UNIVERSITY OF NAIROBI

COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCE

SCHOOL OF MATHEMATICS

FINITE ELEMENT METHOD FOR SOLVING ADVECTION-DIFFUSSION EQUATION IN CLIMATE MODELLING

candidate

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A project submitted in partial fulfilment for a degree of Master Of Science in Applied Mathematics

Declaration

I the undersigned declare that this project is my original work and to the best of my knowledge has not been presented for the award of a degree in any other university.

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Declaration by the supervisors

This project has been submitted for examination with our approval as the supervisors.

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Signature:...............................

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Date:.......................................

Statement

This dissertation has been submitted in partial fulfilment of requirement for a Master of Science degree at the University of Nairobi and is deposited in the University library to be made available to the borrowers under the rules of the library.

Acknowledgment

First and above all,I thank God,the Almighty for providing me this opportunity and granting me the capability to proceed successfully.

Many thanks to my supervisors Dr.Charles Nyandwi and Prof.Ogana who offered continuous advice and encouragement throughout the course of this work.

I acknowledge my sincere gratitude to the University of Nairobi through the School of Mathematics for the scholarship awarded to me to pursue Msc.studies,without which, I could not have met to this far.

To my family members,I take this opportunity to express the profound gratitude from my deep heart to my beloved parents,grandparents and my sibling for their love and continuous support both spiritually and materially.

Tribute to Antipas Kemboi for the financial support you always provided me to make sure that my dreams are met.

To my son Asier Kiptoo who is always the source of my inspiration,thanks for understanding me when I was away several months,your perseverance always kept me going,I will always love you and be there for you.

Finally,I am thankful to my classmates Pauline Achieng,Lucy Ng'anga,Ignace Ntezimman and Patrick Omukuba for the help and support you gave me during my studies.

Thanks to all together with all those whose names have not be mentioned and may God bless you.

Dedication

I dedicate this work to my son Asier Kiptoo,my dad David Mutai and my mum Jackline Mutai.

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Abstract

Our main focus in this project was to solve by Finite element method the advection-diffusion equation as it appear in the climate model developed by Seller in (1969),which incorporate both the atmospheric and oceanic fluid dynamics. Therefore,since the diffusive term is more dominant when the altitude is not involved, we decided to use Budyko climate model for our study.We have used finite element method with quadratic basis function with two,five and ten elements hoping to improve the results or atleast to see if we obtain the same as what was obtained earlier.The results obtained confirm to some differences,the results of Warren and Schneider in (1979) for only two and five elements.Finally for ten elements we have observed very large discrepancy.

Chapter 1

INTRODUCTION

1.1 Climate

Climate has various number of meanings.To geologist or geomorphologist it is an external agent which has many phenomena of interest.For an archaeologist, climate of an earlier time might have lead to a crucial influence upon the people being studied.An agriculturalist may see climate and defines it as the background norm upon which weather of a given year or day can be studied.Therefore for an average person he or she may defines it as moving to a given location in search of good climate condition so as to earn a living.Therefore to some of us when we talk of climate what come to our mind first is temperature,even though rainfall and humidity may also be other factors that need to be considered. Recently increasing atmospheric carbondioxide and other trace of greenhouse gases has led to great impact upon the climate. [24]

Climate is a forcing agent as while as a feature which can be disturbed.This is perceived by use of terms of the feature of the entire system which is either readily or more useful in characterizing the phenomenon of our interest.It is defined as the average of weather at a particular place over a given period of time.This average is performed for a period long enough with respect to the time required for weather prediction.Therefore a satisfactory definition of climate is not easy to obtain because climate system encompasses so many variable.

1.1.1 What is climate modelling?

Climate model can be defined as a mathematical representation of the climate system which involves physical,biological and chemical principles.The equation obtained from these processes are so complex, therefore solution can be obtained by use of numerical method.Thus,climate models provide solutions which are discrete in space and time meaning that results obtained represent average over a given region for a particular period of time.

Climate models therefore use various quantitative method to simulate it's interactions in the atmosphere,ocean,land surface and ice.They have a variety of purposes from the study of dynamics of the weather and climate systems which enables one to project future climate.This models balance or nearly balance the incoming energy from the sun in form of shortwave radiation(visible and ultraviolet) to the earth surface with outgoing energy from the earth surface in form of longwave(infrared) radiation .

Climate models are very important since they have been used quite often to reproduce the main feature that influences the current climate of a particular place and the changes in temperature for some given years.Therefore characteristic of climate of a particular region can be determined by a number of given factors with some of the mechanism which constitute the climatic system.

1.1.2 The climate system

The climate system is composed of the following features the atmosphere,hydrosphere,cryosphere,land surface and the biosphere.

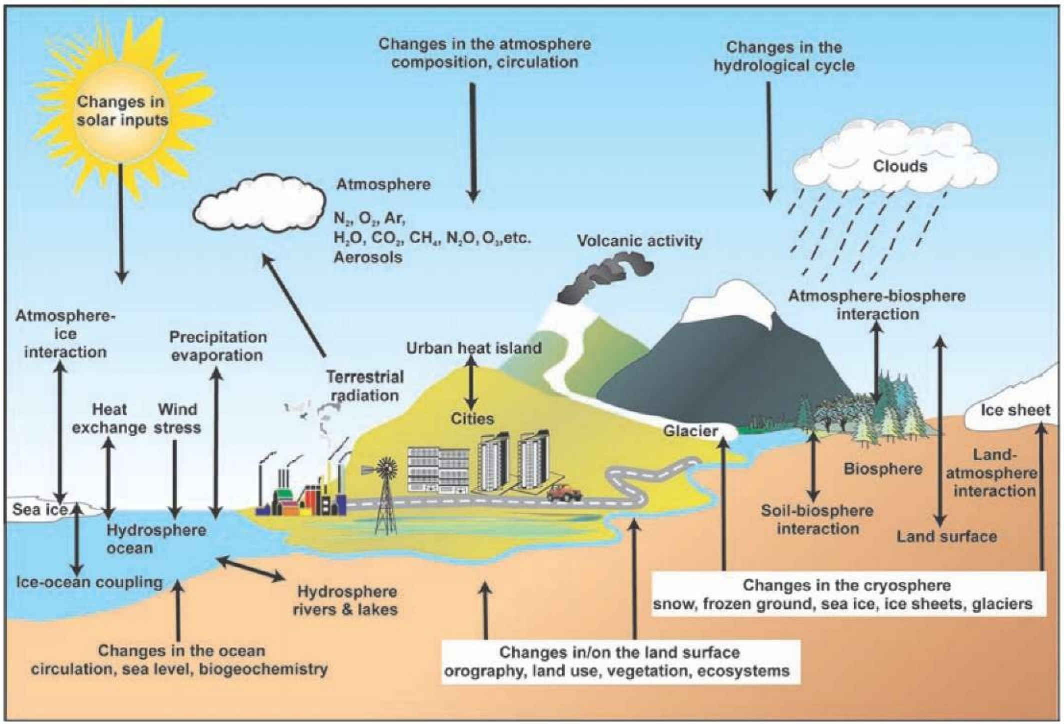


Figure 1.1: An illustration of the components of the climate system with there interactions.

1.1.3 Component of the climate system

Atmosphere

The atmosphere as a set of seven basis equation that governs it with the following seven unknowns;the three component of velocity V that is (u,v,w),the pressure p,the temperature T,the specific humidity q as well as the density ρ. [23]

These seven equations are as follows:

(i) Newton's second law or conservation of momentum $F = MA$

(1-3)

$$\frac{dv}{dt} = -\frac{1}{\rho}\vec{\nabla}\rho - \vec{g} + \vec{F} - 2\Omega * \vec{V}$$

Where $\frac{d}{dt}$ is the total derivatives, including a transport term.

$\vec{g}$ is the apparent gravity vector.

$\vec{F}$ is the force that is due to friction.

Ω is the angular velocity of the earth while the last term is the coriolis force.

(iv) the equation of continuity or the conservation of mass equation is given below;

$$\frac{\partial \rho}{\partial t} = -\nabla.(\rho \vec{v})$$

$\vec{v}$ is the three-dimensional velocity

∇ is the three-dimensional nabla operator

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}.\nabla$ is the material derivatives

(v) The equation of conservation of mass or water vapour equation is given has follows:

$$\frac{\partial \rho q}{\partial t} = -\nabla(\rho \vec{v} q) + \rho(E - C)$$

where E and C represents evaporation and condensation variables of water vapour.

(vi) The thermodynamics equation of the first law or the conservation of energy equation.

Thermodynamic equation expresses that if heat is applied to a parcel of air at a rate of Q per unit mass then this heat can be used to increase the internal energy and to produce work of expansion.

$$Q = C_p \frac{dT}{dt} - \frac{1}{\rho} \frac{d\rho}{dt}$$

where Q is the heating rate per unit mass of air

C_p is the specific heat of air.

(vii) The equation of state also plays an important role in the atmosphere;

The atmosphere is assumed to be a perfect gas.Thus we have the equation.

$$P = \rho R_g T$$

where p is the pressure of air

ρ is the density of air

T is the temperature

R is the gas constant

Computing the heating rate requires a detailed analysis of the radiation in the atmosphere that accounts for the longwave and shortwave radiation.

other components are

(i) Ocean

(ii) Sea ice

(iii) land surface

(iv) ice sheet

(v) marine biogeochemistry

The equations governing the climate system are differential equations that can be expressed both as partial differential equation and ordinary differential equations.Therefore it is necessary to ensure that these equations obtained are well-posed that is the problem must have a unique solution which depends on the initial and boundary conditions.

The transport of energy as well as matter in fluids are determined in nature by diffusion and advection.These processes induce flux of energy and matter in which the mathematical description is derived by continuum mechanics.All processes that involves the climate system are influenced by the advective and diffusive transport of substances. [9]

1.1.4 Diffusion

Diffusion is defined as a process that is caused by thermal motion of molecules.It is also refer to as Brownian motion.This is caused by a positive thermodynamics temperature $T > 0$ where the molecules are in constant motion.A good example is the diffusion of a given perfume open in an empty room.

For example if a bottle of perfume is opened in an empty room and allowed to evaporate into the surrounding air,the whole room will soon be scented. [9]

We then consider the one dimensional case where the concentration gradient involving y and z direction are zero so as to obtained the equation as shown below,

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

here C refers to a given concentration gradient while the quantity D is the coefficient of diffusion.

1.1.5 Advection

Advection is defined as the horizontal transport of a given property in the atmosphere or ocean such as heat,humidity or salinity [9].For example,if we consider a one dimensional case we have advection equation given by the equation below;

$$\frac{\partial C}{\partial t} = -\frac{\partial (uC)}{\partial x}$$

Where u is the velocity and C is the concentration gradient.

