \exp\left(-\frac{x^2}{4k(t-s)}\right) h(s)\, ds$$

Inhomogeneous heat equation Problem on $(-\infty,\infty)$ homogeneous initial conditions

$$\begin{cases} u_t = ku_{xx} + f(x,t) & -\infty < x < \infty, 0 < t < \infty \\ u(x,0) = 0 & IC \end{cases}$$

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) f(y,s)\, dy\, ds$$

Problem on $(0, \infty)$ with homogeneous Dirichlet boundary conditions and initial conditions

$$
\begin{cases}
u_t = ku_{xx} + f(x,t) & 0 \le x < \infty, \ 0 < t < \infty \\
u(x,0) = 0 & IC \\
u(0,t) = 0 & BC
\end{cases}
$$

$$
u(x,t) = \int_0^t \int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} \left(\exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) - \exp\left(-\frac{(x+y)^2}{4k(t-s)}\right) \right) f(y,s) \, dy \, ds
$$

Chapter five

5.0 Numerical Methods

A numerical method for the solution of two-phase flow equations has been developed. The method is based on the commonly used principles for the solution of two-phase flow: a staggered mesh, an upwind treatment of the convection terms, and the implicitness of the transfer terms. The purpose was to develop a robust and fast method for the analysis of nuclear reactors. Consequently, the equations are solved in a one-dimensional form, but the principles of the method are also applicable for multidimensional solutions.

Direct methods compute the solution to a problem in a finite number of steps. These methods would give the precise answer if they were performed in infinite precision arithmetic. Examples include Gaussian elimination, the Quadrature factorization method for solving systems of linear equations, and the simplex method of linear programming. In practice, finite precision is used and the result is an approximation of the true solution (assuming stability).

In contrast to direct methods, iterative methods are not expected to terminate in a number of steps.

Starting from an initial guess, iterative methods form successive approximations that converge to the exact solution only in the limit. Chapra, Steven C., Canale, Raymond P.,[1985].

A convergence criterion is specified in order to decide when a sufficiently accurate solution has (hopefully) been found. Even using infinite precision arithmetic these methods would not reach the solution within a finite number of steps (in general). Examples include Newton's method, the bisection method, and Jacobi iteration. In computational matrix algebra, iterative methods are generally needed for large problems

However Arnoldi iteration reduces to the Lanczos iteration for symmetric matrices. The corresponding Krylov subspace method is the minimal residual method (MinRes) of Paige and Saunders. Unlike the unsymmetric case, the MinRes method is given by a three-term recurrence relation. It can be shown that there is no Krylov subspace method for general matrices, which is given by a short recurrence relation and yet minimizes the norms of the residuals, as GMRES does.

Another class of methods builds on the unsymmetric Lanczos iteration, in particular the BiCG method.

These use a three-term recurrence relation, but they do not attain the minimum residual, and hence the residual does not decrease monotonically for these methods. Convergence is not even guaranteed.The third class is formed by methods like CGS and BiCGSTAB.

These also work with a three-term recurrence relation (hence, without optimality) and they can even terminate prematurely without achieving convergence. The idea behind these methods is to choose the generating polynomials of the iteration sequence suitably.None of these three classes is the best for all matrices; there are always examples in which one class outperforms the other. Therefore, multiple solvers are tried in practice to see which one is the best for a given problem.

5.01 Method of False Transients

he lattice Boltzmann method (LBM) was used to solve the energy equation of a transient conduction–radiation heat transfer problem. The finite volume method (FVM) was used to compute the radiative information. To study the compatibility of the LBM for the energy equation and the FVM for the radiative transfer equation, transient conduction and radiation heat transfer problems in 1-D planar and 2-D rectangular geometries were considered.

In order to establish the suitability of the LBM, the energy equations of the two problems were also solved using the FVM of the computational fluid dynamics.

