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General Relativity: The Core Concepts and Its Significance in Astrophysics

by

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Declaration

I the undersigned declare that this project is my original work and to the best of my knowledge has not been presented for the award of a degree in any other University.

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Declaration by Supervisor

This project has been submitted for examination with my approval as supervisor.

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Dedication

To all my family members, with love and appreciation.

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In the name of the only One, who makes all things possible.

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Abstract

The aim of this study is to understand the core or founding principles of the General Theory of Relativity in order to appreciate its sequential development as well as its importance in explaining various astrophysical phenomena in our universe.

Contents

Chapter 1

Introduction

The Theory of General Relativity has for very long been considered a very formidable one to the extent that, nearly half a century after its inception, many thought that the only two people who fully understood it were the co founders: *Albert Einstein* and *Marcel Grossmann.* However, it is my belief that this theory is not significantly more difficult to grasp than any others such as electromagnetism or quantum mechanics.

> The better we understand the process of scientific knowledge acquisition, the better we will be able to create conditions in which young scientists can follow the lead of innovators like Albert Einstein. *Donald Salisbury[12]*

Einstein's theory of General Relativity (GR) is a theory of gravity which asserts that matter causes the four dimensional space-time in which we live to be curved, and that our perception of gravity is a consequence of this curvature. The central idea of this theory is summed up in an elegant-though extremely complicated-set of equations called the Einstein's Field Equations (or EFE), appropriately. They can be written in a beautifully simple form

$$G = 8\pi T \tag{1.0.1}$$

The beauty of general relativity is that this simple formula explains gravity more accurately than Newtonian physics and is entirely consistent with large scale experiments.

> "Only by a mixture of physical reasonableness, mathematical simplicity and aesthetic sensibility can one arrive at Einstein's field equations. The general theory of relativity is, in fact, an example of the power of speculative thought." *Prof S. Chandrasekhar*

Physical Theories

One of the central challenges of physics is-and has always been-to predict how things move. We embark on this study by examining the main physical theory regarding motion that existed in the early twentieth century, the commonly known laws of motion attributed to the 16th century scientist *Sir Isaac Newton*. Here, we realize that what is one of the founding principles of GR, *the principle of equivalence*, had its base as a curious secondary property of Newton's laws. However Newton's idea of complete separability of space

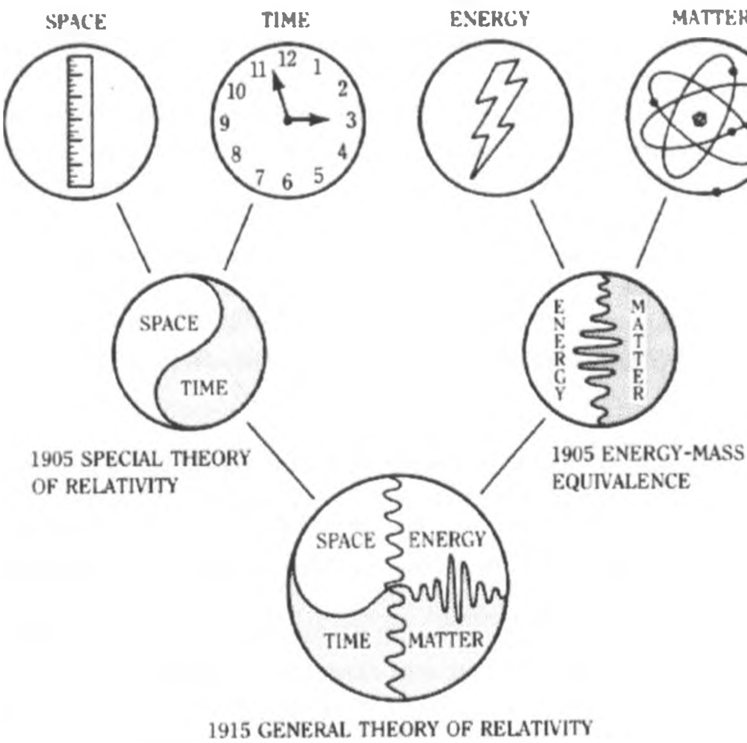

Figure 1.1: This figure shows the time line of progression of theories from Newtonian absoluteness of space, time, matter and energy to the unification of all four in GR.(Source: http://abyss.uoregon.edu/js/21st_century_science/lectures/lec07.html)

and time and the concept of absoluteness of the same break down when they are subjected to critical analysis. Einstein was convinced that there was no absolute space or time but rather that space-time was one continuum. Moreover, another theory, the one of Maxwell's electrodynamics was also conflicting with the idea of absoluteness. In 1905, Einstein presented his Special Theory of Relativity based on two postulates from which all results could be derived.

It soon became clear that while this theory could encompass mechanics and electromagnetism, gravity lay beyond its reach. The effort to reconcile special relativity with Newtonian gravitation theory turned out to be exceptionally demanding, and it led to the General Theory which transcends both these starting points. The situation with GR, is it doesn't overturn Newton's laws, rather it extends them. It turns out that in the kinds of situations that Newton was looking at, general relativity reduces down to Newton's laws.

Solutions of the Einstein's Field Equations

We have mentioned that the EFE's form the heart of GR. These equations are nonlinear partial differential equations and, as such, difficult to solve exactly. Nevertheless, a number of exact solutions are known, most of which have physical consequences such as the Schwarzschild solution, which implies the existence of *black holes*. The onslaught of GR brought with it numerous predictions about the universe we live in. An example *par excellence* of this was the prediction of the perihelion advance of Mercury, which solved an outstanding problem of Astronomy at the time.

Apart from this, GR's predictions also provide techniques for astronomical exploration. The Doppler Shift, the bending of light rays as they pass a massive object and other predictions are the primary techniques astronomers use to ascertain presence of objects which cannot be seen.

With the advent of high resolution telescopes, including the Hubble Space Telescope, there is no end to the astrophysical observations which continue to prove that it is indeed relativistic gravity, and not classical gravity, that rules the motion of heavenly bodies.

The Great Scientists

We will Sprinkle throughout the pages of this write-up, names of some of the scientists whose contributions were vital in the build up of the General Theory. The aim of this is not to down-play the extraordinary brilliance of Albert Einstein, but rather to appreciate all the other great minds and to clarify the misconception that Einstein was an isolated genius who created his new world through sheer inspired imagination![12]

Chapter 2

Brief Overviews of Classical/Newtonian Physics

2.1 Elementary Foundations of Newtonian Mechanics

The basis of Newtonian mechanics is contained in three fundamental postulates:

1. *The Law of Inertia*

   'A body in which no external force is acting will persevere in a state of rest or rectilinear motion.'

   The concept of mass or inertia is simply the capacity which is inherent in every material object to resist a change of motion. The greater the mass of the object the greater will be the force required to produce a given change of motion or acceleration. The quantity of motion (or momentum) is defined by $m(dx^i/dt) = mV = p$, where m is the mass and V the velocity. Thus the first law can also be stated that in the absence of external forces, the momentum is constant(or zero).

2. *The Law of Motion*

   'The change of motion of a body is proportional to the force which acts on it.'

   It is this law which is regarded as the central hypothesis of Newtonian mechanics. Mathematically this may be expressed as: .

   $$\mathbf{F} = m\frac{d^2x^i}{dt^2} = \frac{d\mathbf{P}}{dt} \tag{2.1.1}$$

   The substance of this law is that force is that which is required to produce a change of motion.

3. *The Law of Equality of Action and Reaction*

'When two bodies interact, for the force exerted by the first body on the second there is an equal and opposite force exerted on the second body by the first'

The foregoing laws together with the various theorems which are deducible from them constitute an instance of what philosophers of science call a physical model[1]. The most striking feature of this model is that all its hypotheses are in the form of *vector equations.*The vectorial character of these laws allow us to infer certain feature of the nature of the space in which they are maintained to hold.

In the first place, a vector is independent of position. Accordingly, whether a mechanical experiment is performed at $P(X_1^i)$ or at $P(X_2^i)$, the outcome should be unaffected. The implication of this is that from a mechanical point of view, space is *homogeneous*.i.e all points in space are equivalent. In the second place, a vector is independent of direction. This implies that the action of a force is independent of the direction in which it is acting. Hence from a mechanical standpoint, space is *isotropic*, i.e. the same in all directions. These two features of Newtonian space imply that the laws of Newtonian mechanics are covariant with respect to the inhomogeneous rotation group, which maybe labelled O_{i3}. Further, the principal invariant of O_{i3} is length or distance. Therefore, Newtonian space contains a length concept which is invariant with respect to the transformations O_{i3}, which reflects its symmetry. Such an invariant definition of distance is called a *metric*. The metric of Newtonian space is given by:

$$(ds^2) = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 = \delta_{ij} dX^i dX^j (i,j = 1,2,3) \qquad (2.1.2)$$

Just as Newton's laws reveal their independence of location and direction, they also reveal their independence of temporal location. Thus just like *space*, *time* in Newton's world, has the status of an *absolute* object or substance.

2.2 Newton's Law of Universal Gravitation

Newton determines that gravity controls the motion of objects in the Universe (i.e. Newton's apple). Galileo was the first to notice that objects are "pulled" towards the center of the Earth, but Newton showed that this same force (gravity) was responsible for the orbits of the planets in the Solar System. This idea is very familiar to us now, but the idea that the planets ought to behave according to the same rules as objects down here on Earth was a whole new concept at the time.

According to Newton, Objects in the Universe attract each other with a force **F** that varies directly as the product of their masses and inversely as the square of their distances. This is mathematically written thus:

$$\mathbf{F} = \frac{G m_1 m_2}{R^2} \tag{2.2.1}$$

where

$$\begin{aligned} G =& \text{gravitational constant} \\ m_1 =& \text{mass of first object} \\ m_2 =& \text{mass of second object} \\ R =& \text{the distance between the two objects.} \end{aligned}$$

All masses, regardless of size, attract other masses with gravity. You don't notice the force from nearby objects because their mass is so small compared to the mass of the Earth.