1.2 Advection-diffusion equation

To obtain advection-diffusion equation we combine both the advection and diffusion equation as mention above.This is because the two processes describe physical phenomena in which particles,energy,or other quantities are transferred.For one dimensional case, we have the following equation shown below; [9]

$$\frac{\partial C}{\partial t} = -\frac{\partial (uC)}{\partial x} + \frac{\partial}{\partial x}(D\frac{\partial C}{\partial x}) + P$$

The general equation in 3 dimensional is given has follows;

$$\frac{\partial C}{\partial t} = -\vec{\nabla}.(u\vec{C}) + \vec{\nabla}.(D\vec{\nabla}C) + P$$

Where ∇ represent gradient operator and $\vec{\nabla}.$ is the divergence operator.

Here C is the concentration gradient.

D refers to coefficient of diffusion .

$\vec{u}$ is the average velocity of the quantity in motion.

p refers to the source or sink of the our quantity C.

Given that C is the mass density. Furthermore, the diffusion variable and the sources or sinks terms vanishes in our equation,then we have a special type of the equation shown below;

$$\frac{\partial \rho}{\partial t} = -\nabla.(\vec{u}\rho)$$

This is the general form of continuity equation.These equations represents the basis for the mathematical description of processes in the climate system where the solution obtained constituents the climate modelling.

1.2.1 Application

Advection-diffusion equation(ADE) benefits from wide areas of application in such different disciplines [21]:

- As environmental engineering
- Mechanical engineering
- soil science
- petroleum engineering
- chemical engineering
- Heat transfer

ADE interprets the spreading of scalar or non-scalar quantities under specified initial and boundary conditions.This equation can either be solve by analytical or numerical methods.Heat transfer equation in climate modelling is our main research area of study and we will solve the equation obtained by using finite element method.

1.3 Background of the problem

Energy balance climate model(EBMs) are the simplest climate models developed earlier by varies authors.They were introduced at the same time by both Budyko [1] and seller's [2].These models are easy to understand and therefore one can obtain both analytical and numerical solution.Budyko brought up a type of energy balance model by using the zonal and annual averaging but allowing the latitudinal dependence of surface temperature, albedo and meridional heat transport.

These authors came up with one layer thermodynamic models of the earth's average surface temperature fields where they consider a balance between the net energy coming in from the sun into a strip around the earth and the energy going outside the strip.They were described in terms of solar radiation that is absorbed by the earth surface, terrestrial radiation that is emitted into the atmosphere and the divergence of heat for the given strip.

For zero-dimensional EBMs,the temperature T is describe as global since it depend on time only with not other variable in space included.Therefore the global temperature of the earth depend on the following parameters,the planetary albedo and the reflection on the land and sea surfaces.

Mathematically,for this case temperature T is expressed as an ordinary differential equation (ODE) where $\frac{\partial T}{\partial t}$ is determined by the balance between the energy coming in from the sun in form of shortwave radiation and the outgoing longwave radiation emitted from the earth surface as shown by equation given below;

$$C\frac{\partial T}{\partial t} = Q(1-\alpha(T)) - (A+BT) \tag{1.1}$$

Where $Q = Q_0/4$ is the solar constant.

In this project, we are going to consider the (EBMs) that take into account the space variable x as a function of temperature T.It is important to take into consideration advection -diffusion equations that take into account change in temperature from pole to equator at the same time respecting the overall radiative balance.We consider the following diffusive symmetric annual energy balance climate models shown in the equation below;

$$C(x)\frac{dT}{dt} = QS(x)(1-\alpha(x,\mu)) - (A+BT) + \frac{d}{dx}(K(x)(1-x^2)\frac{dT}{dx}) \tag{1.2}$$

where Q is the averaged global solar insolation flux,

K is the diffusive coefficient of the horizontal heat transport per latitude belt.

$I = A + BT$ is the outgoing longwave radiation where A and B are the empirical data for the radiation coefficient.

Eq.(1.2) is called a one-dimension energy balance climate model.In the given equation we assumed that the temperature T depend on the space x only through a latitudinal variable $x \in (0, 1)$ which is symmetric across the equator with $x = 0$ at the equator and $x = 1$ at the north pole.The variable x is the sine of the latitude.

To solve Eq.(1.2) it is important to consider following homogeneous boundary conditions with $x = 0$ at the equator and $x = 1$ at the North pole.

$$(1 - X^2)^{\frac{1}{2}} \frac{\partial T}{\partial t}|_{x=0,1} = 0 \tag{1.3}$$

Whereby at the equator and at the poles there is no heat transport. This model has very many parameters.Therefore variation in this parameters lead to a temperature change in each latitude zone. [29]

One of the heat transport processes not considered in Budyko model is convection or(advection).Thoug these are very important processes which involve the movement of heat by transport of energetic mass in the atmosphere.This is because the convective process could have included many parameters,as well as constants which could have complicated our climate model equation.Due to the following reason the convective term is not taken into consideration in our model.Thus we have use diffusive energy balance climate model. [22]

In the past various method have been use to solve the equation including the analytical studies that was done by Warren and Schneider (1979) and North (1981) by using two-mode approximation method [7, 36] .Use of numerical schemes in solving advection-diffusion equation in one dimension energy balance climate model has been on the rise.Thus there has been a need to obtain efficient solution of this one dimension energy balance climate model which in this study is by use of Finite element method.

1.4 Problem statement

This involves the study of the numerical solution of steady state diffusive zonally symmetric mean annual energy balance climate model given below,

$$-\frac{d}{dx}(k(x)(1-x^2)\frac{dT}{dx}) + (A+BT) = QS(x)(1-\alpha(x,\mu))$$

subject to the boundary conditions;

$$(1-x^2)^{\frac{1}{2}}\frac{dT}{dx}|_{x=0,1} = 0$$

This is because the solutions to obtained and their efficiency will help other readers to comprehend how the solution of this equation has been done over years.

1.5 Main objectives

1 To understand climate system and to study how to derive one dimensional energy balance climate model given by the equation,

$$C(x)\frac{dT}{dt} = QS(x)(1-\alpha(x,\mu)) - (A+BT) + \frac{d}{dx}(K(x)(1-x^2)\frac{dT}{dx})$$

2 To study Galerkin finite element method use in solving this energy balance climate model.

1.6 Specific objectives

1. To understand the main concept of FEM.

2. To list down steps used in solving problems by finite element method.

3. To determine the numerical solution for diffusive steady state symmetric one-dimensional energy balance climate model equation using finite element method.

4. To obtained the algebraic linear systems of the problem by assemblying mass matrix and stiffness matrix.Furthermore, to determine the load vectors.

5. To obtain solution of the linear systems of equation using Gaussian elimination method.

1.7 Literature review

Numerical solution of one dimension energy balance climate model based on either analytical or numerical method has not been dealt with in a wider perspective for a nonlinear advection - diffusion equation in climate modelling.This is because the problem is based on complex geometries and unstructured meshed point.

Gerald R.North in (1975) solved explicitly the ordinary diffusive thermal heat transfer climate model equation using hypergeometric functions and they used the results obtained to study ice sheet latitude as a function of solar constant and stability analysis about the equilibrium points [6].

Gerald R.North,Louis, H,David, P. and Bruce, W in (1979) did a variational evaluation of Budyko-Seller model with a purpose to present a functional of the temperature field which takes on the extreme value and they found that the stable solution represents local minimum point while saddle point refer to unstable solution.They achieved this by use of a spectral example. [41]

Stephen,W. and Stephen H.Schneider in (1979) further solved the one dimensional energy balance climate model equation and they realist that annual change in radiation as a function of averaged surface temperature is to be analyzed with data obtained seasonal in order to evaluate there validity in term of climate changes experiment.They found that the temperature coefficient for the different zones examined,differs from each other by as much as a factor of 2 [36]. The spectral method earlier used to solved climate models equation provided a framework in which numerical techniques can be used to solved this energy balance climate model.

B.William and Gustar (2011) in [8] did a research on the equation of energy balance climate model versus the economic by taking into account climate change using two-mode approximation method.They developed a two-mode solution of the problem by given the human forcing function in terms of $h(x,t)$ and they discovered that the discontinuous function in terms of absorption create a strong nonlinearity where a small change in T_0 leads to a large change which bring about damages in some given latitudes.Thus because of the nonlinear property it was not easy to obtained the analytical solution.They then used climate parametrization by North (1975) to solved nonlinearity of the problem [6].

Bermejo,Carpio and Diaz (2009) in [8] solved two-dimensional climate model on nonlinear parabolic problem by using compact Riemannian manifold with no boundary conditions for the surface temperature.This numerical analysis by finite element method was solved only for the spherical earth where the elements used was in terms of quasi-uniform spherical triangles.They also studied existences,uniqueness and the stability of the approximate solution they obtained.

Rahmat,Ariwahjoedi and Suleiman (2011) [43] solved zero-dimension climate model by using of numerical method known as Newton-Raphson and Steepest Descent method.They realist that this method offers very good approximate solution for averaged surface temperature of the earth and the atmosphere. Prof.T.Stocker(2014) in [9] solved the one dimensional advection equation by explaining how to using difference method of centered in time,centered in space (CTCS) and he observed that for a given time step from $t = 0$ to $t = \Delta t$ the CTCS scheme does not give good results and he then solved the same equation by a method known as forward in time,centered in space (FTCS). This scheme requires the computation of the new time step by using the solution obtained in the previous steps.He found that the amplitude increase with time,therefore $|C_{m,n}| \longrightarrow \infty$ for $n \longrightarrow \infty$.Therefore solution explodes using this scheme.

He then applied the method of forward in time,upstream in space (FCUS) and he realized that the method does not produce good numerical solution but leads to strong damping and dispersion of the solution.He then solved the equation by using implicit trapezoidal scheme and he obtained system of linear equation in which the matrix was to be inverted in order to solve for the new time step given by equation shown below;

$$AC_n + BC_{n+1} = o \tag{1.4}$$

The solution at time $n + 1$ was given by

$$C_{n+1} = -B^{-1}AC_n \tag{1.5}$$

The matrices obtained were sparse,thus to obtained this solution without inverting the matrix was expensive.He then showed that the FTCS scheme was alway unstable and he introduced a diffusive term by use of a method called Lax scheme to stabilized the FTCS method.

This scheme was not good enough to give appropriate solutions since it had some disadvantage and he introduced the Lax-Wendroff scheme to address this problem though the scheme overestimated the values at maximum and show some oscillation of temperature which was trailing.

Thus the current study is meant to solve diffusive steady state one-dimension climate model equation with an efficient method that can solve problems arising by use of finite difference method so that we obtained accurate solutions.Hence the use of finite element method.

1.8 Project outline

This project is organized in the follows parts,chapter 2 deals mainly in derivation of advection-diffusion equation in climate modelling.This equation is categorized as follows,first there is explanation of zero-dimensional climate model and finally one-dimensional climate model including it's parameters.

Chapter 3 discusses the finite element method and it's procedures.Additionally,the selection of interpolation functions for finite element method and coefficient matrices construction for elements.Chapter 4 solving the problem by finite element method using quadratic element.