The FVM used in the radiative heat transfer was employed to compute the radiative information required for the solution of the energy equation using the LBM or the FVM (of the CFD). Chai, J., Patankar, V. [1993],

To study the compatibility and suitability of the LBM for the solution of energy equation and the FVM for the radiative information, results were analyzed for the effects of various parameters such as the scattering albedo, the conduction–radiation parameter and the boundary emissivity. The results of the LBM–FVM combination were found to be in excellent agreement with the FVM–FVM combination. The number of iterations and CPU times in both the combinations were found comparable. A method is described which permits in the entire practical N tu-range the numerical reduction of transient matrix heat transfer test data resulting from single blow experiments. Two particular cases are analyzed in detail:

First the idealized case where the upstream fluid temperature follows a step change, and second the more realistic case where the upstream fluid temperature follows a decreasing exponential function, which is an acceptable assumption whenever a fast

response electrical heater is employed to produce the upstream fluid temperature change. Discrete direct curve matching as formulated in this paper consists of minimizing a suitable distance function defined on discrete sets of data points. In an appendix to the present paper, the implementation of a direct curve matching method is presented for the more general case of an "arbitrary" upstream fluid temperature change

5.02 Mesh Definition

The heat equation is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function $U(x,y,z,t)$ of three spatial variables (x,y,z) and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

or equivalently

$$\frac{\partial u}{\partial t} = k \nabla^2 u$$

Where k is a constant.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. Crank, J.; Nicolson, P. [1947]. In statistics, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

5.03 Finite difference approximation

A finite-difference method for solving the time-dependent Navier Stokes equations for an incompressible fluid is introduced. This method uses the primitive variables, i.e. the velocities and the pressure, and is equally applicable to problems in two and three space dimensions. Test problems are solved, and an application to a three-dimensional convection problem is presented.

Equation systems describing one-dimensional, transient, two-phase flow with separate continuity, momentum, and energy equations for each phase are classified by use of the method of characteristics. Little attempt is made to justify the physics of these equations. Many of the equation systems possess complex-valued characteristics and hence, according to well-known mathematical theorems, are not

well-posed as initial-value problems (IVPs). Real-valued characteristics are necessary but not sufficient to insure well-posedness. In the absence of lower order source or sink terms (potential type flows), which can affect the well-posedness of IVPs, the complex characteristics associated with these two-phase flow equations imply unbounded exponential growth for disturbances of all wavelengths.

Analytical and numerical examples show that the ill-posedness of IVPs for the two-phase flow partial differential equations which possess complex characteristics produce unstable numerical schemes. These unstable numerical schemes can produce apparently stable and even accurate results if the growth rate resulting from the complex characteristics remains small throughout the time span of the numerical experiment or if sufficient numerical damping is present for the increment size used. Other examples show that clearly nonphysical numerical instabilities resulting from the complex characteristics can be produced. These latter types of numerical instabilities are shown to be removed by the addition of physically motivated differential terms which eliminate the complex characteristics. Duwairi, H.M. [2005]

5.04 Stability and Convergence

The nth iterate minimizes the residual in the Krylov subspace Kn. Since every subspace is contained in the next subspace, the residual decreases monotonically. After m iterations, where m is the size of the matrix A, the Krylov space Km is the whole of Rm and hence the GMRES method arrives at the exact solution. However, the idea is that after a small number of iterations (relative to m), the vector xn is already a good approximation to the exact solution.

This does not happen in general. Indeed, a theorem of Greenbaum, Pták and Strakoš states that for every monotonically decreasing sequence $a_1, a_2, a_3, \ldots a_{m-1}, a_m = 0$, one can find a matrix A such that the $\|r_n\| = a_n$ for all n, where r_n is the residual defined above. In particular, it is possible to find a matrix for which the residual stays constant for m − 1 iterations, and only drops to zero at the last iteration.In practice, though, GMRES often performs well. This can be proven in specific situations. If A is positive definite. Where λmin(M) and λmax(M) denote the smallest and largest eigenvalue of the matrix M, respectively .Borjini, M.N., Mbow, C., Daguenet, M. [1999].

Conclusion and Recommendations

Interactions between matter and energy began in the Big Bang and continue today in everything from the microscopic jiggling of atoms to the gargantuan collisions of galaxies. Understanding the universe therefore depends on becoming familiar with how matter responds to the ebb and flow of energy. it is evident that many physical phenomena can be modeled using partial differential equations in particular heat transfer. In many cases analytical solutions are not enough thus we rely on numerical solutions to obtain more information on the inherent problems .In this paper we have observed that the application of numerical methods are limited to the cases where the functions under consideration are well behaved. To have a general way of solution we have to device new methods of discretizing the boundary conditions so as we can get solutions that are in line with the experimental results.

Thus we need to undertake more research on this topic to further our knowledge so that we can effectively utilize our limited resources for the betterment of humanity.

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