The Newtonian law of gravitation incorporates a truly remarkable feature. In the law of motion $\mathbf{F} = m\mathbf{a}$, the mass is that property of a body which is manifested by its resistance to change of motion. It is called the body's *Inertial mass* . On the other hand the role of mass in (2.2.1) is quite different. It is associated with the body's capacity to be a source of gravitational force. Accordingly it is called the *gravitational mass.* From a strictly logical standpoint, the inert mass and the gravitational mass might be independent quantities. Nevertheless, extremely accurate experiment has shown a precise proportionality between them which amounts to numerical equality. Within the context of Newtonian mechanics this equality is left as a remarkable coincidence. Einstein, however had a different view as we shall see in a subsequent chapter.

Chapter 3

The Principle of Special Relativity

3.1 Preamble

The special theory of relativity describes how objects move through space and time. It shows that time is not a universal quantity which exists on its own, separate from space. Rather, future and past are just directions, like up and down, left and right, forward and back, in something called space-time. You can only go in the future direction in time, but you can go at a bit of an angle to it. That is why time can pass at different rates.[4]

In a Newtonian Universe, there should be no difference in space or time regardless of where you are or how fast you are moving. In all places, a meter is a meter and a second is a second. You should be able to travel as fast as you want, with enough acceleration.

By the late 1800's, it was becoming obvious that there were some serious problems for Newtonian physics concerning the need for absolute space and time when referring to events or interactions (frames of reference). In particular, the newly formulated theory of electromagnetic waves (theory of electrodynamics of *James Clerk Maxwell*)required that light propagation occur in a medium.

In the 1890's, two physicists (*Michelson and Morley*) tried to measure the Earth's velocity around the Sun with respect to Newtonian absolute space and time. This would also test how light waves propagated since all waves must move through a medium. For light, this medium was called the aether.

The experiment attempted to measure the movement of the earth through the aether by its effect on the motion of light. To their astonishment and everyone's dismay they could not detect any change whatsoever. It was not that the null reading was disappointing, it was simply incomprehensible. If light and matter both move through space how is it possible for the earth to be hurtling through space at 18.5 miles per second and have no effect on light?

At this point, Newtonian physics was in a profound conceptual crisis. The Michelson-Morley result meant either of two conditions. Either light and matter have unrelated motions and move to different space references; or light and matter move through space alike as Newton assumed, and light's velocity is always constant because changes in the physical state of bodies in motion prevent measuring any effect on the motion of light.

If the first supposition is correct, then there is a medium for light and we have to readjust the way we think of space. If the second supposition is correct, then space is an unreactive void and we have to change our thinking about the effect of motion on the physical conditions of matter. It was this latter supposition that Einstein followed in developing his theory of relativity.

Einstein kept Newton's physics and his assumption that light and matter move alike through a space void. The difference is he dismissed Newton's universal space as a background reference and made all motion relative. The consequences of this theory are what are known generally as "relativistic effects." in which length contracts, mass increases, and time slows, as objects move faster.

Thus was released, in the paper of 1905, the theory of Special Relativity in which Einstein provided the method of reconciling mechanics and electrodynamics, two largely disjoint disciplines, in both of which he was a self-taught master. As articulately put by the philosopher of science, Roger B. Angel [1], one is struck by the exquisite simplicity of this paper in that although the central ideas have rich physical implications, they are not based on new physical assumptions. In fact all of the physical hypotheses necessary for Einstein's new theory had been available in the nineteenth century. What was essentially new was Einstein's profound revision of the classical concepts of space and time.

3.2 The Foundational Postulates

Some of the relativistic effects mentioned above are the Doppler Effect, Aberration of light, Time Dilation and Length Contraction. All these effects have been successfully tested experimentally, the latter two having been tested extensively using different clocks and measuring rods. It is because of this that earlier generations of philosophers thought that the special theory had its basis in the hypotheses about the behaviour of clocks and measuring rods. However, as we shall see, this is incorrect. In actuality, to formulate his Special theory of Relativity, Einstein presented two fundamental postulates from which all the results could be derived.

1. The laws of physical phenomena are same in all the inertial frames of reference,

2. The velocity of light (in free space) is the universal constant independent of the motion of the source.

These two claims had until then been considered incompatible. Einstein's standpoint was that in fact there was no formal logical incompatibility between them. There remained, however the task of finding an appropriate set of transformations between electromagnetic inertial frames i.e frames with respect to which Maxwell's laws hold in their basic form.

3.3 The Lorentz Transformations

We use a simplified form [1] of Einsteins own procedure to derive these transformation laws. We require that the optical wave front equation $(x^1)^2 + (x^2)^2 + (x^3)^2 = c^2t^2$ be covariant when referred to a second frame which has a uniform velocity with respect to the original one.

For ease of interpretation we use the coordinate variables X, Y, Z. Consider two inertial frames K and $\overline{K}$ whose corresponding axes are always parallel and the respective origins coincide at $t = 0$. The relative velocity in the x-direction between the two frames is v. With respect to K, the law of interest is of the form

$$x^2 + y^2 + z^2 = c^2t^2 \tag{3.3.1}$$

$\overline{K}$ is also an inertial frame. Accordingly the form of the wave-front equation must, with respect to $\overline{K}$ be

$$\overline{x}^2 + \overline{y}^2 + \overline{z}^2 = c^2\overline{t}^2 \tag{3.3.2}$$

To determine the appropriate group of transformations for (3.3.1), we first take into consideration the symmetry and relativity which require that in our restricted case, we must have the mappings

$$y \rightarrow \overline{y} = y, z \rightarrow \overline{z} = z. \tag{3.3.3}$$

Secondly, considerations of relativity require that the uniformity of expansion of the wave-front in one inertial frame entail its uniformity in every inertial frame. The preservation of uniformity provides the valuable clue that the transformation be linear. Using this property and the fact that distances should be transformed into distances and times into times, it follows there cannot be relative acceleration between the two frames. From all these qualitative considerations which have some rigorous mathematics behind them, the required form of the general transformation can thus be written :

$$\begin{aligned} x &\rightarrow \overline{x} = f(x) - g(vt) \\ y &\rightarrow \overline{y} = y \\ z &\rightarrow \overline{z} = z \\ t &\rightarrow \overline{t} = j(t) + h(x/v). \end{aligned} \tag{3.3.4}$$

Applying the transformations (3.3.4) to (3.3.1) we get

$$f^2(x) - 2g(vt)f(x) + g^2(vt) + y^2 + z^2 = c^2j^2(t) + 2c^2j(t)h(x/v) + c^2h^2(x/v). \tag{3.3.5}$$

Choosing f, g, j as follows

$$f(x) = x; j(t) = t; g(vt) = vt$$

and substituting into (3.3.5) yields

$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 + 2c^2th(x/v) + c^2h^2(x/v). \tag{3.3.6}$$

Setting $h(x/v) = -vx/c^2$ in this equation gives

$$x^2(1 - v^2/c^2) + y^2 + z^2 = c^2t^2(1 - v^2/c^2). \tag{3.3.7}$$

Dividing this by the factor $(1-v^2/c^2)^{\frac{1}{2}}$, we get the required multiplicative factor and our required mappings are:

$$
\begin{aligned}
x \rightarrow \overline{x} =& (x - vt)/(1 - v^2/c^2)^{\frac{1}{2}} \\
y \rightarrow \overline{y} =& y \\
z \rightarrow \overline{z} =& z \\
t \rightarrow \overline{t} =& (t - vx/c^2)/(1 - v^2/c^2)^{\frac{1}{2}}.
\end{aligned} \tag{3.3.8}
$$

Equations (3.3.8) constitute the homogeneous Lorentz Transformation, which is applicable not only to the wave-front equation but also to the more general basic equations of Maxwell's electrodynamics. Further this transformation proves that there is no contradiction between the universality of c and the principle of relativity. However, the compatibility requires that electrodynamics be Lorentz covariant.

The most startling aspect of the Lorentz transformation is that *time is relative.* That is to say in relativistic physics, given the time of an event in a frame K, it will not have the same value of time with respect to $\overline{K}$.

In virtue of the ubiquity of the factor ($\gamma = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}$), it follows that if $v > c$, lengths and durations would acquire imaginary values to which no physical interpretation can be given. In particular the equations become singular or indeterminate when $v = c$. Thus according to Special Relativity, the velocity of light is not only a universal constant, but moreover is the upper bound for all physical velocities with respect to initial frames of reference.

3.3.1 The Role of factor γ in the Concept of Mass

The factor γ is the key thing, which tells you how relativistic things are.

- When $v << c$, $\gamma = 1$ and we have Newton's laws.
- As $v \rightarrow c$, $\gamma = \infty$, and we have relativistic effects.
- In the intermediate case, we have the Post Newtonian Approximation.

Example 1. *The relativistic mass is related to the Newtonian mass by the following expression*

$$
m = \frac{1}{(1 - v^2/c^2)^{\frac{1}{2}}} m_0 \tag{3.3.9}
$$

1. *When $v << c$, $\gamma = 1$ and $m = m_0$.*
2. *As $v \rightarrow c$, $\gamma = \infty$, mass becomes infinite and this proves that c is the upper limit of velocity.*

3. When $v/c \neq 0$ but is $<< 1$, we obtain a case of the mass-energy equivalence ($E = mc^2$).

Thus, it is seen that for velocities which are very small as compared to c, the velocities of everyday life, may be ignored and the Lorentz transformation becomes effectively identical with the Galilean Transformation (which was used for Newtonian Mechanics). This explains why relativistic effects went undetected for so long.