Chapter 5 details the numerical solution obtained by using Galerkin finite element method by showing it's results.Finally,there is conclusion and recommendation for future work.

Chapter 2

DERIVATION OF ADVECTION DIFFUSSION EQUATION IN CLIMATE MODELLING

2.1 What is climate modelling?

climate model can be defined as a mathematical representation of the climate system in terms of physical,biological and chemical principles.The equation obtained from these principles are complex thus it requires the use of numerical method in order to determine the approximate solution.For a climate model to be complete some of the component required are the solar radiation,the Earth's radius and it's period of rotation,the landscape,the ocean and some rock as well as soil properties. [23]

2.2 Components of energy balance on the surface of earth

The sun is the main source of energy in the atmosphere.Therefore the atmosphere is considered as a vacuum, thus there is no medium for energy to be transfered through kinetic collision and therefore cannot be lost by escape of energetically heated particle since the atmosphere is bound to the surfaces. Therefore, radiation is the only process of losing energy from the earth surface.Radiative processes can be explained in terms of the principle of blackbody radiations which refers to the energy reaching the top of the earth atmosphere surface from the sun, with an average of;

$$Q = 342.5W/M^2$$

which corresponds to a quarter of the energy received $Q_o = 1370W/M^2$.

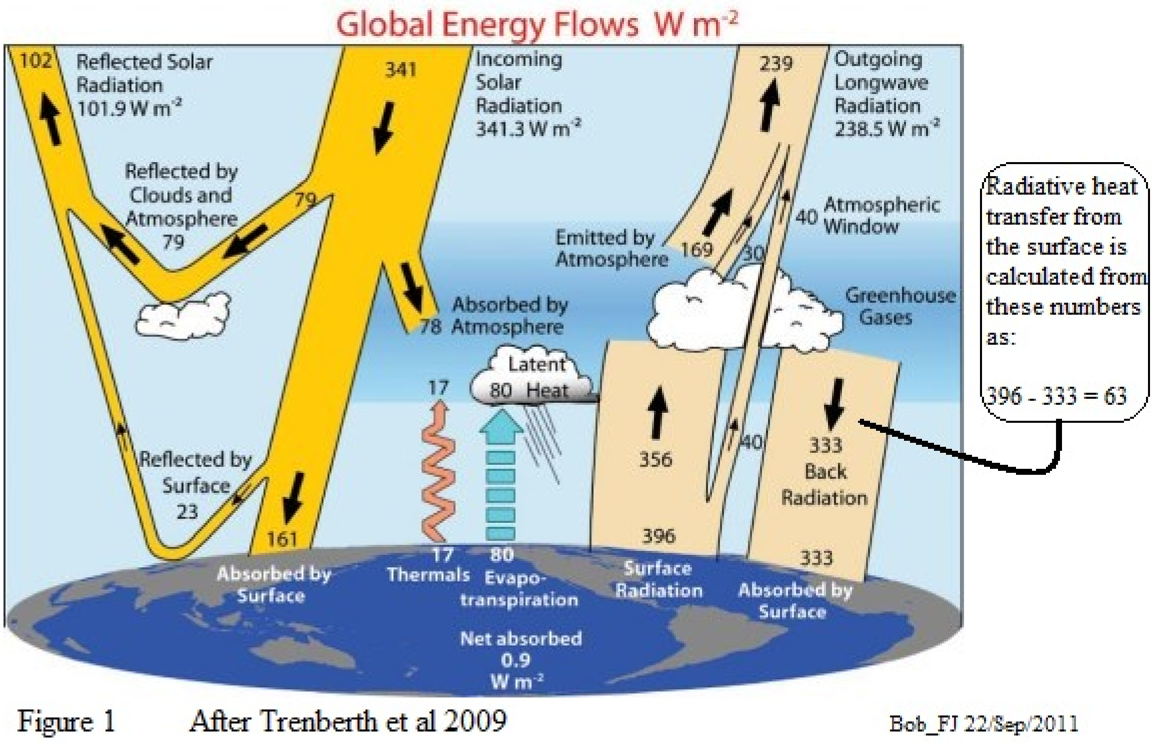


Figure 2.1: Global energy flow in the atmosphere and earth surface. From the Source of Kiehl and Trenberth,1997 :,with a title Earth's Annual Global mean energy budget.AM.Met.Soc.78, $197 - 208$

2.2.1 A Simple energy balance of climate

Radiation budget of the earth

The major characteristic of a physical- chemical systems of the climate systems are expressed in terms of its energy budget.This energy budget is governed by the shortwave radiation R_i which coming into the top of the atmosphere from the sun and the longwave radiation R_o which escaping back into the atmosphere.The balance obtained between R_i and R_o gives us the mean temperature of the surface.

The energy distribution in terms of height,latitude and longitudes influences the distribution of temperature in the system.The study of spatially zero-dimension $(O - D)$ model has the global temperature as one of the only variable.

The solar radiation dependences on reflection by ice-albedo feedback and infrared absorption on temperature.Thus average reflectivity of the earth-atmosphere system is called the planetary albedo.

Global radiation budget

Climate is said to be global if it is determine by a radiation balance in the planet.The earth absorbs incoming solar radiation from the sun which causes its warming and cooling by radiating energy back into space through long wavelengths whereas some of it is absorbed by the atmosphere.

The variable for the global surface temperature is given below;

$$T = T(t)(c)$$

This radiation is either absorbed by the atmosphere 22% or transmitted to and absorbed by the ground 45% or reflected back to space 33%. A particular important role in both the absorption and reflection of solar radiation is played by clouds:they cover on average 50% of the earth surface.Clouds are very important in reflecting large amount of the short-wave radiation while some are reflected by the earth's surface and therefore absorbed by the atmosphere.

2.3 The zero-dimensional energy balance model

If we characterize a column of the earth-atmosphere system a single number,say the sea level temperature,we develop models leads to only horizontal dimension which is refers to as zero-dimensional climate model.This model is the simplest of all the earth's climate system. [6] The earth obtain heat by absorbing solar radiation from the sun and is cooled by radiating thermal energy through longwave radiation back into space.The shortwave radiation coming into the top of the atmosphere from the sun is given has, $Q = \frac{Q_0}{4}$

The solar constant, $Q_0 = 1370W/M^2$ refers to the radiative flux with a disk of earth's radius $R = 6300KM$ perpendicular to the sun's rays. This number 4 is obtained by taking the average of this energy over the earth's spherical surface area given by the equation below;

$$A = 4\pi R^2$$

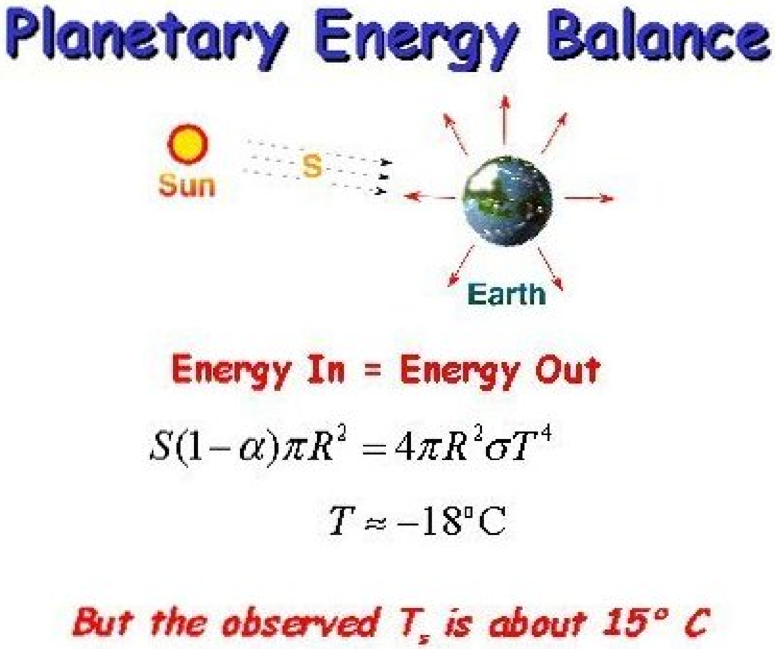


Figure 2.2: Heat absorbed and emitted by the earth

A disk of radius R from the sun ray's has an area of πR^2. Parts of the incoming radiation is reflected back to space by either clouds or snow as well as ice cover on the ground.

If we assume a steady state condition,the shortwave radiation absorbed by the earth surface is given by the equation;

$$F_{sw} = Q(1-\alpha)\Pi R^2 \tag{2.1}$$

this equation should be the same as longwave radiation given below;

$$F_e = 4\Pi R^2 \sigma T_e^4 \tag{2.2}$$

At the equilibrium temperature we have the equation

$$(1-\alpha)Q = \sigma T_e^4 \tag{2.3}$$

here $\sigma = 5 \cdot 67 * 10^{-8}(W/M^2K^4)$ refers to the Stefan-Boltzmann coefficient and $Q = \frac{Q_0}{4}$. Water vapour and CO_2 in the Earth's atmosphere are the main greenhouse gases which absorb a lot of longwave radiation emitted from the surface.

2.3.1 Emitted radiation

The energy lost through longwave radiation is modelled in terms of a linear equation given below. [20]

$$I(x) = A + BT(x) \tag{2.4}$$

this equation represents values of Stephan-Boltzmann principles of blackbody radiation and the effect of greenhouse gas on the earth's atmosphere.

where $A = 211.2Wm^{-2}$ and $B = 1.55Wm^{-2}$.This Eq. (2.4) is assumed to hold for each latitude x so that I(x) and T(x) are considered to be functions in x. [6]

Here $I(x)$ is the outgoing infrared radiation flux (Wm^{-2}).

$T(x)$ the surface temperature(sea level)temperature.

2.3.2 Ice-Albedo feedback

Given that the extent of the snow and ice cover on the earth surface is large it causes low temperatures since it reflected more of the solar radiation back to space.The greater the Snow and ice cover on the surface,leads to high albedo reflectivity.This implies a positive feedback. For example the snow and sea ice cover of the Northern hemisphere causes the Planetary albedo feedback. [9]

Therefore longwave radiation increases given that temperature also increases.A small change of the temperature from equilibrium,let's to a slight warming which results in cooling through increased longwave radiation.If temperature drops future below the freezing point snow and ice will cover the ground causing an increase of albedo on the surface.Thus temperature will drop further since more of the sunlight is reflected back to the space.

2.3.3 Water vapour feedback

The water vapour feedback plays an important role in the climate system because it is the main natural greenhouse gas.Therefore a warm atmosphere can be able to hold more water vapour than a cold atmosphere.Thus with this additional water molecules in the warm atmosphere it causes an improvement of natural greenhouse effect through increasing of long-wave radiation.

2.3.4 Cloud feedback

Low cloud affect shortwave radiation from the sun through albedo while the high cloud affects longwave radiation emitted by the surface of the earth.

2.3.5 A model for global temperature

The variable perceived most widely as defining climate is temperature T.It is also most important in determining the component of radiation balance.From $Eq(2.3)$ we obtain the equation governing the model which is, [4]

$$C\frac{\partial T}{\partial t} = F_{sw} - F_{lw} \tag{2.5}$$

with heat capacity $C > 0$ and $\frac{\partial T}{\partial t}$ is the rate of change of fixed latitude.

It expresses the approximate radiation balance between absorbed radiation and emitted radiation.Therefore we have the equation;

$$C\frac{\partial T}{\partial t} = (1 - \alpha(T))Q - (A + BT) \tag{2.6}$$

where $Q = \frac{Q_0}{4}$ is the solar insolation receive from the sun and α is the surface albedo.

Any slight imbalance between F_{sw} and F_{lm} leads to changes in the temperature of the system.This type of model was derive from the work of Budyko (1969) and Seller(1969).

2.4 One-dimensional energy balance climate model

2.4.1 Horizontal heat transport

The derivation of zero- dimensional energy balance model was based on vertical heat transfer.The result we have obtained will guide us to derive one-dimensional climate model.This model is a latitude- dependence one.There are three type of heat transfer:radiative,conductive and convective.In radiactive transfer,energy passes between the medium's molecule by electromagnetic radiation due to photo emission.

In conduction,energy passes from one molecules to another by thermal agitation that is,energy is transported from parts of the system with higher temperature to those of lower temperature.

The solar radiation Q varies strongly with latitudes.The circulation of the Earth's orbital

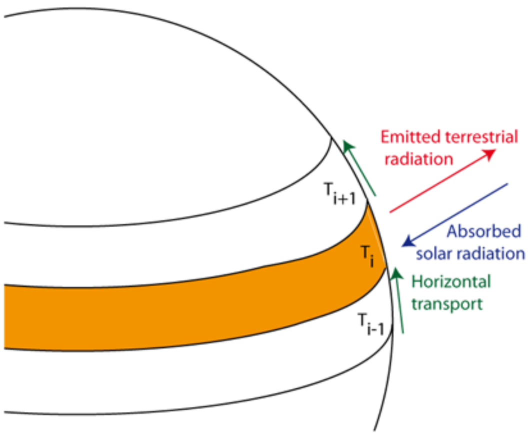


Figure 2.3: This is a representation of one-dimensional EBM whereby temperature is given as an averaged over a zone of latitude.

parameters given together with a high albedo for a given higher latitude has lead a great differences in the amount of solar received at a particular latitude.

The net radiation receives at the top of the atmosphere between the absorbed solar radiation and the outgoing longwave radiation given as follows $F_{sw} - F_{lw}$ is always positive at low latitude and has a negative value at high latitude. Transport of energy and matter in any given fluids is determined by both diffusion and advection processes.These processes leads to an induction of fluxes of energy whereby in mathematical terms it is derived by a process called continuum mechanics. [9] **Diffusion** is a random process which take place at all times leading to a net transport when given some certain conditions.This process is initiated by thermal motion of molecules. **Advection** is caused by ambient flow which involves the transport energy.All processes involving the climate systems are influenced by both advection and diffusion through the transport of mass,energy and momentum.

2.4.2 Meridional heat transport

Meridional heat transport takes into account the use of a diffusive parametrization [3].Thus low latitude areas the earth while receives more heat than what its emits back to space,but at high latitude it looses more energy through longwave radiation than what its receives from the sun. Therefore given an integral of the net radiation from one pole to another its yields a meridional heat transport by the climate from areas of low to high latitudes.Hence

$$F_m = -CK\vec{\nabla}T = -CK\frac{\partial T}{\partial x} \tag{2.7}$$

where K refers to diffusive coefficient and x refers to latitudinal variable of the North-South directions in terms of Cartesian coordinates
where $\bigtriangledown$ is the gradient operator and F_m is meridional heat flux on a given latitude. Conduction is the main form of heat transport in the earth hence temperature will change in time only due to the divergence of the conductive heat flux at a given latitude.The equation for temperature change at any given latitudinal belt is given in the equation below;

$$C\frac{\partial T}{\partial t} = -\vec{\nabla}F_m + F_{sw} - F_{lw} \tag{2.8}$$

here the first term on the right hand sides is the divergence of the meridional heat transport. The variables$T(\theta)$,$C(\theta)$ and $K(\theta)$ depend entirely on latitude θ which ranges from $-\frac{\Pi}{2}$ at the south pole to $\frac{\Pi}{2}$ at the north pole. Therefore since the earth is spherical we have to write our equation obtained in Eq.(2.8) in terms of Laplace operator has given

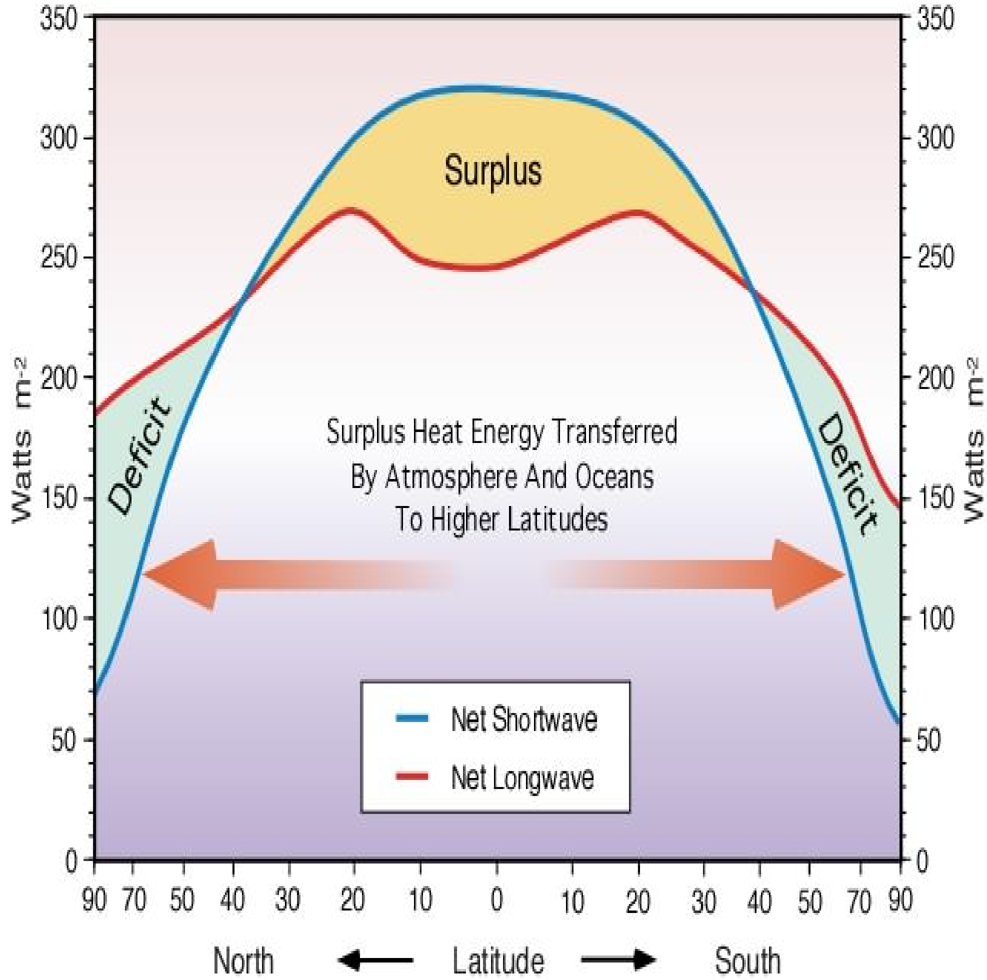


Figure 2.4: Mean latitudinal distribution of the earth's radiations From:Briggs,Smithson and Fall,Fundamental of physical Geography,(1989) Toronto:Copp Clarke and Pitmans Canadian Edition.Copp Clark Pitman Ltd

below in the equation;

$$\vec{\nabla}T_m = -\vec{\nabla}CK\vec{\nabla}T = \frac{-1}{R^2COS\theta}\frac{\partial}{\partial\theta}(CKCOS\theta\frac{\partial T}{\partial\theta}) \tag{2.9}$$

2.4.3 Derivation of this energy balance climate model.

We have consider that the earth is a sphere [16].Let R be the radius,latitude to be θ,the length of the latitude belt be $2\Pi RCOS\theta$ with latitude $Rd\theta$ and the area of latitude belt,$dA = 2\Pi R^2COS\theta d\theta$.Let the flux per unit length of longitude be given by the equation;

$$F = \frac{-D}{R^2}\frac{\partial T}{\partial x} = -D\frac{\partial T}{\partial x} \tag{2.10}$$

Flux is from hot to cold regions and increase in magnitude if the meridional temperature gradient increases.Total flux per latitude θ is given by

$$F = \tilde{F} \times 2\Pi RCOS\theta = -2\Pi DCOS\theta\frac{dT}{d\theta} \tag{2.11}$$

We assume that shortwave radiation = longwave radiation at the latitude.This means that,

$$SW\downarrow + F|_\theta = LW\uparrow + F|_{\theta+d\theta}$$

This radiation fluxes are per unit area so need to be multiplied by the area of the latitude belt.On rearranging and substituting for area we have

$$(SW\downarrow - LW\uparrow)\cdot 2\Pi R^2COS\theta d\theta = F|_{\theta+d\theta} - F|_\theta \tag{2.12}$$

Using the Taylor's of first order we get,

$$F|_{\theta+d\theta} = \frac{\partial F}{\partial\theta}|_{d\theta}d\theta$$

We know that
$SW\downarrow = Q(1-\alpha)S(\theta)$
And since we have linearize the longwave radiation we have
$LW\uparrow = A + BT$
Where T is the surface temperature

$$(1-\alpha)QS(\theta) - (A+BT)\cdot 2\Pi R^2COS\theta d\theta = \frac{dF}{d\theta}|_{d\theta}d\theta \tag{2.13}$$

substituting the value of F from Eq.(2.11) into Eq.(2.13) we have the equation,

$$(1-\alpha)QS(\theta)-(A+BT)=\frac{-1}{2\Pi R^2COS\theta}\frac{d}{d\theta}(2\Pi DCOS\theta\frac{dT}{d\theta}) \tag{2.14}$$

Using the transformation of variables.Let $x = sin\theta$ Then we have,

$$\frac{d}{d\theta}=\frac{dX}{d\theta}\cdot\frac{d}{dX}=cos\theta\frac{d}{dX} \tag{2.15}$$

which gives

$$(1-\alpha(\theta))QS(\theta)-(A+BT)=\frac{-D}{R^2}\frac{d}{dX}(COS^2\theta)\frac{dT}{dX} \tag{2.16}$$

since the temperature changes with time from Eq.(2.16) and using the fact that,
$COS^2\theta = 1 - SIN^2\theta$
we obtained one dimensional energy balance climate model equation by using the fact that $X^2 = SIN^2\theta$

$$C(x)\frac{dT}{dt}=(1-\alpha(x))QS(x)-(A+BT)+\frac{d}{dx}(K(x)(1-X^2)\frac{dT}{dX}) \tag{2.17}$$

This equation summarizes radiactive,that is vertical heat flux and advective, that is horizontal heat fluxes.The shortwave radiation and longwave radiation have become functions of X and T rather than of the single,global variable T that was model for the zero dimensional model. This Eq.(2.17) is a non-linear second order partial differential equation.It's solution can be obtained by solving the given equation subject to the Neumann boundary conditions of zero heat flux at the pole and equator.The conditions at the poles is natural,while that at the equator is equivalent to assuming the symmetry of the two hemispheres;such a symmetry assumption is reasonable for the simplicity of the model.Since we let $x = sin\theta$ and we know that $0 \leq \theta \leq \frac{\Pi}{2}$ then,
$x = +1, x = -1$ at the poles
$x = 0$ at the equator

By defining variable $x = sin\theta$ the equation governing steady state solution can be written as follows

$$\frac{d}{dx}(k(x)(1-x^2)\frac{dT}{dx})+(1-\alpha(x))QS(x)-(A+BT)=0 \tag{2.18}$$

Where $S(x)$ represents the mean annual distribution of radiation at each latitude.This equation has solution of the given form $T = T(x,t)$ which is subject to following boundary conditions.