3.4 Four-Vector Formalism and Space-Time Geometry

Einstein did not drop the concepts of space and time but rather took their relativistic reinterpretation. The approach derives from the contributions of his older contemporary and teacher *Hermann Minkowski*. It was he who first clearly perceived that special relativity may be more adequately viewed against the background of four-dimensional spacetime. In short, he realized that the various relativistic phenomena should be coordinated with a new kind of geometry.

We have already seen that the wave-front equation is Lorentz covariant. A similar expression which also has the property of Lorentz covariance is

$$s^2 = x^2 + y^2 + z^2 - c^2t^2. \tag{3.4.1}$$

This expression defines the space-time interval between two events. One may regard the Lorentz transformation as a rotation in four-dimensional spacetime. With respect to one set of coordinates there is more spatial separation between events and more temporal separation, with respect to another set there is less, but in all cases the total space-time separation is invariant. Such considerations occasioned the famous remark of Minkowski, in a paper written in 1908, in which special relativity was given its four dimensional formulation.

> Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

In the Minkowski formalism time is treated as a fourth independent coordinate in addition to the X^i. Specifically he sets $X^4 = ict$, where i is the imaginary number $\sqrt{-1}$, so that $i^2 = -1$. The differential form of (3.4.1) may be represented as

$$(ds)^2 = \delta_{\mu\nu} dX^\mu dX^\nu. \tag{3.4.2}$$

This is the expression of the spacetime interval. It is the *metric form* of special relativity. If the norm of a four-vector, in this case ds^2 imparts information about the causal structure of the spacetime. When $ds^2 < 0$, the interval is time-like and the square root of the absolute value of ds^2 is an incremental proper time. Only time-like intervals can be physically traversed by a massive object. When $ds^2 = 0$, the interval is light-like, and can only be traversed by light. When $ds^2 > 0$, the interval is space-like and the square root of ds^2 acts as an incremental proper length. Space-like intervals cannot be traversed, since they connect events that are out of each other's light cones. Events can be causally related only if they are within each other's light cones.

In classical physics, it is natural to depict the trajectory or path of an object in terms of the distance $s(t)$ expressed as a function of time. In relativistic physics the natural parameter for the trajectory of a particle is its proper- time τ.

Spacetime is not merely a generalisation from three to four dimensions. Although space and spacetime are both flat and infinitely extended continuums, space is a manifold of points whereas spacetime is a manifold of events(which refers to a point in time). Their profound difference lies in the definitions of their metrics. The metric of space is given by

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{3.4.3}$$

whereas the metric for spacetime is

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dt)^2. \tag{3.4.4}$$

The important distinction here is not in the difference of dimensionality but the difference of *signature*. No matter what type of coordinate system happens to be chosen in which to express the metric of space, the metric coefficients will all have positive signs. Thus one of the invariant or intrinsic properties of Euclidean space is its signature, which is represented by $(+++)$. A metric of this kind is said to be *positive definite.* In contrast, the invariant signature of Minkowski spacetime is $(+++-)$. A signature of this kind is called *indefinite.* A pertinent difference between the two types of metric is that whereas the separation or interval between two non-coincident points is always positive in the first case, in the latter, it may be positive, negative or even zero. In fact, such a metric is often called a *pseudo-metric*, since it violates some of the basic properties which are normally imposed on metric functions.

In this chapter, we have seen that in order for the postulates of the special theory to hold, the laws of nature must be Lorentz covariant. It was also seen that the new concept of space and time (the continuum) could be modelled by Minkowski spacetime. This new formalism enabled the construction of various geometric objects on the spacetime manifold which saw space and time being expressed as a four-vector, momentum and energy combined to form another four-vector etc. These spacetime objects have been used to produce a relativistic analogue to classical mechanics which led to a series of truly remarkable physical consequences, all of which have been experimentally confirmed. Although spacetime may be treated as a mere formal convenience in special relativity, it proves to be a conceptual necessity in the deeper context of general relativity.

Chapter 4

The Path to General Relativity

General relativity is essentially a space-time theory of gravity. Its laws must be formulated in four-dimensional terms. That, in itself poses no great problem. However, whereas the four-dimensional manifold of special relativity was flat, that of general relativity is curved. Further, its curvature varies from region to region according to the distribution of matter. The basic task of experimental relativity is to determine the nature and degree of curvature in the various parts of the universe.

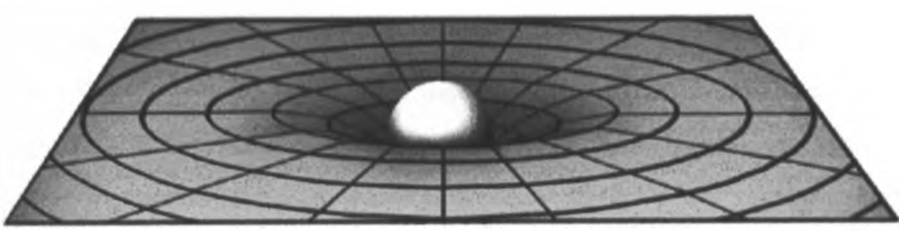

Figure 4.1: A depiction of matter curving space-time as asserted by GR. (Source:http://abyss.uoregon.edu/js/21st_century_science/lectures/lec07.html)

Furthermore, Einstein's standpoint on gravity is fundamentally different from the classical viewpoint. As concisely summarised by Stephen Hawking [4], the general theory of relativity explains that, "gravity was not just a force that operated in a fixed background of space-time. Instead, gravity was a distortion of space-time, caused by the mass and energy in it. Objects like cannonballs and planets try to move on a straight line through space-time, but because space-time is curved, warped, rather than flat, their paths appear to be bent."

4.1 The Heuristic Principles

Einstein utilised three heuristic principles in the torturous path to the field law of gravitation.

- Firstly, it should be generally covariant.
- Secondly, it should approximate the Newton-Poisson field law in the limit of low velocities and weak fields.
- Finally, it should admit the laws of special relativity in the restricted case of vanishing gravitation.

The first principle is known as *The Principle of General Covariance* and it simply means that the field laws should hold in all reference frames. Since the space-time of general relativity is curved and its curvature varies from region to region, forming field laws that are generally covariant was no easy task even for Einstein!

Let us define the curvature of a surface. Given a point P on the surface, its Gaussian or intrinsic curvature can easily be measured. Thus, a scalar curvature field may be defined on any smooth, two-dimensional manifold. However, when one generalises from two to n dimensions, it is no longer possible to represent the curvature by a scalar field. What is now required is a tensor field called the *curvature tensor*. This important generalisation was provided by Riemann.

Since in the above-mentioned sense, there can be no pre-determined geometric structure to the universe, it follows that there can be no privileged coordinate system, either Cartesian or Minkowskian, for the description of the universe. Consequently, the laws of physics should be formulated in such a way that they reflect no particular property of one coordinate system or another. The mathematical formalism that was employed to achieve complete independence of any given system of coordinates is called *tensor analysis*. General relativity is formulated completely in the language of tensors. Einstein had learned about them, from the geometer *Marcel Grossmann*, who was his friend and later, his colleague.

4.2 Tensor Analysis

Like vectors, tensors are geometric objects having a linear homogeneous law of transformation. Specifically, scalars and vectors are just two (special) kinds of tensors. A scalar is said to be a tensor of rank zero while a vector is a tensor of rank one. The metric tensor $g_{\mu\nu}$ is an example of a tensor of rank two.

4.2.1 Different Tensors and their Transformation Laws

The covariant components of a tensor are denoted by subscripts. For example, A_{ij} is a covariant tensor of rank two and its transformation law is given by:

$$\overline{A}_{pq} = \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}A_{ij}. \tag{4.2.1}$$

Similarly, the contravariant components of a tensor are denoted by superscripts, for example,A^{ij} is a contravariant tensor of rank two. Its transformation law is given by:

$$\overline{A}^{pq} = \frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}A^{ij}. \tag{4.2.2}$$

The transformation law for a tensor of a given rank and type follows directly from (4.2.1) and (4.2.2). For example, the mixed tensor A^i_j would transform as follows:

$$\overline{A}^p_q = \frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^q}A^i_j. \tag{4.2.3}$$

The profound significance of (4.2.3) is that the transformation is linear and homogeneous. Thus, if each component of a tensor has a particular value at a given point of the manifold as evaluated with respect to a given coordinate system, the value of each component of the same tensor at the same point as evaluated with respect to a second arbitrary coordinate system, is obtained by pre-multiplying the component in the first system by a set of numbers, the results of which are then summed. The important consequence of this is that if a tensor vanishes in one coordinate system, i.e all its components are zero, then it must vanish in every system. This, in turn, implies that a tensor equation which holds in one coordinate system must hold in every coordinate system. Thus, tensors are precisely the sort of mathematical entity needed for general relativity.

4.2.2 The Metric Tensor

Mathematically, spacetime is represented by a 4-dimensional differentiable manifold M and the metric is given as a covariant, second-rank, symmetric tensor on M, conventionally denoted by g. Moreover, the metric is required to be nondegenerate.

Physicists usually work in local coordinates (i.e. coordinates defined on some local patch of M). In local coordinates x^μ, the metric can be written in the form

$$g = g_{\mu\nu} dx^\mu dx^\nu. \tag{4.2.4}$$

The factors dx^μ are one-form gradients of the scalar coordinate fields x^μ. The metric tensor is thus a linear combination of tensor products of one-form gradients of coordinates such that

$$g_{\mu\nu} = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu}. \tag{4.2.5}$$

The coefficients $g_{\mu\nu}$ are a set of 16 real-valued functions (since the tensor g is actually a tensor field defined at all points of a spacetime manifold). In order for the metric to be symmetric, we must have

$$g_{\mu\nu} = g_{\nu\mu}. \tag{4.2.6}$$

With the quantity dx^μ being an infinitesimal coordinate displacement, the metric acts as an infinitesimal invariant interval squared or line element. For this reason one often sees the notation ds^2 for the metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{4.2.7}$$

4.2.3 Christoffel Symbols

The Christoffel symbols are nothing but functions of the first partial derivatives of the metric tensor $g_{\mu\nu}$. They are not tensors as they differ in their transformation properties from a tensor. The Christoffel symbol of the first kind is defined as

$$[\mu\nu, \sigma] = \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right) \tag{4.2.8}$$

and the Christoffel symbol of the second kind is defined as

$$\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right). \tag{4.2.9}$$

From (4.2.8) and (4.2.9) it follows that

$$\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\lambda\sigma} [\mu\nu, \sigma]. \tag{4.2.10}$$

4.2.4 Covariant Differentiation

The covariant derivative is a derivative of tensors that takes into account the curvature of the manifold on which these tensors are defined, as well as dynamics of the coordinate basis vectors. In Cartesian coordinates, the covariant derivative is simply a partial derivative ∂_a. The covariant derivative is also known as the semi-colon derivative and is written as $A_{;a} = \nabla_a A = D_a A$.

The rule for covariant differentiation for a covariant tensor of rank two is

$$A_{pq;r} = A_{pq,r} - \left\{ \begin{array}{c} s \\ pr \end{array} \right\} A_{sq} - \left\{ \begin{array}{c} s \\ qr \end{array} \right\} A_{ps}. \tag{4.2.11}$$

The corresponding rule for the contravariant tensor is

$$A^{pq}_{;r} = A^{pq}_{,r} - \left\{ \begin{array}{c} p \\ rs \end{array} \right\} A^{sq} - \left\{ \begin{array}{c} q \\ rs \end{array} \right\} A^{ps}. \tag{4.2.12}$$

Note:*The covariant differentiation of tensors of higher rank is done in a similar manner.*

4.2.5 The Riemann-Christoffel Tensor and Ricci Tensor

Of particular interest to us is the *Riemann Christoffel Tensor* also known as the *Riemann Curvature Tensor.* This tensor is a function of the metric tensor and of its first and second derivatives. Its significance is as a precise measure of the curvature of the manifold in the region where it is evaluated.

The Riemann curvature tensor is defined

$$R^{\lambda}_{\mu\nu\sigma} = \frac{\partial}{\partial x^{\nu}} \left\{ \begin{array}{c} \lambda \\ \mu\sigma \end{array} \right\} - \frac{\partial}{\partial x^{\sigma}} \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} + \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} \left\{ \begin{array}{c} \lambda \\ \alpha\nu \end{array} \right\} - \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} \left\{ \begin{array}{c} \lambda \\ \alpha\sigma \end{array} \right\}, \tag{4.2.13}$$

where $\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}$ are the Christoffel symbols of the second kind.

In the four-dimensional space-time manifold, the Riemann curvature tensor would have 256 distinct components. However, due to various symmetries there are, in fact, only 20 distinct components. To know the values of these is to know everything about the local geometry of the manifold. As we shall see in a following section, this tensor supplies the clue to the formulation of the law of gravitation of general relativity.

If we contract λ and σ in this tensor and using the identity

$$\left\{ \begin{array}{c} \mu \\ \mu\lambda \end{array} \right\} = \frac{\partial log\sqrt{g}}{\partial x^{\lambda}}, \tag{4.2.14}$$

we get,

$$R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma} = \frac{\partial^2 log\sqrt{g}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial}{\partial x^{\sigma}}\left\{ \begin{array}{c} \sigma \\ \mu\nu \end{array} \right\} + \left\{ \begin{array}{c} \alpha \\ \mu\sigma \end{array} \right\}\left\{ \begin{array}{c} \sigma \\ \alpha\nu \end{array} \right\} - \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\}\frac{\partial log\sqrt{g}}{\partial x^{\alpha}} \tag{4.2.15}$$

a tensor called *Ricci Tensor*, which is a symmetric tensor.

The ***scalar curvature (Ricci scalar)*** of a Riemannian manifold M

$$R = g^{\mu\nu}R_{\mu\nu} \tag{4.2.16}$$

is a map $M \rightarrow \mathrm{R}$ that characterizes the intrinsic curvature of the manifold at every $x \in M$.

4.2.6 Energy-Momentum Stress Tensor

In General Relativity, the energy-momentum stress tensor (or simply stress tensor),$T^{\mu\nu}$ of a perfect fluid is given as:

$$T^{\mu\nu} = (\rho + p/c^2)u^{\mu}u^{\nu} - pg^{\mu\nu} \tag{4.2.17}$$

and the vanishing of its divergence is expressed as:

$$T^{\mu\nu};\mu = 0. \tag{4.2.18}$$

In relativity, energy and momentum are the temporal and spatial parts of a single 4-vector p^{μ}. In the stress tensor this has been replaced by the 4-momentum density ρu^{μ}.

The stress tensor is a symmetric tensor which represents the energy content of the fluid and which, when taken over to curved space-time, acts as the source to the gravitational field.

Since in relativity, we lose the distinction between mass and energy, all forms of energy should produce a gravitational field. Moreover, energy is not a scalar but only the zeroth component of the 4-momentum.

As a result, the matter content of spacetime is concisely summarised in the stress tensor $T^{\mu\nu}$ and with suitable definitions of $T^{\mu\nu}$, (4.2.18) is valid for all fluids and fields (not just perfect fluids).

Remark 1. *In his arduous journey to the General Theory, Einstein (1915) had attempted to learn how to construct adequately covariant objects by studying the first and second Beltrami operators acting on scalars. Incidentally, in the process he came remarkably close to developing an understanding of covariant derivatives that, as will surprise most relativists, was first enunciated later by the Italian mathematician* Tullio Levi-Civita *in 1917 [12]. Later in that year, Levi-Civita initiated a correspondence with Einstein to correct mistakes Einstein had made in his use of tensor analysis. The correspondence lasted 1917-18, and was characterized with mutual respect, with Einstein at one point writing:*

> I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the likes of us have to make our way laboriously on foot. *Einstein, to Levi-Civita on tensor analysis.*

This brief account on tensors gives us sufficient machinery to embark on the formulation of the relativistic field law of gravity. However, we need to understand why gravity is not a force, but the manifestation of the curvature of spacetime.

4.3 Geodesics

Objects move in a straight line, in the absence of an external force. This is Newton's first law. What do you mean by going in a straight line if you're in a curved space? In this case, a straight line is the shortest distance between two points.

Geodesics are the special intrinsic curves on the surface that are analogous to the straight lines in the Euclidean space as they are curves of extreme length. A geodesic is also defined as the path of shortest distance on a surface between two given points on it.

For a relevant example, imagine going between two points on the Earth's surface. Suppose you want to take a flight from New York City to Rome, Italy. These two cities are at nearly the same latitude. However, when the plane takes off from New York, it won't be headed due East. If the pilot chose this route, the plane would end up in Africa, or would have to be turning left for the whole trip. (See for yourself with a little toy car and a globe.) Instead, the pilot heads a little North of East-by 33 degrees, if we ignore winds. This way, the plane can go in a straight line, and end up in Rome. That is, the plane can follow a geodesic to make the flying easier.

Now we can visualize the trip. The whole time, the plane can just keep flying straight and level. The pilot does not need to be turning either left or right, but will end up in Rome nonetheless. It turns out that this is also the fastest route; a "straight line" is still the shortest distance between two points, even if it is a straight line that curves. This is, basically, the path that flights from New York to Rome actually follow. We have just described geodesics in space; the plane's path follows a geodesic along the two-dimensional surface of the Earth.

The first curvature vector P of a parameterized curve C on a Riemannian manifold M is:

$$P_i = \frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \frac{dx^p}{ds}\frac{dx^q}{ds} \tag{4.3.1}$$

A curve on M whose first curvature is zero is called a geodesic. Thus, a geodesic is a curve that satisfies the system of second order differential equations

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \frac{dx^p}{ds}\frac{dx^q}{ds} = 0 \tag{4.3.2}$$

4.4 The Principle of Equivalence

In chapter one, we saw that classical mechanics requires that we distinguish between inertial and gravitational mass. The truly remarkable feature of this property is that their ratio is equal to one and the same universal constant for all bodies, regardless of their size or density. This allows us to ascribe the same number to the two types of mass of a given body i.e. to make this ratio equal to unity. However, this remained as an unexplained curious theoretical principle of Newtonian Mechanics.

Einstein was motivated by the conviction that nothing in nature is arbitrary. Accordingly, he took the position that the two kinds of mass could be equal only if they were, in fact, one and the same. This claim of numerical identity for two seemingly independent properties of matter entailed a profound revision of the conceptual framework in which they occur.

Einstein accounts for the close link between inertia and gravitation in the following way [1]. Suppose that K is an inertial frame of reference with respect to which a certain body B is in uniform motion. Let $\overline{K}$ be accelerating uniformly with respect to K. The body B, which is not influenced by external forces will, of course, accelerate with respect to $\overline{K}$. However, it is obvious that the actual acceleration is wholly independent of the composition or state of B itself, since it is due solely to the acceleration of the system of reference $\overline{K}$.