NEUMANN BOUNDARY CONDITION

(i) since there is no heat transport at the pole. [17]

$$(1-x^2)^{\frac{1}{2}}\frac{d}{dx}T_1(x)|_{x=1}=0$$

(ii) No heat transport across the equator

$$(1-x^2)^{\frac{1}{2}}\frac{d}{dx}T_0(x)|_{x=0}=0$$

This is a two -point boundary value problem.Thus climate is now characterized in terms of annual, average and global mean surface temperature T.The steady-state equation becomes an nonlinear (ODE) of T which can therefore be solved numerically.

2.5 Budyko's model as a difference equation.

To obtained a complete picture of the concepts pertaining ice albedo feedback mechanism , we need to look at both space of the function used and the value of the ice line.Therefore given that the two factors are achieved then Budyko model can be treated in terms of dynamical systems hence becoming a mathematical equation to be studied. [35]

If we assume that all functions depending on latitude are symmetric in the equator then we need only the Northern Hemisphere ranging from $(0 \leq \theta \leq \frac{\pi}{2})$.Therefore at any time t, we let $T(x,t)$ be the surface temperature on a given circle of latitude θ with $x = sin\theta$.Thus the following equation gives us diffusive zonal symmetric energy balance climate model equation;

$$C\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(K(x)(1-x^2)\frac{\partial T}{\partial x}) + QS(x)\beta(x,\mu) - (A+BT) \tag{2.19}$$

where Q refers to amount of energy which is received at top of the atmosphere from the sun.The $S(x)$ is the distribution of energy per a given latitude zone which can be computed by taking into consideration Earth's orbital elements.Authors including Tung [34]and North [17],use Legendre polynomial approximation function given by the equation below to obtained this values;

$$S(x) = 1 - 0.482.\frac{3x^2-1}{2} \tag{2.20}$$

Therefore the fraction of the radiative energy by planet at a given latitude x,when the ice line is given at a value μ,is expressed in the equation shown below;

$$\beta(x, \mu) = 1 - \alpha(x, \mu) \tag{2.21}$$

We assume that the surface considered is covered by water or ice with only one ice line μ.The albedo function $\alpha(x, \mu)$ is smooth and it dependence on the ice-line.If the ice line is given at μ and albedo at a latitude x then we have the following equation shown below; [35]

$$\alpha(x, \mu) = \frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_2 - \alpha_1}{2}.tanh[M.(x - \mu)] \tag{2.22}$$

where $\alpha_1 =$ represents either ocean or water albedo for areas free of ice.

$\alpha_2 =$ represents the ice albedo for area covered with ice.

The variable M represent the gradient of a given albedo function near the specified ice line and is always expressed as a fixed quantity.We obtain albedo at the given ice line by determining average of the ice and ocean albedo values.At the equator of the given ice line,the albedo is always approximated in terms of ocean albedo,while at the poles we expressed in terms of ice albedo.These values α_1 and α_2 have no dimensions.Using the parameters the ice line has been taken as $\mu = 0.95$. [17]

Therefore the values of $\beta(x, \mu)$,takes the following step function given below;

$$\beta(x, \mu) = \begin{cases} \alpha_1 = 0.32 & x > \mu \\ \alpha_2 = 0.62 & x < \mu \end{cases}$$

The value of diffusive coefficient $K(x)$ is given by;

$K(x) = \frac{D}{R^2}$ here R refers to the radius of the earth surface.

$$D(\theta) = (1.5 + 2.5\cos\theta) * 10^7 \tag{2.23}$$

Chapter 3

FINITE ELEMENT METHOD

3.1 Introduction

An equation which involving derivatives of one or more dependent variables with respect to one or more independent variables are called a differential equation.Differential equation is used to solve problem pertaining real life matters which cannot be solved directly,therefore solution can be obtained approximately by using numerical methods. [11]

Thus, a differential equation can be expressed as a relation between an unknown function U with it's derivatives u^k,and the unknown is given by the equality below $1 \leq K \leq N$,where K and N are both integers.They are either in terms of partial differential equations(PDES) or ordinary differential equations(ODES). [10]

A differential equation which involving derivatives with respect to a single independent variables is called an ordinary differential equations(ODES).They are broadly classified according to the order of the highest derivative of the dependent variable with respect to the independent variable that appears in the equation.If given the function $U(x)$ which depends on only one variable $x \epsilon \Re$, the equation is then is called an ordinary differential equation (ODES).

A differential equation which involves partial derivatives with respect to two or more independent variables is called partial differential equations(PDES).

For example,

$$U_t(x,t) - U_{xx}(x,t) = 0$$

is a good example since it has two independent variables and the equation is a homogeneous (PDE) of second order.

Thus, if given the function $U(x,t)$ that depends on more than one variable,the differential equation is then called a partial differential equation(PDE).A solution of any given differential equation may be functions of the form; e.g $U(x), U(x,t)$ or $U(x,y)$.

The solution U cannot be expressed in the form of elementary functions but numerical methods are the best way for solving the differential equation through construction of the approximate solutions.

3.1.1 Ways of classifying a partial differential equations

Both ODES and PDES can be broadly classified in terms of linear and non-linear equations.A linear partial differential equation can be expressed in a form where all the partial derivatives appear in linear form and none of the coefficient terms depends on the dependent variable. [11]

For example

$$U_{yy}(x,y) + U_{xx}(x,y) = f(x,y)$$

is an example of linear non-homogeneous PDE which is of second order.

A non-linear PDE can be described as a partial differential equation which involves non-linear term.For example

$$yU_{yy}(x,y) + U_{xx}(x,y) = 0$$

is an good example of non-linear homogeneous PDE which is of second order.

3.1.2 Classification based on discriminant

Given a general second order PDE with constant coefficients in two independent variable as shown below [10]

$$Au_{xx}(x,y) + 2Bu_{xy}(x,y) + Cu_{yy}(x,y) + Du_x(x,y) + Eu_y(x,y) + Fu(x,y) = G \quad (3.1)$$

The discriminant is given as $d = AC - B^2$. Thus classification based on this discriminant is as follows;

1 If $d = AC - B^2 > 0$,we have an elliptic equation.

2 If $d = AC - B^2 = 0$,we have parabolic equation

3 If $d = AC - B^2 < 0$,we have hyperbolic equation

Laplacian equation is a good example of elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The heat equation given below,

$$U_t(x,t) - U_{xx}(x,t) = 0 \tag{3.2}$$

is one of the parabolic equation.

The wave equation shown below

$$U_{xx}(x,y) + U_{yy}(x,y) = 0 \tag{3.3}$$

reperesents good example of hyperbolic equation.

3.1.3 Types of problems governing differential equation

1 INITIAL VALUE PROBLEMS(IVP)

An initial value problem refers to one where the dependent variables and possible the derivatives are specified initially(at time $t = 0$)or at the same value of independent variables.

For example,for a given time dependent differential equation which is of second order the initial values for $t = 0$ given by $U(x,0)$ and $U_t(x,0)$, are generally specified.That is,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, U(x,t_0) = f(x)$$

.

2 BOUNDARY VALUE PROBLEMS(BVP)

A boundary value problem is one where the dependent variable with its derivatives are specify at the extreme ends of the independent variable.

Therefore, to determine solution U uniquely for a differential equation we must have the boundary conditions imposed at the boundary points

THERE ARE THREE TYPES OF BOUNDARY CONDITIONS

1 DIRICHLET BOUNDARY CONDITIONS

Given a stationary heat equation in one dimension has shown below

$$\frac{\partial^2 u}{\partial x^2} = f(x)$$

in (a, b),the homogeneous Dirichlet boundary conditions is given has follows

$$u(a) = u(b) = 0$$

The value of the given function u is always specified on the boundary.The dependent variable of PDE are prescribed in the domain at different points.

2 NEUMANN BOUNDARY CONDITIONS

A Neumann boundary condition which is imposed on either ODE or PDE always specifies the derivatives of values of a solution it can take on the boundary of the given domain.The heat equation in one dimension is a good example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial u}{\partial n} = 0$$
$$u(x, 0) = u_0(x), x\varepsilon\Re$$

3 MIXED BOUNDARY CONDITIONS

If we combine both Dirichlet and Neumann boundary condition we shall obtained mixed boundary condition hence it is a linear combination of the two terms:i.e A stationary heat equation is a good example

$$\frac{\partial^2 u}{\partial x^2} = f(x), 0 < x < 1$$
$$u(0) = 0, a(1)\frac{\partial u}{\partial x} = g_1$$

when we shall introduce finite element method ,we will show how these conditions are important when obtaining weak formulation of the problem.

3.1.4 Methods of solving differential equation

We consider the type of the physical problem and various corresponding mathematical formulation that will allow use to apply different variety of method so as to obtain solution .The common well know methods are the finite difference method,finite volume method and finite element method.

Finite difference method expresses every occurrence of partial derivative in terms of discrete approximation using a grid points.It is not very high in accuracy and does not implement more complex geometry.

Finite element method is quite different from the finite difference when it comes to the discretization of the domain of interest of the PDES. It is used now days to solve all kinds of PDES.Therefore we will apply this method to the steady state one dimensional energy balance climate model equation.

3.2 What is finite element method?

This is a numerical technique used in solving problems which are described in terms of partial differential equation (PDE).The physics of phenomena encounter in mathematics application is often modelled under the form of a boundary value problem.This method has various procedures for solving PDES and is used in a variety of applications as follows;

(i) in structural mechanics

(ii) dynamics

(iii) heat transfers

(iv) fluid flow

(v) electric and magnetic fields

(vi) electromagnetic

The equation describing the evolution in time are called initial value problems and consist of coupling of PDE in time with a boundary value problem in space.