Now from a classical standpoint, the acceleration of the body would provide a basis for an observer at rest in $\overline{K}$ to be able to detect the acceleration of his own reference frame. Einstein pointed out that such a viewpoint is incorrect. Although the acceleration of B is independent of its intrinsic nature, one may not conclude that it is merely the kinematic consequence of choosing $\overline{K}$ as a frame of reference. Specifically, this is because the same type of acceleration could have been imparted to the body by a suitably chosen gravitational field. Hence, the observer in $\overline{K}$ could claim to be attached to an inertial frame by holding the body of interest to be under the influence of a gravitational field. In short, inertial effects, which are the result of the choice of reference frame, are of the same kind as those which are due to gravity. So an observer who occupies a frame of reference which is apparently inertial and who suddenly experiences a series of phenomena which might lead him to suspect that it has ceased to be inertial would always have at his disposal the possibility of postulating the existence of a suitable field of gravity as the cause of the phenomena.

The *principle of equivalence* is simply the claim that inertial fields and gravitational fields, together with their effects, are indistinguishable. The same name also refers to the closely related principle of the equivalence of gravitational and inertial mass. Einstein took the position that if no experiment can reveal a difference between the two kinds of mass, then it is redundant to postulate that two masses exist. Accordingly, the basis for the principle of equivalence is that gravitational and inertial mass are not merely equal but identical! Since gravitational mass is that property of a body which responds to the gravitational field while inertial mass may be thought of that property of the body which responds to an inertial field, it follows that if the two properties are really one, then the two fields are really one, in the sense that they must be of the same kind.

To further explain this, suppose that one is in a lift in which a weight is suspended from the ceiling by an extension spring. If the gravitational field acting on the weight were suddenly to increase, the spring would be observed to undergo an increase of length. This would be an indication that the weight has become heavier. However, precisely the same extension of the spring would be observed if the lift were to undergo a sudden upwards acceleration. But in this case, the lengthening of the spring would be accounted for by the action of a field of inertia instead of one of gravity.

A truly striking feature of the principle of equivalence is its heuristic significance. This is almost certainly explicable by its being a factual principle of nature. Very simply, in order to ascertain, at least qualitatively, the effects of gravitation on a certain kind of phenomenon, one has merely to consider how that phenomenon would be described when referred to an appropriately accelerating frame of reference. Then in virtue of the equivalence of inertial and gravitational descriptions, one immediately determines how that phenomenon will be affected by a gravitational field.

The most celebrated example of the application of this principle is probably Einstein's prediction of the bending of light-rays by a sufficiently strong gravitational field. Imagine that a light-ray is beamed from one wall of a lift to the opposite one. Now if the lift is made to ascend very rapidly the effects which take place inside it are indistinguishable from the effects that would be produced by a sudden increase in the field of gravity. It is fairly clear that the beam of light will not hit the opposite wall at the point one would expect if the lift were stationary or moving uniformly, but at a somewhat lower point. In other words, the light ray is curved when referred to the accelerating frame. It follows on the basis of the principle of equivalence that light-rays must be curved or deflected from their straight path by a gravitational field. This consideration enabled Einstein, several years before he had formulated his general relativistic law of gravitation, to predict that starlight will be deflected as it passes the massive sun.

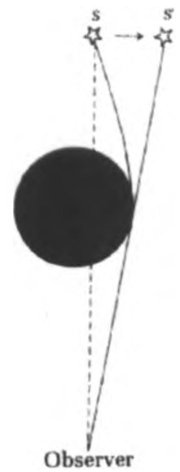

Figure 4.2: The bending of starlight as it passes the a massive object e.g the sun.(Source:http://abyss.uoregon.edu/js/21st_century_science/lectures/lec07.html)

4.5 Essence of the Theory: Einstein's Field Equations

The Einstein's Field Equations (EFE)are the fundamental equations of Einstein's general theory of relativity. Einstein and Marcel Grossman had realised that the metric tensor $g_{\mu\nu}$ describing the geometry of spacetime seemed to depend on the amount of gravitating matter in the region in question.(and so adopted the kernel letter g for gravity)[8].

The metric tensor contains two separate pieces of information:

1. The relatively unimportant information concerning the specific coordinate system used(e.g. spherical, polar, cylindrical etc).

2. The important information regarding the existence of gravitational potentials.

It is seen that in a nearly Cartesian coordinate system, g_{00} was essentially the Newtonian potential. In a more general coordinate system, this Newtonian potential would be dispersed throughout the $g_{\mu\nu}$ so there is a sense in which all the components $g_{\mu\nu}$ can be regarded as gravitational potentials. Since the matter content of spacetime is concisely summarised in the stress tensor $T^{\mu\nu}$, if matter causes the geometry, then it might be tempting to put that

$$g^{\mu\nu} = \kappa T^{\mu\nu}, \tag{4.5.1}$$

where κ is some coupling constant.

This looks plausible because both $g^{\mu\nu}$ and $T^{\mu\nu}$ are symmetric and $g^{\mu\nu};\mu = 0$ is in agreement with $T^{\mu\nu};\mu = 0$. However (4.5.1) does not reduce to Poisson's equation in the Newtonian limit.

Since $g^{\mu\nu}$ are the gravitational potentials, it is clear that what is needed in place of it in (4.5.1) is a symmetric tensor involving the second derivatives of $g_{\mu\nu}$.

Einstein, in 1915, published his belief in the equation for the relationship between metric tensor and matter as:

$$R^{\mu\nu} = \kappa T^{\mu\nu}, \tag{4.5.2}$$

where $R^{\mu\nu}$ is the contravariant Ricci Tensor. Again this looks plausible since $R^{\mu\nu}$ is symmetric and contains second derivatives of $g_{\mu\nu}$. However, $R^{\mu\nu}$ does not satisfy $R^{\mu\nu};\mu = 0$ and later in the same year Einstein modified the equation to

$$G^{\mu\nu} = \kappa T^{\mu\nu}, \tag{4.5.3}$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein Tensor, and $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar.

The expression on the left of (4.5.3) represents the curvature of spacetime as determined by the metric and the expression on the right represents the matter/energy content of spacetime. The EFE can then be interpreted as a set of equations dictating how the curvature of spacetime is related to the matter/energy content of the universe. These equations form the core of the mathematical formulation of general relativity.

The Einstein field equations are a system of second order coupled non-linear partial differential equations for a Riemannian metric tensor on a Riemannian manifold. One possibility is that the tensor field $T_{\mu\nu}$ is specified and that these equations are then solved to obtain g. A noteworthy case of this is the vacuum Einstein equations, in which $T_{\mu\nu}$=0.

Another possibility is that $T_{\mu\nu}$ is given in terms of some other fields on the manifold and that the Einstein equations are augmented by differential equations which describe those fields. In that case, one speaks of Einstein-Maxwell equations, Einstein-Yang-Mills equations, and the like depending on what these other fields may happen to be.

It should be noted that, on account of the Bianchi identity, there is an integrability condition $\nabla_\mu(g)T^{\mu\nu} = 0$. (Here,$\nabla_\mu(g)$ denotes covariant differentiation with respect to the Levi-Civita connection of the metric tensor $g_{\mu\nu}$). When choosing $T_{\mu\nu}$, these conditions must be taken into account in order to guarantee that a solution is possible.

Chapter 5

Relativistic Astrophysics

Relativistic Astrophysics deals with those aspects of the physical explanation of astronomical phenomena which depend on the theory of Relativity. Since its inception, General Relativity has largely relied on astronomical evidence for its support. While there is no end to the need to test any theory, Relativity can be taken as satisfactorily established.

5.1 Experimental Proofs of GR

The first and most exemplary observational proof for GR was the prediction of the *perihelion advance of Mercury* which solved an outstanding problem of astronomy at the time. It had long been known that the perihelion of Mercury, the point at which Mercury is closest to the sun, is gradually advancing so that the planetary orbit is not quite closed. Newtonian Mechanics proved incapable of providing a satisfactory explanation.

However, when Einstein created his new theory of gravity, it gave in a most natural way, the prediction for Mercury's orbit that is a little bit different from the Newtonian case in just the right way to explain this problem that people had been trying to solve for fifty years unsuccessfully. Thus, this was the first empirical verification, of general relativity.

Apart from this, there are predictions of *black holes, gravitational waves and gravitational lensing* among others. Strong evidence exists for each of these phenomena today.

In this chapter, we will examine three ways in which GR plays an important role in the field of Astrophysics.

1. Solutions of EFE give predictions of phenomena,

2. Techniques from GR, and

3. GR as the main source of explanations.

5.1.1 The Schwarzschild Solution and Black Holes

In the past fifteen years or so black holes have been converted into a standard topic in observational astrophysics. There are dozens, probably hundreds of objects we can point to in the sky and say, "yes those things are black holes."

But what is a black hole? A black hole is simply something in which the escape velocity is greater than or equal to the speed of light. If the escape velocity is greater than the speed of light, then light cannot escape this object, so one is not able to see it.

Now the escape velocity, V_{esc}, i.e. the speed required to escape the gravitational field of an object; supposing that the object has mass equal to M and radius equal to r, is given as

$$V_{esc} = (\frac{2GM}{r})^{1/2}. \tag{5.1.1}$$

If this is equated to c and r made the subject of the formula, we have

$$r_s = (\frac{2GM}{c^2}). \tag{5.1.2}$$

The s in the subscript is due to *Karl Schwarzschild* who discovered it and hence it is called the Schwarzschild radius.

Therefore, another definition of a black hole is something in which the radius of the object is less than the Schwarzschild radius-because if the radius is less, then the escape velocity will be even greater.

The Schwarzschild solution is the most general spherically symmetric, vacuum solution of the Einstein field equations. The Schwarzschild black hole or static black hole is a black hole that has no charge or angular momentum. The Schwarzschild black hole is characterized by a surrounding spherical surface, called the *event horizon*, which is situated at the Schwarzschild radius, often called the radius of a black hole.