In this method we consider a given domain of interest which is represented in form of an assembly of what is called finite element and therefore we approximate the given functions in terms of nodal values to be determined.Iterative procedures are used so as to obtained efficient the numerical solution of a problem in matrix equation.

This method use approximate technique to solve solution of differential equation by using a piecewise interpolating polynomials.A numerical method arises due to the need of converting a continuous problem into a discrete form.The continuous problem always have infinitely many unknowns which cannot be solved using a computer to obtained exact solution .Thus we have to approximated by using discrete form of the problem which has many number of unknowns in terms of finite elements.The more we increased the number of unknowns, we then improve the accuracy of the solution.

Functions are therefore expressed in terms of basis functions in which the equation to be considered is solved in terms of weak form.Functions is approximated by the equation shown below;

$$U_h = \sum u_i \varphi_i x.$$

here h is the step length and u_i is the unknown to be determined.

FEM is a one of the numerical method which resembles FDM although it is a general and powerful method when used in application to solve real world problems that constitute complicated physical geometry and as well as boundary conditions. Galerkin's method is one way of approximating solution of a given PDES and ODES. This approximate values ares [10]

1 Easy to obtain differentiation and integration of the problem.

2 This method is spanned by a set of orthogonal basis function defined in a given finite dimensional vector space.

3.3 Steps in Finite Element Method

- DISCRETIZE THE CONTINUUM

We divide the domain of interest into smaller regions called elements.This elements contain inside certain number of points called nodes and is based on partition of an interval (a,b) into subinterval given below

$$a = x_0 < x_1 < x_2 < \ldots < x_m < x_{m+1} = b$$

where $h_j = x_j - x_{j-1}$ and $j = 1, \ldots, m+1$ is a partition of the interval (a,b) into m+1 subinterval.For each given node point $x = \xi_i$ we can associate basis function in the form $\varphi_i(x)$,$i = 0, 1, 2, \ldots, q$. Thus we obtained $q + 1$ basis functions.

- SELECT THE TYPE OF INTERPOLATION POLYNOMIAL TO USE

We select the kind of function we will take to describe the variation of the function inside each element.This is done by use of polynomials.

A polynomial given in the form $P \in P^q(a, b)$, which has the value of $P_i = P(\xi_i)$ at the given nodes $X = \xi_i$ for each value of $i = 0, 1, 2, \ldots, q$, expressed in the form of Lagrange basis is given by the equation;

$$P(x) = p_0\varphi_0(x) + p_1\varphi_1(x) + \ldots + p_q\varphi_q(x) \tag{3.4}$$

where $\varphi_i(x)$,$i = 0, 1, 2, \ldots, q$ is the basis function. This polynomial can either be piecewise linear, quadratic or cubic.

- THE FORMULATION

  Given the PDE to solve we must find a system of algebraic equation for each element.

  This is done by multiplying the initial problem of the PDE with a test function v in a certain vector space V and we integrate by part over [a,b] thus we obtain weak formulation of the problem.

  If we assume that for a typical dependent variable U written in the form

  $$U = \sum u_i \varphi_i \tag{3.5}$$

  we then substitute it to the weak formulation of the problem so as to obtain equation for each element in the form given below;

  $$[A^e][U^e] = [b^e] \tag{3.6}$$

- ASSEMBLYING THE EQUATIONS FOR DIFFERENT ELEMENTS

  We have to assemble the equation for all elements.If we have a total of $q+1$ nodes in the system then we must build up a global matrix [A] of size $(q+1) \times (q+1)$ and a global vector [b] of size $(q+1) \times 1$.

  Thus FEM reduce a problem into matrix form of the equation shown below,

  $$[A][U] = [b] \tag{3.7}$$

- SOLVE THE SYSTEM OF EQUATION

  One can use whatever method to solve this matrix.The more the number of nodes the better the quality of the solution,it can either be a sparse matrix or a band matrix.

  We can solve this by using direct method, iterative method,LU factorization and QR factorization.

3.4 Formulation

3.4.1 Strong formulation

The set of governing PDE's with a given boundary conditions is refer to as a classical or a strong formulation of the problem.

Example 1. *Consider a one dimension heat equation given by:*

$$\frac{\partial u}{\partial t} = C\frac{\partial^2 u}{\partial x^2}, 0 \leq X \leq 1$$

subject to the following initial conditions

$U(x, 0) = U_0(x)$

with the given boundary conditions

$U(0) = U(1) = 0$

This is called strong formulation.

3.4.2 Weak formulation

We then reformulate the strong form of the problem into the weak form.This form is always a variation formulation of the given problem where we multiply the initial value problem with a test function and we integrate by part over a given domain to obtained weak formulation of the problem.This form relaxes the given problem whereby we obtain the approximate solution instead of an exact solution that satisfies the strong form on a given average of a domain.It is called a weaker statement of the problem because given a solution of the strong form will always satisfy the weak form, but vice versa is not always true.

Example 2. *Consider Poisson equation in one dimension.*

$$\nabla^2 U = P_0, 0 \leq x \leq 1$$

subject to the following boundary conditions

$u(0) = u(1) = 0$

We then multiply the given equation with a test function v and integrate by part over the domain Ω.

$$\int_{\Omega} (\nabla^2 u - P_0)v = 0$$

We then define a space of function and call it H^1. This is a functional space where all the function are bounded. Let also define X be a sub-space of H^1 where we can determine our solution U.Let $u, v \in X$.We know that from calculus we can obtain the equation given has follows;

$$\nabla(V\nabla U) = \nabla V.\nabla U + V\nabla^2 U$$

Then we can be able to write the equation in the form given below;

$$\int_{\Omega} V\nabla^2 U = \int_{\Omega} \nabla(V\nabla U) - \int_{\Omega} \nabla V.p\nabla U$$

By use of Gauss's theorem on the formula $\nabla(V\nabla U)$ given below we obtained the following equation,

$$\int_{\Omega} (V\nabla U) = \int_{\Omega} V\nabla U.\hat{n}ds = 0$$

We then reduce our equation to

$$\int_{\Omega} V\nabla^2 U = -\int_{\Omega} \nabla V.\nabla U$$

Finally we obtained the equation has follows;

$$-\int_{\Omega} \nabla V.\nabla U = \int_{\Omega} P_O V dA$$

This is the weak formulation or variational problem of our equation given.The choose of the test function should satisfies the type of the boundary condition given in terms of Dirichlet,Neumann or Mixed boundary conditions.

For our example given homogeneous Dirichlet boundary condition;

$U(0) = U(1) = 0$

So we let,

$V(0) = V(1) = 0$

This integration by part results in boundary contribution, which can be replaced by either using boundary conditions or using the restriction imposed on the test functions V.

3.5 Polynomial interpolation

3.5.1 Element type

Given a continuous function we can approximate the function by an approximate function and before we do that we need to consider the type of element we have to use.It is important to take into consideration a given type of element which is suitable for the problem to be solve numerically.For example,

(i) ONE- DIMENSIONAL ELEMENTS

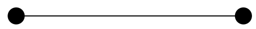


Figure 3.1: Linear

Figure 3.2: Quadratic

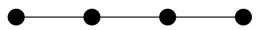


Figure 3.3: Cubic

(ii) TWO- DIMENSIONAL ELEMENTS

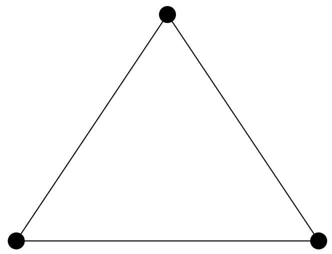


Figure 3.4: Linear triangular

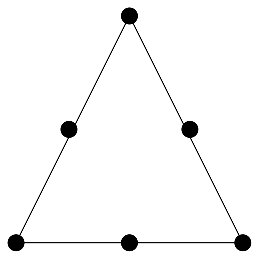


Figure 3.5: Quadratic triangular

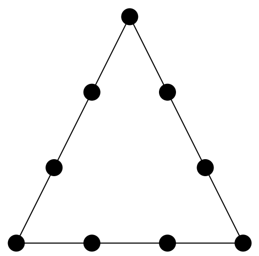


Figure 3.6: Cubic triangular

The regions is divided into smaller parts called nodes.The value of the fields variable computed at the nodes are used to approximate the values at non- nodal points (that is,the interior mesh points) by interpolation of the nodal values.

Element type depend on the following

1 shape .

2 the number and type of nodes of the elements.

3 type of the nodal variable of the elements.

4 type of the interpolating function.

Shape function or basis function are associated with element.

3.6 Basis function

3.6.1 Vector space of a piecewise linear function on given interval

Given an interval $I = [a, b]$,we let $I_h a = x_0 < x_2 < x_3 \ldots < x_{N-1} < x_N = b$ to be a partition of our interval I into subintervals $I_j = [x_{i-i} - x_i]$ of $h_i = x_i - x_{i-1}, i = 1, 2, \ldots, N$.We then let $V_h = v|v$ be a continuous piecewise linear function on I_h then V_h is a vector space which has the hat functions given by $(\varphi)_{j=0}^N$ known as basis function.Let v be our approximation so we can be express as a linear combination of $\varphi_i(x)'s$. [10]

Then we can write $v(x)$ in the form,

$v(x) = \sum_{i=0}^N v(x_i)\varphi_i(x), \forall v \in v_h$

Let $x = x_j$ then $\phi_i's$ has the property that

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

i.e

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{h_{i+1}}, & x_i \leq x \leq x_{i+1} \\ 0, & x \notin [x_{i-1}, x_{i+1}] \end{cases}$$

These functions are refer to as basis function since they are linearly independent thus it is not always easier to make one out of a combination of another one.

The graph below shows an example of linear basis function with three elements.