This is a a point in space where the escape velocity is equal to the speed of light. The reason it is called that is because, if nothing can go faster than the speed of light, what it means is that any event that takes place inside this imaginary sphere can't radiate any information about what's going on to the outside, because the escape velocity is greater than the speed of light. Light can not escape it, and because nothing else can go faster than the speed of light, nothing else can escape. So, no information of any kind can come from inside the event horizon to the outside.

Any non-rotating and non-charged mass whose radius is smaller than the Schwarzschild radius forms a black hole. The solution of the Einstein field equations is valid for any mass M, so, according to GR, a Schwarzschild black hole of any mass could exist if nature is kind enough to form one.

A Schwarzschild black hole has a Schwarzschild metric, and cannot be distinguished from any other Schwarzschild black hole except by its mass. This metric is given as

$$ds^2 = \frac{1}{1 - 2M/r}dr^2 + r^2 d\Omega^2 - (1 - 2M/r)dt^2, \tag{5.1.3}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$.

Here, for a mass M, the Schwarzschild radius $r_s = 2M$. We see immediately that something strange happens when the radius of the star, $r_* = 2M$, and we look at two cases.

Case 1. ***Not-So-Dense Stars:*** *Radius of the star, $r_* > 2M$. If we recall that the Schwarzschild metric is only valid for outside a star; that is, $r > r_*$, we find that $r > 2M$ as well, and so $1 - 2M/r$ is positive, and never zero. (If $r < 2M$, we are inside the star, and the Schwarzschild metric no longer applies).*

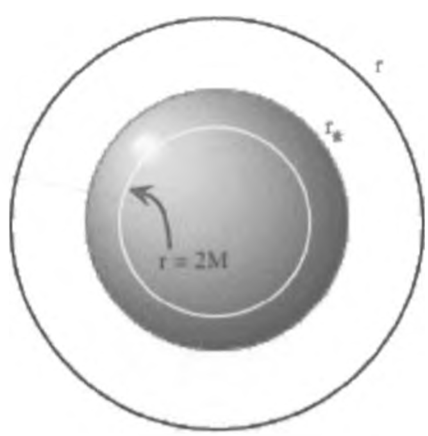

Figure 5.1: A stable star whose radius is larger that its Schwarzschild radius $r = 2M$ (Source: http://people.hofstra.edu/stefan_warner/diff_geom/)

Case 2. *Extremely Dense Stars:* *Radius of the star,* $r_* < 2M$*. Here, two things happen: First, as a consequence of the equations of motion, it can be shown that in fact the pressure inside the star is unable to hold up against the gravitational forces. In fact, if the star is so dense that its mass is greater than 1.4 times the mass of the sun,(*the Chandrasekhar limit*), even the electron degeneracy pressure at the centre of the star is insufficient to hold the star up, and the star collapses. In fact, it collapses to a singularity, a point with infinite density and no physical dimension, a black hole. For such objects, we have two distinct regions, defined by* $r > 2M$ *and* $r < 2M$*, separated by the event horizon,* $r = 2M$*, where the metric goes infinite.*

Figure 5.2: A star that has collapsed into a black hole surrounded by the event horizon. (Source: http://people.hofstra.edu/stefan_warner/diff_geom/)

Since the Schwarzschild metric is only expected to be valid for radii larger than the radius R of the gravitating body, there is no problem as long as $R > r_s$. For ordinary stars and planets this is always the case. For example, the radius of the Sun is approximately 700,000 km, while its Schwarzschild radius is only 3 km.

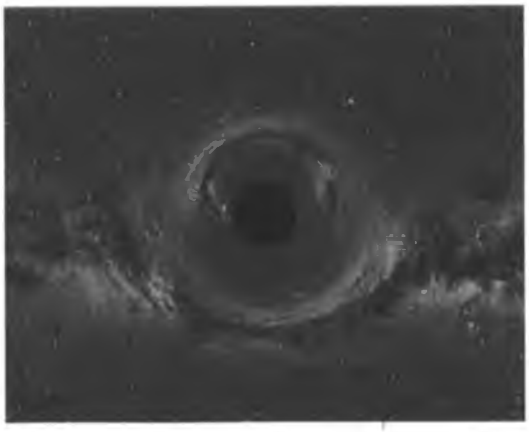

Figure 5.3: An artist's simulated view of a Black Hole's Event Horizon. (Source:http://en.wikipedia.org/wiki/black_holes)

5.1.2 The Kerr Solution and Frame Dragging

The Kerr metric is an exact solution of the Einstein field equations. The Kerr metric is a generalization of the Schwarzschild metric, which describes the geometry of spacetime around an uncharged, perfectly spherical, and non-rotating body. However, the exact solution for an uncharged, rotating body, the Kerr metric, remained unsolved until it was discovered by Roy Kerr.

According to this metric, such rotating bodies should exhibit frame dragging. Roughly speaking, this effect predicts that objects coming close to a rotating mass will be entrained to participate in its rotation, not because of any applied force or torque that can be felt, but rather because the curvature of spacetime associated with rotating bodies. At close enough distances, all objects - even light itself - must rotate with the body; the region where this holds is called the ergosphere.

The Kerr metric describes the geometry of spacetime in the vicinity of a mass M rotating with angular momentum J

$$\begin{aligned} c^2 d\tau^2 = {} & (1 - \frac{r_s r}{\rho^2}) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\ & - (r^2 + \alpha^2 + \frac{r_s r \alpha^2}{\rho^2} \sin^2\theta) \sin^2\theta d\phi^2 + \frac{2 r_s r \alpha \sin^2\theta}{\rho^2} c dt d\phi, \end{aligned} \tag{5.1.4}$$

where the coordinates r, θ, ϕ are standard spherical coordinate system, and r_s is the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}$$

and where the length-scales α, ρ and Δ have been introduced for brevity

$$\begin{aligned} \alpha &= \frac{J}{Mc} \\ \rho^2 &= r^2 + \alpha^2 \cos^2\theta \\ \Delta &= r^2 - r_s r + \alpha^2. \end{aligned}$$

The Kerr metric has two surfaces on which it appears to be singular. The inner surface corresponds to a spherical event horizon similar to that observed in the Schwarzschild metric. This occurs where the purely radial component g_{rr} of the metric goes to infinity. Solving the quadratic equation $1/g_{rr} = 0$ yields the solution

$$r_{inner} = \frac{r_s + \sqrt{r_s^2 - 4\alpha^2}}{2}. \qquad (5.1.5)$$

Another singularity occurs where the purely temporal component g_{tt} of the metric changes sign from positive to negative. Again solving a quadratic equation $g_{tt} = 0$ yields the solution

$$r_{outer} = \frac{r_s + \sqrt{r_s^2 - 4\alpha^2 \cos^2\theta}}{2}. \qquad (5.1.6)$$

Due to the $cos^2\theta$ term in the square root, this outer surface resembles a flattened sphere that touches the inner surface at the poles of the rotation axis, where the colatitude θ equals 0 or π. The ergosphere lies between these two surfaces. Within this volume, the purely temporal component g_{tt} is negative, i.e., acts like a purely spatial metric component. Consequently, particles within this ergosphere must co-rotate with the inner mass, if they are to retain their time-like character.

Figure 5.4: The ergosphere of a rotating body is where frame-dragging is experienced. (Source: http//en.wikipedia.org/wiki/kerr_metric)

An excerpt from the website *Windows to the Universe,[17]* confirms existence of Frame-Dragging

November 6, 1997:

Satellite observations of Black Holes confirm frame-dragging effect 80 years after prediction. Einstein predicted the effect, called "frame dragging", 80 years ago. Like many other aspects of Einstein's famous theories of relativity, it is so subtle that no conventional method could measure it.

Using recent observations by X-ray astronomy satellites, including NASA's Rossi X-ray Timing Explorer,(NASA:1997) a team of astronomers is announcing that they see evidence of frame dragging in disks of gas swirling around a black hole.

Einstein also predicted that the rotation of an object would alter space and time, dragging a nearby object out of position compared to predictions by the simpler math of Sir Isaac Newton.

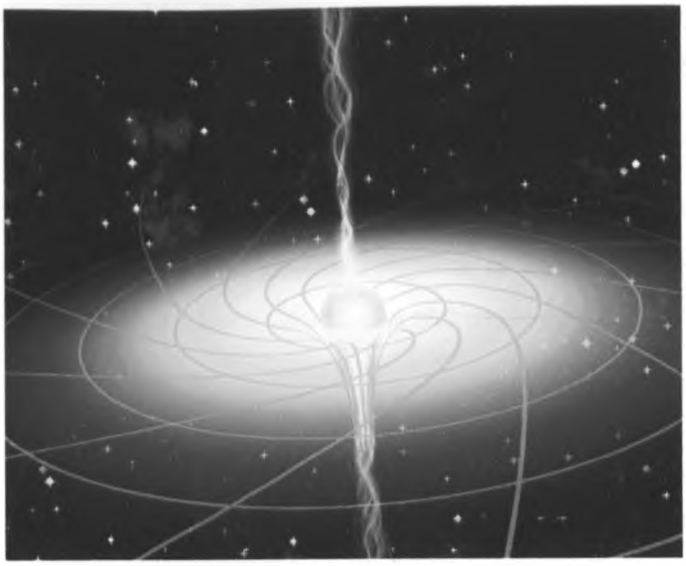

Figure 5.5: An artist's concept of frame dragging shows a black hole's rotation twisting the fabric of space and time. As material crowds in through the accretion disk, some gases are squeezed outward to become superluminal jets. Although superluminal jets have been observed, the rest of the image is speculative since no one has observed a black hole in such detail. (Source: Joe Bergeron of Sky and Telescope magazine).

5.2 Techniques from GR

General Relativity predicts how objects would behave around massive gravitating bodies. Astronomers use these predictions as methods for ascertaining presence of various astronomical bodies as well as for calculating the dimensions of such things. In this section we will look at two interesting techniques provided by GR.

5.2.1 Gravitational Lensing

A relatively new method used by astronomers is to look for a change in the way light behaves, due to gravitational lensing. It is somewhat akin to how you avoid running through sliding glass doors. If the glass is very clean and you get no reflections, how can you tell the door is there? Your eyes perceive that the light from outside is bent, and infer that something is bending it, namely the door. Note that you did not see light FROM the door, but you saw the effect made by the door on light from objects behind it. Similarly, we can use telescopes to look at light from very distant objects behind the clusters of galaxies we are interested in to study the clusters.

The possibility that the path of light could be bent by the gravity of a large object was predicted by Einstein's Theory of General Relativity, and this effect was observed soon after the theory was published. Because people normally think of glass or plastic lenses as bending light, we call any massive object that bends light rays a "gravitational lens." Astronomers now use specialized electronic cameras on large telescopes to very carefully measure how much the light from background objects is bent. By analyzing the amount of bending, we can then determine the mass of whatever is doing the bending.

Scientists have proposed (*2006: Windows to the Universe*) the existence of "dark matter" halos around individual galaxies and clusters of galaxies. Dark matter is material which affects the objects around it through gravity, but which emits no light of any wavelength that we can detect. Astronomers suggest that we see only the "tip of the iceberg" when we use our large telescopes to look out into space. In the cluster Abell 2218, distant blue

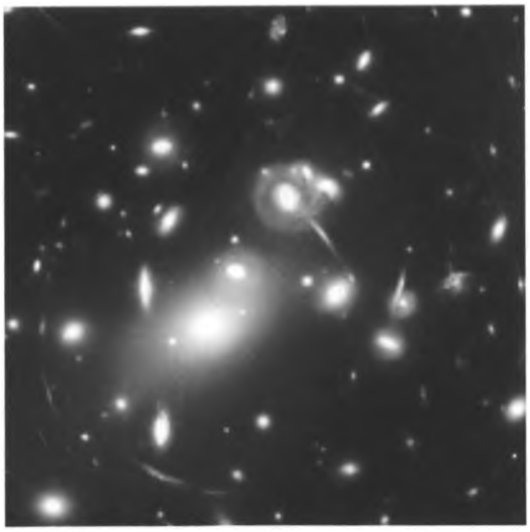

Figure 5.6: Gravitational Lensing in the Galaxy Cluster Abell 2218 (source: NASA/ A. Fruchter/ STScI http://www.windows.ucar.edu/tour/link=/the_universe/Lensing)

galaxies behind the large cluster of galaxies are "squished" into a circular shape around the middle of the foreground cluster. By measuring the amount of distortion in the more distant blue galaxies, we can determine that there is indeed "dark" matter in the cluster! In fact, we can even measure how much mass there is that we can not see – this galaxy cluster happens to have nearly 400 trillion times the sun's mass in "dark" matter.

5.2.2 Gravitational Microlensing

Astronomers have discovered an extrasolar planet only three times more massive than the Earth, the smallest yet observed orbiting a normal star. The star itself is not large, perhaps as little as one twentieth the mass of our Sun, suggesting to the research team that relatively common low-mass stars may present good candidates for hosting Earth-like planets.

Led by David Bennett of the University of Notre Dame, the international research team presented its findings in a press conference Monday, June 2, 2008, at 11:30 a.m. CDT at the American Astronomical Society Meeting in St. Louis, Mo.

"Our discovery indicates that that even the lowest mass stars can host planets," says Bennett. "No planets have previously been found to orbit stars with masses less than about 20 percent that of the Sun, but this finding indicates that even the smallest stars can host planets."

The astronomers used a technique called gravitational microlensing to find the planet, a method that can potentially find planets one-tenth the mass of the Earth.

The *gravitational microlensing* technique, which comes from *Einstein's General Theory of Relativity,* relies upon observations of stars that brighten when an object such as another star passes directly in front of them (relative to an observer, in this case on Earth). The gravity of the passing star acts as a lens, much like a giant magnifying glass. If a planet is orbiting the passing star, its presence is revealed in the way the background star brightens. "This discovery demonstrates the sensitivity of the microlensing method for finding low-mass planets, and we are hoping to discover the first Earth-mass planet in the near future," said Bennett.

With support from the National Science Foundation (NSF), Bennett has been one of the pioneers in using gravitational microlensing for detecting low mass planets. He has been working with collaborators around the world to find a number of planets that are ever closer in size to the Earth.

For the most recent discovery, the research collaborators took advantage of two international telescope collaborations: Microlensing Observations in Astrophysics (MOA), which includes Bennett, and the Optical Gravitational Lensing Experiment (OGLE).

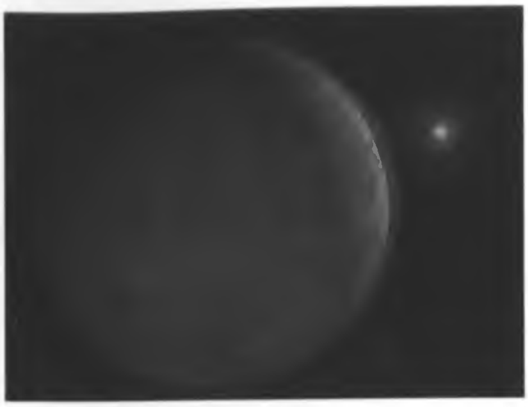

Figure 5.7: Artist's conception of the newly discovered planet orbiting a brown dwarf 'star' (Source: NASA's Exoplanet Exploration Program)

5.3 Does GR have all the Answers?

It is well known and documented that the theory of General Relativity is the most popular fall-back for explaining most astronomical phenomena. In this section, we recount some studies involving three different phenomena to see whether indeed GR has stood up to the tests.

5.3.1 The Inclination of Planetary Orbits

A. Qadir and H. Rizvi (1988) in their paper: *A Relativistic Explanation of the Inclination of Planetary Orbits [11]*, explored whether Relativity could provide an explanation for the fact that the planetary orbits are nearly, but not exactly in the same plane and they all nearly coincide with the solar equatorial plane.

The standard model for planetary formation was taken; that initially there was a large, roughly spherical cloud of gas and dust in which there was a high intrinsic angular momentum. Then a collapse of the cloud was initiated. The high angular momentum caused the shrunken cloud to collapse to a disc. As the collapse proceeded some matter stayed out in a thin disc while the rest continued to collapse. Sometime after the sun was ignited, accretion occurred and the planets formed from the rings in a short period. The total age of the solar system is taken to be about 4.5 billion years and the actual process of planetary formation only about 0.1 billion years!

The solar equatorial plane was taken as the reference point. Putting the solar thickness as the limit, eliminates the possibility of the present inclinations being the original ones. As such the argument was that the planets must have been driven out of their initial orbits (which were very nearly in the solar equatorial plane) and a relativistic explanation was sought for this.

One suggestion is that the inclination arose due to the effect of the Kerr nature of the gravitational field of the sun. This was largely because the Kerr metric does not have coplanar geodesics except in the equatorial plane. Since it would be impossible to try out all possible initial conditions over the 4.5 billion years for the solar system in full general relativistic detail, what was done was to appeal to the ψN-formalism to provide the Kerr correction to the Newtonian gravity.

Using the ψN-formalism, the potential for a gravitational source was given as:

$$\Phi = \frac{-G(mr - Q^2/2c^2)}{(r^2 + a^2\cos^2\theta/c^2)} \tag{5.3.1}$$

where $a = s/m$, s being the total intrinsic angular momentum of the gravitating source and m its mass.

Taking the derivative of the potential given by (5.3.1) with respect to θ having set $Q = 0$, we get

$$F = \frac{Gmra^2 \sin 2\theta\mu}{(r^2 + a^2\cos\theta^2/c^2)}, \tag{5.3.2}$$

where μ is the planetary mass. Here F is the covariant polar component of the force vector. This force would act out of the plane of the orbit towards the rotation axis. Of course the polar angle of the planet goes on altering as it moves in its orbit. When the planet crosses the equatorial plane this force is zero. However, when it goes to the maximum or minimum value of θ, at either end, the force acts in the same way. The total force acting on the orbital plane is the average of this force over the entire orbit, which after some approximations is calculated to be:

$$F = \frac{2Gma^2\theta\mu}{c^2r^2}, \tag{5.3.3}$$

where θ, the angle of inclination of the planet's orbital plane to the solar equatorial plane, is taken to be small.

Since F is the covariant component of the force, it has units of mass times length squared per unit time squared. Thus, the second derivative of the angle of inclination at any instant is given by

$$\ddot{\Theta} \approx \frac{2Gma^2}{c^2r^5}\Theta, \tag{5.3.4}$$

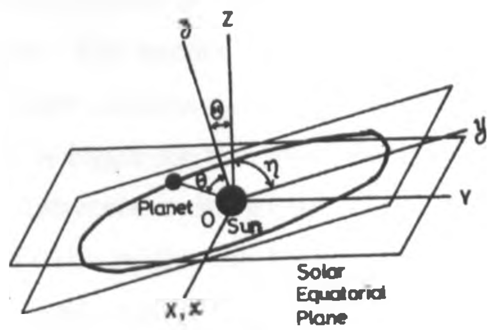

Figure 5.8: A presentation of the orbital geometry; XOY is the solar equatorial plane and xOy is the orbital plane of the planet. The polar angle of the planet lies between η and $\pi - \eta$. The inclination of the orbital plane is Θ.

having divided out by the extra length squared. Writing a in terms of the total angular momentum of the sun, S, we get

$$\ddot{\Theta} = \frac{2GS^2}{mc^2r^5}\Theta = K^2\Theta. \tag{5.3.5}$$

It has been previously discovered that K depends linearly on the solar angular momentum.