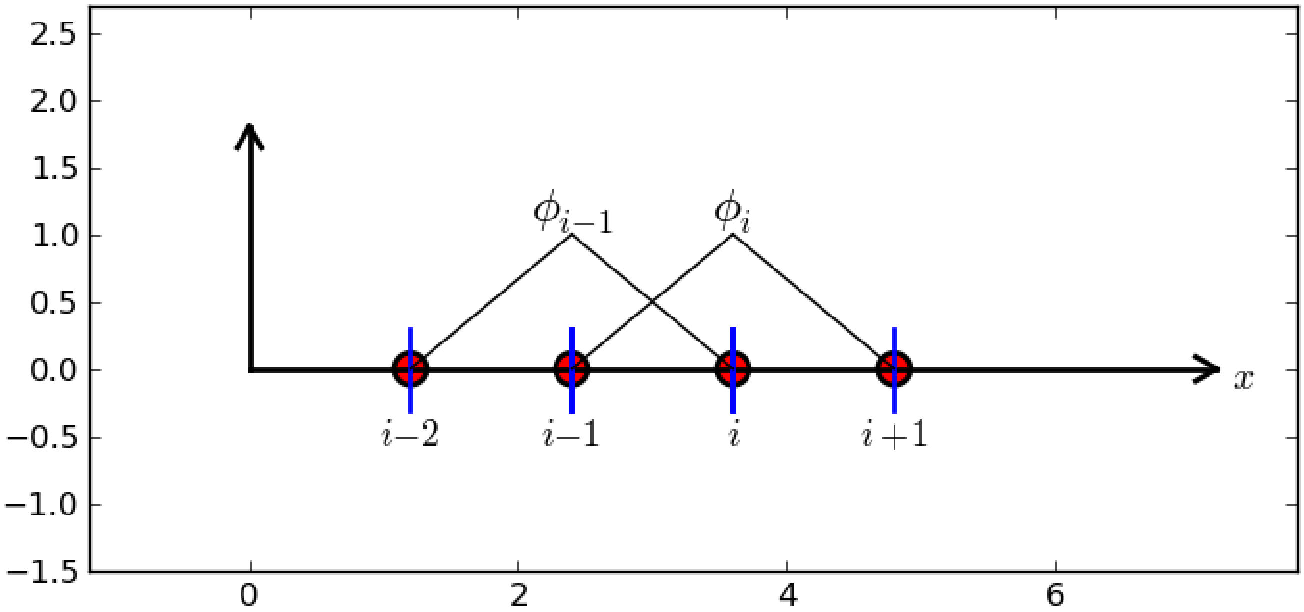


Figure 3.7: Linear hat function

3.6.2 Lagrange interpolation

Definition 1 (Definition of a cardinal functions)**.** *[10].Lagrange basis is refer to as the set of polynomial* $\lambda_{i\,i=0}^{q}$ *which are associated with the given* $q+1$ *distinct points in the given interval* $a = x_0 < x_1 < \ldots < x_q = b$ *where [a,b] are closed and are determined by the following requirement, for* $i = j$ *we obtained the following* $\lambda_i(x_j) = 1$ *with a value of zero for* $i \neq j$ *thus we have* $\lambda_i(x_j) = 0$*.*

$$\lambda_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})\downarrow(x-x_{i+1})\ldots(x-x_q)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})\uparrow(x_i-x_{i+1})\ldots(x_i-x_q)}$$

By use of arrows $\downarrow,\uparrow$ we wanted to highlight the fact that $\lambda_i(x) = \Pi(\frac{x-x_i}{x_i-x_j})$ cannot be expressed in the following singular factor shown $\frac{x-x_i}{x_i-x_i}$

Example 3. *If we let* $q = 2$*,then we obtained an interval of the form* $a = x_0 < x_1 < x_2 = b$*,here we have*

$$i = 1, j = 2 \Longrightarrow \delta_{12} = \lambda_1(x_2) = \frac{(x_2-x_0)(x_2-x_2)}{(x_1-x_0)(x_1-x_2)} = 0$$

$$i = j = 1 \Longrightarrow \lambda_1(x_1) = \frac{(x_1 - x_0)(x_1 - x_2)}{(x_1 - x_0)(x_1 - x_2)} = 1$$

A polynomial $P(X) \in P^q(a,b)$ with the following values $P_i = P(X_i)$ at the given nodes $x_i, i = 0, 1, 2, \ldots, q$,can therefore be represented in terms of Lagrange basis shown below;

$$P(x) = p_0\lambda_0(x) + p_1\lambda_1(x) + \ldots + p_q\lambda_q(x)$$

Definition 2. *let [a,b] be a closed interval with the partition shown* $a \leq \xi_0 < \xi_1 < \ldots < \xi_q$ *where* $q + 1$ *are distinct interpolation nodes on the given interval.Thus* $\Pi_q f \in P^q(a,b)$ *interpolate* $f(x)$ *at the given nodes* ξ_i*,if given that*

$$\Pi_q f(\xi_i) = f(\xi_i), i = 0, 1, \ldots, q$$

therefore the Lagrange formula $\Pi_q f(x)$ *can be written as follows;*

$$\Pi_q f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) + \ldots + f(\xi_q)\lambda_q(x)$$

Example 4. *let the variable* x *be given has follows* $x = \xi_i$*, then our linear Lagrange basis functions for* $q = 1$ *is given by the equations shown below;*

$$\lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - \xi_0}$$

$$\lambda_1(x) = \frac{x - \xi_0}{\xi_1 - \xi_0}$$

thus the Lagrange formula given that $q = 1$ *is shown below;*

$$\Pi_q f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x)$$

One-dimensional element has the following basis function.

- Linear basis function
- Quadratic basis function
- Cubic basis function

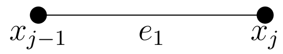


Figure 3.8: Linear element

We will only give a explanation of linear and quadratic basis function. Lets consider a polynomial of one-dimensional case given by the equation shown below;

$$P(x) = \alpha_0 + \alpha_1(x) + \alpha_2(x^2) + \ldots + \alpha_n(x^n)$$

This is a polynomial of degree n. To obtain a linear basis function we consider a linear element as shown above with two nodes and one element. Lets consider a polynomial of degree (1) in (a, b)

$$P(x) = \alpha_0 + \alpha_1(x)$$

Using these two nodes we can obtain polynomial at each node.Therefore we have the following

$$P(x) = \alpha_0 + \alpha_1(x_{j-1}) = f(x_{j-1})$$

$$P(x) = \alpha_0 + \alpha_1(x_j) = f(x_j)$$

We can find that

$\alpha_0 = f(x_{j-1})\frac{x_j}{x_j - x_{j-1}} + f(x_j)\frac{-x_{j-1}}{x_j - x_{j-1}}$

$\alpha_1(x) = f(x_{j-1})\frac{-x}{x_j - x_{j-1}} + f(x_j)\frac{x}{x_j - x_{j-1}}$.

Then our polynomial becomes

$$\begin{aligned} P(x) = \alpha_0 + \alpha_1 x = f(x_{j-1})\frac{x_j - x}{x_j - x_{j-1}} &+ f(x_j)\frac{x - x_{j-1}}{x_j - x_{j-1}} \\ &= f(x_{j-1})\lambda_{j-1}(x) + f(x_j)\lambda_j((x) \end{aligned}$$

Hence for the value of $x_{j-1} \leq x \leq x_j$ we have that $j = 1$.
Therefore we can imposed restriction on $\lambda_{j-1}(x)$ and $\lambda_j(x)$ on the given piecewise linear basis function $\phi_{j-1}(x)$ and $\phi_j(x)$. These values ϕ_{j-1} and ϕ_j are linear basis function.

REMARKS
It is noted that,instead of global coordinate x,one can use normalized coordinate ξ in Lagrange interpolation function.

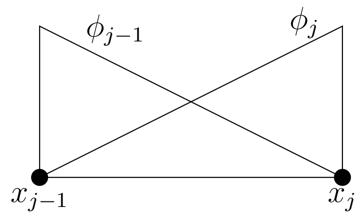


Figure 3.9: Linear Lagrange basis function

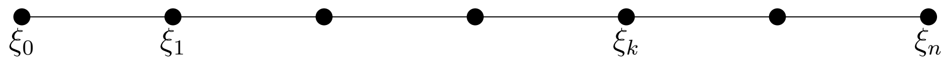


Figure 3.10: Normalized coordinate

1-dimensional quadratic element

Example 5. *If we wish to have an approximate solution of a function $U(x)$ defined in the following closed interval [a,b] then we expressed the set of basis function ϕ_i as shown below;*

$$u(x) = \sum_{i=1}^{n} c_i \phi_i$$

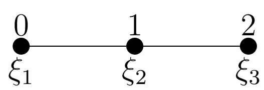
where i refer to the number of a given nodes points. Let our local coordinate be given by

$$\xi = \frac{x - x_j}{x_{j+1} - x_j}$$

thus the elements can also be expressed in the form $x = [0, 1]$.

Given our function of interest in terms of $u(x)$ we can obtained the approximate solution by using the quadratic basis function given in the equation below;

$$u(\xi) = c_1 + c_2\xi + c_3\xi^2$$



Our three nodes points therefore can be defined as shown $\xi_{1,2,3} = 0, \frac{1}{2}, 1$ and we obtained the values of u_1, u_2 and u_3 has follows;

$u_1 = c_1$

$u_2 = c_1 + 0.5c_2 + 0.25c_3$

$u_3 = c_1 + c_2 + c_3$

By solving the equations simultaneously we obtained the following equations;

$c_2 = -3u_1 + 4_2 - u_3$
$c_3 = 2u_1 - 4u_2 + 2u_3$
$c_1 = u_1$

We can therefore express our approximated function as a sum of all the basis function obtained by the values at the given 3 nodes.Thus we have the following equation;

$$u(\xi) = u_1 + (-3u_1 + 4u_2 - u_3)\xi + (2u_1 - 4u_2 + 2u_3)\xi^2$$

$$u(\xi) = u_1(1 - 3\xi + 2\xi^2) + u_2(4\xi - 4\xi^2) + u_3(-\xi + 2\xi^2)$$

We finally obtained

$$u(\xi) = u_1N_1(\xi) + u_2N_2(\xi) + u_3N_3(\xi)$$

where $N_1\xi, N_2\xi, N_3\xi$ are quadratic basis functions.

Accuracy can be improve by the following properties; [14]:

- decreasing the size of the height between elements as small as possible.
- by increasing the given order of the interpolating polynomials.
- by doing both of the properties at the same time.

3.7 Quadrature rule

3.7.1 Simpson's rule

Given an interval [a,b] we can therefore approximate its integral as shown $I = \int_a^b f(x)dx$ by use of a partition of I into a given subintervals so that in each subinterval of a function f is approximate by a degree 2 polynomial. This rule use the values of f at the two given end points a and b with the midpoint in the form $\frac{a+b}{2}$.We determines the values of $f(a), f(b)$ and $f(\frac{a+b}{2})$ at each point.