To compute the relativistic effect of the change of the angle of inclination of the orbital plane we take the general solution of (5.3.5)

$$\Theta(t) = A\cosh K(t-T) + B\sinh K(t-T). \tag{5.3.6}$$

Thus, we would have

$$\Theta(T) = A\dot{\Theta}(T) = B/K. \tag{5.3.7}$$

Thus for the relativistic effect to act, to move the angle of inclination, there must be either an initial perturbation A at some time T, or an initial angular velocity B/K, or both. Thus the *relativistic explanation* postulates that the planet stays in its original orbit, in the equatorial plane of the sun, for $t < T$ and is then perturbed out of this plane by some mechanism. At this stage the relativistic driving mechanism takes over and pushes the planet increasingly out of the solar equatorial plane so that at the present time the angle of inclination is the observed value.

The study was restricted to the inner planets, namely Mars, Earth, Venus and Mercury, largely because of the larger angular momentum of the outer planets in comparison to that of the sun. Lengthy calculations of the value of the planetary constant K, showed a remarkable uniformity in the value of K multiplied by the period over which the planet has been driven into its present inclination. The results continued to be stable under change of initial conditions. There remains a variation in $K(t - T)$ such that for the same perturbation it is larger for the 'outer' inner planets, Mars and Earth. It may be that this difference is due to the influence of the outer planets.

The conclusion of the study was that it is apparent from the general behaviour of things, that relativistic gravity and not classical gravity, rules the motion of the heavenly bodies.

5.3.2 Speed of Gravitational Waves

In his paper: *The Confrontation between General Relativity and Experiment*, Clifford M. Will (2003) questions whether Einstein was right to guess that the speed of gravity was equal to the speed of light, in his theory of GR. He writes:

According to GR, in the limit in which the wavelength of gravitational waves is small compared to the radius of curvature of the background spacetime, the waves propagate along null geodesics of the background spacetime, i.e. they have the same speed, c, as light. In other theories, the speed could differ from c because of coupling of gravitation to background gravitational fields. For example, in some theories with a flat background metric η, gravitational waves follow null geodesics of η, while light follows null geodesics of g (Will, 1993). Another way in which the speed of gravitational waves could differ from c is if gravitation were propagated by a massive field (a massive graviton), in which case v_g would be given by, in a local inertial frame,

$$\frac{v_g^2}{c^2} = 1 - \frac{m_g^2 c^4}{E^2} \approx 1 - \frac{1}{2}\frac{c^2}{f^2 \lambda_g^2}, \tag{5.3.8}$$

where m_g, E and f are the graviton rest mass, energy and frequency, respectively, and $\lambda_g = h/m_g c$ is the graviton Compton wavelength ($\lambda_g >> c/f$ assumed). An example of a theory with this property is the two-tensor massive graviton theory of Visser (Visser, 1998).

This clip from **Windows to the Universe**, [17] proves that Einstein was right!

To make his most well-know theory, the General Theory of Relativity, physicist Albert Einstein had to guess that the speed of gravity was equal to the speed of light. Now, scientists have found that he was likely right!

His theory assumes that the force of gravity acts at the same speed as the speed of light. Until recently this assumption had not been tested thoroughly. While the speed of light has been measured as 299,800 km/sec, scientists found that it is very difficult to accurately measure the speed of gravity. Einstein's assumption appeared to be true, based on indirect evidence, but it had never been comprehensively tested before now.

Scientists had the opportunity to test this assumption of Einstein's theory when, on September 8, 2002, Jupiter passed in front of a distant quasar called J0842. They released their findings at the meeting of the American Astronomical Society in Seattle, WA in January 2003. They reported that the planets gravity bent the quasars radio waves, making the quasar appear to move in an elliptical shape through the sky according to the observations of many radio telescopes in the United States and Germany. From the shape of the quasars motion, the scientists calculated that the speed of gravity is nearly the same as the speed of light!

5.3.3 Dark Energy

In the early 1990's, one thing was fairly certain about the expansion of the Universe. Theoretically, it continues to expand but gravity was certain to slow the expansion as time went on. Then came 1998 and the Hubble Space Telescope (HST) observations of very distant supernovae that showed that, a long time ago, the Universe was actually expanding more slowly than it is today. So the expansion of the Universe has not been slowing due to gravity, as everyone thought, it has been accelerating. No one expected this, no one knew how to explain it. But something was causing it.

Theorists still do not know what is causing this, but they have given it a name. It is called dark energy. We know how much dark energy there is because we know how it affects the Universe's expansion. Other than that, it is a complete mystery. It turns out that roughly 70 percent of the Universe is dark energy. Dark matter makes up about 25 percent. The rest - everything on Earth, everything ever observed with all of our instruments, all normal matter, adds up to *less than 5 percent* of the Universe.

One explanation for dark energy is that it is a property of space. Albert Einstein was the first person to realize that empty space is not nothing. The first property that Einstein discovered is that it is possible for more space

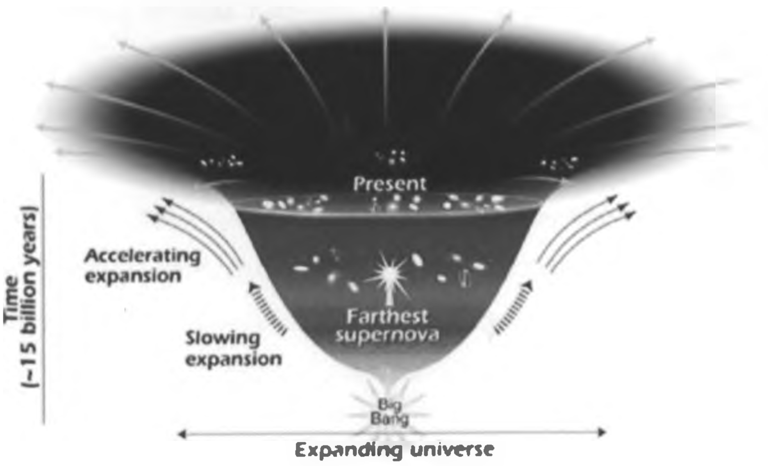

Figure 5.9: This diagram shows changes in the rate of expansion since the Universe's birth 14 billion years ago. The more shallow the curve, the faster the rate of expansion. The curve changes noticeably about 7.5 billion years ago, when objects in the Universe began flying apart at a faster rate. Astronomers theorize that the faster expansion rate is due to a mysterious, dark energy that is pulling galaxies apart. Credit: NASA/STSci/Ann Feild

to come into existence. Then one version of Einstein's gravity theory, the version that contains a cosmological constant, makes a second prediction: 'empty space' can possess its own energy. Because this energy is a property of space itself, it would not be diluted as space expands. As more space comes into existence, more of this energy-of-space would appear. As a result, this form of energy would cause the Universe to expand faster and faster. Unfortunately, no one understands why the cosmological constant should even be there, much less why it would have exactly the right value to cause the observed acceleration of the Universe.

Another explanation for how space acquires energy comes from the quantum theory of matter. In this theory, 'empty space' is actually full of temporary ('virtual') particles that continually form and then disappear. But when physicists tried to calculate how much energy this would give empty space, the answer came out wrong by a lot. The number came out 10^{120} times too big. It's hard to get an answer that bad.

Another explanation for dark energy is that it is a new kind of dynamical energy fluid or field, something that fills all of space but something whose effect on the expansion of the Universe is the opposite of that of matter and normal energy. Some theorists have named this *antimatter*. But, if antimatter is the answer, we still do not know what it is like, what it interacts with, or why it exists.

A last possibility is that Einstein's theory of gravity is not correct. That would not only affect the expansion of the Universe, but it would also affect the way that normal matter in galaxies and clusters of galaxies behaved. But if it does turn out that a new theory of gravity is needed, what kind of theory would it be? How could it correctly describe the motion of the bodies in the Solar System, as Einstein's theory is known to do, and still give us the different prediction for the Universe that we need? There are candidate theories, but none are compelling. So the mystery continues [7].

Further Research

We have seen in the preceding chapter, that Dark Energy remains an unsolved phenomenon in the docket of the relativistic astrophysicist. Likewise, there are other open areas for research in this field:

- **Applying New Curvature Tensors in Solutions to the EFE**
  In his papers, *Relativistic Significance of Curvature Tensors (1982)* and *Physical Properties of some Curvature Tensors (2007)*, Pokhariyal has defined several new curvature tensors on the pattern of Weyl's projective curvature tensor and has studied the physical and geometrical properties of some. These tensors have, as yet, not been used in solving the Einstein's field equations and thus present a lucrative area for research.

- **Looking for Relativistic Explanations of Astrophysical Phenomena**
  As in the previously outlined study of A. Qadir and H. Rizvi[11], there is need to look for other relativistic effects in astrophysics which do not merely make a detectable difference but are fundamental to the phenomenon observed. Further, they need not only be for exotic astronomical objects but also for some ignored aspect of the more common objects.

- **Delving Deeper into the Genesis of GR**
  D. Salisbury in his review (2009) of the book: *Jurgen Renn(ed): The genesis of general relativity[12]*, writes:
  These volumes are the result of over two decades of effort, by most of the leading scholars in the field, to understand the proeess that culminated in Einsteins publication of the general theory of relativity. There is an obvious utilitarian reason why this study should be of interest to us, *the better we understand the process of scientific knowledge acquisition, the better we will be able to create conditions in which young*

scientists can follow the lead of innovators like Albert Einstein.
Further, how many of us know of the work of, for example, Max Abraham, Gustav Mie, Gustav Herglotz, and Gunnar Nordstrom, just to name a few of the significant scientists who feature in these volumes? A study of this book would undoubtedly provide the stimulus for perceiving the universe in the manner of our predecessors.

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