The area given by the equation $y = f(x)$ refers to the approximate values under the graph expressed by a polynomial of degree 2 given by the equation $P_2(x)$ where these values are represented as follows;$P_2(a) = f(a), P_2(\frac{a+b}{2}) = f(\frac{a+b}{2})$ finally $P_2(b) = f(b)$.To determine $P_2(x)$ by using the Lagrange interpolation for $q = 2$ we let $x_0 = a, x_1 = \frac{a+b}{2}$ and $x_2 = b$ and we obtained the equation shown below;

$$P_2(x) = f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x) + f(x_2)\lambda_2(x).$$

$$\begin{cases} \lambda_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \\ \lambda_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \\ \lambda_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \end{cases}$$

Thus

$$I = \int_a^b f(x)dx \approx \int_a^b P_2(x)dx = \sum_{i=0}^{2} f(x) \int_a^b \lambda_i(x)dx$$

We can easily compute the integrals

$$\int_a^b \lambda_0(x)dx = \int_a^b \lambda_2(x)dx = \frac{b-a}{6}, \int_a^b \lambda_1(x)dx = \frac{4(b-a)}{6}.$$

Hence

$$I = \int_a^b f(x)dx \approx \frac{b-a}{6}[f(x_0) + 4f(x_1) + f(x_2)].$$

This is the simple Simpson's rule. [10]

3.7.2 Composite Simpson's rule

This rule is based on the following approximate values of the integral given below;

$$I = \int_a^b f(x)dx.$$

- We divide the closed interval [a,b] into uniform N subintervals shown below;

  $a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < X_N = b$

- Then we can expressed the integral given above as follows;

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \ldots + \int_{x_{N-1}}^{x^N} f(x)dx = \sum_{k-1}^{k} \int_{x_{k-1}}^{x_k} f(x)dx$$

here each subinterval is given as $I_k = [x_{k-1}, x_k]$,where $k = 1, 2, \ldots, N$

For our simple Simpson's rule on each subinterval we have

$$\int_a^b f(x)dx = \int_{x_{k-1}}^{x_k} f(x)dx \approx \sum_{k=1}^{N} \frac{h}{6}[f(x_{k-1}) + 4f\frac{x_{k-1} + x_k}{2} + f(x_k)]$$

This is the composite Simpson's rule. [10]

3.8 Matrix assemblying

We assemble mass and stiffness matrix for all the elements as shown below in the diagram and with the help of Gaussian elimination method we can obtain the solution of our problem.[12]

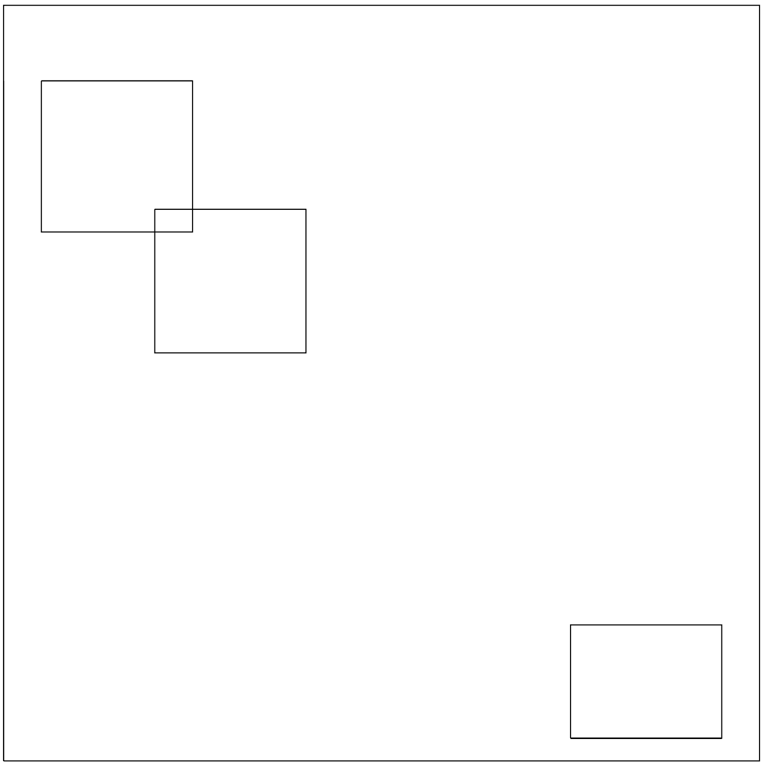


Figure 3.11: Shows matrices assembly for more element

Chapter 4

FEM FOR SOLVING ONE DIMENSIONAL STEADY STATE ENERGY BALANCE CLIMATE MODEL

4.1 Derivation of the problem

We consider the Galerkin finite element method for solving one dimensional steady state energy balance climate model equation.From Eq.(2.19) to obtained steady state equation we let $\frac{\partial T}{\partial t} = 0$ and we have the following equation;

$$-\frac{\partial}{\partial x}(K(x)(1-x^2)\frac{\partial T(x)}{\partial x}) + A + BT(x) = QS(x)(1-\alpha(x,\mu)) \tag{4.1}$$

subject to the boundary conditions

$$(1-x^2)^{\frac{1}{2}}\frac{\partial T(x)}{\partial x}|_{x=0,1} = 0 \tag{4.2}$$

The Galerkin problem is based on the variation formulation,where we multiply Eq. (4.1) with a given test function v and integrate over the open interval $(0, 1)$.

From Eq. (4.1) let

$$u(x) = (1-x^2)\frac{\partial T(x)}{\partial x} \tag{4.3}$$

Then we have the following equation by substituting the value of Eq.4.3 into Eq.4.1;

$$-\frac{\partial(K(x)u(x))}{\partial x} + A + BT(x) = QS(x)(1 - \alpha(x, \mu)) \tag{4.4}$$

We re-arrange Eq.(4.4) to obtain

$$-\frac{\partial u(x)}{\partial x} + \frac{B}{K}T(x) = \frac{QS(x)(1 - \alpha(x, \mu)) - A}{K} \tag{4.5}$$

Here $\frac{\partial K}{\partial x}u = 0$ since $K(x)$ are constant values of $x \in (0, 1)$.

We multiply Eq. (4.5) with a test function say v and integrate over the interval $(0, 1)$ as shown below;

$$\int_0^1 (-\frac{\partial u}{\partial x} + \frac{B}{K}T(x))v(x)dx = \frac{QS(x)(1 - \alpha(x, \mu)) - A}{K}\int_0^1 v(x)dx \tag{4.6}$$

We then perform the integration by part on the first term on the left of Eq.(4.6) and we obtained the following equation.

$$-\int_0^1 \frac{\partial u}{\partial x}v(x)dx = -u(x)v(x)|_0^1 + \int_0^1 u(x)v'(x)dx \tag{4.7}$$

Therefore applying the following boundary conditions in Eq.(4.2) we obtained the equation shown below,

$$-\int_0^1 u'(x)v(x)dx = \int_0^1 u(x)v'(x)dx \tag{4.8}$$

By substituting Eq.(4.8) into Eq.(4.6) we have,

$$\int_0^1 u(x)v'(x)dx + \frac{B}{K}\int_0^1 T(x)v(x)dx = \frac{QS(x)(1 - \alpha(x, \mu)) - A}{K}\int_0^1 v(x)dx \tag{4.9}$$

Here $S(x)$ solar insolation constant and $(1 - \alpha(x, \mu))$ are constant values depending on $x \in (0, 1)$. Therefore by Substituting Eq.(4.3) into Eq.(4.9) we obtained the following equation shown below;

$$\int_0^1 (1 - x^2)\frac{\partial T}{\partial x}v'(x) + \frac{B}{K}\int_0^1 T(x)v(x)dx = \frac{QS(x)(1 - \alpha(x, \mu) - A}{K}\int_0^1 v(x)dx \tag{4.10}$$

This is the weak formulation of the problem.

We then determine $T_j = T(x_j)$ using the approximate values of $T(x)$ at the nodes x_j,$1 \leq j \leq M$.

Using the basis function $\varphi_j(x)$,we write

$$T(x) = \sum_{j=1}^{M} T_j \varphi_j(x) \tag{4.11}$$

which implies that

$$T'(x) = \sum_{j=1}^{M} T_j \varphi_j'(x) \tag{4.12}$$

Thus by substituting the values of Eq.(4.11) and Eq.(4.12) into Eq.(4.10) we obtained the following equation.

$$\begin{aligned} \sum_{j=1}^{M} T_j \int_0^1 (1-x^2)\varphi_j'(x)v'(x)dx + \frac{B}{K}\sum_{j=1}^{M} T_j \int_0^1 \varphi_j(x)v(x)dx \\ = \frac{QS(x)(1-\alpha(x,\mu)) - A}{K} \int_0^1 v(x)dx \end{aligned} \tag{4.13}$$

Since $v(x)$ can also be expressed as a linear combination of the basis functions $\varphi_i(x)$,we then write the equation for v(x) in the following form;

$$v(x) = \sum_{i=1}^{M} \varphi_i(x) \tag{4.14}$$

By substituting the values of Eq.(4.14) into Eq.(4.13) we have the following equation;

$$\begin{aligned} \sum_{i,j=1}^{M} T_j \int_0^1 (1-x^2)\varphi_j'(x)\varphi_i'(x)dx + \frac{B}{K}\sum_{i,j=1}^{M} T_j \int_0^1 \varphi_j(x)\varphi_i(x)dx \\ = \frac{QS(x_i)(1-\alpha(x_i,\mu) - A)}{K} \sum_{i=1}^{M} \int_0^1 \varphi_i(x)dx \end{aligned} \tag{4.15}$$

This set of equation can be expressed in matrix form shown below;

$$A\vec{T} = \vec{F} \tag{4.16}$$

while the value of A is given in the equation shown $A = S + M$

Here

$$S = S_{ij} = \int_0^1 (1 - x^2)\varphi_j'(x)\varphi_i'(x)dx \tag{4.17}$$

$$M = M_{ij} = \frac{B}{K}\int_0^1 \varphi_j(x)\varphi_i(x)dx \tag{4.18}$$

$$F = \frac{QS(x_i)(1 - \alpha(x_i, \mu)) - A}{K}\int_0^1 \varphi_i(x)dx \tag{4.19}$$

Where S is refers to the stiffness matrix of the equation.

M is refers to the mass matrix of the equation.

F is refers to the load vector of the equation.

From Eq.(4.16) we then solve the temperature per latitude belt for two ,five and ten elements.

4.2 Evaluation of stiffness matrix S,mass matrix M and load vector F two elements.

From the following equations Eq.(4.17),Eq.(4.18) and Eq.(4.19) we evaluated values for these matrices by using quadratic basis function derived below.

4.2.1 Quadratic basis function for two elements.

We let $h_i = 0.25$

We partition the interval $I = (0, 1)$ into subinterval,

$0 < 0.5 < 1$

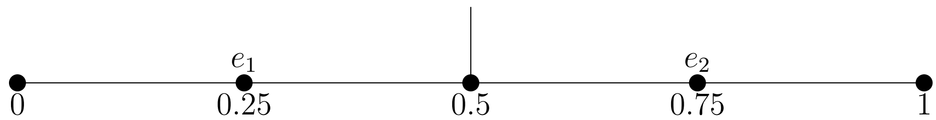


Figure 4.1: Quadratic finite elements for two elements

(1) Quadratic basis functions for element one (e_1) are:

$$\varphi_0^{(1)}(x) = \frac{(x-0.25)(x-0.5)}{(0-0.25)(0-0.5)} = \frac{x^2 - 0.75x + 0.125}{0.125}$$

$$\varphi_1^{(1)}(x) = \frac{x(x-0.5)}{(0.25-0)(0.25-0.5)} = \frac{x^2 - 0.5x}{-0.0625}$$

$$\varphi_2^{(1)}(x) = \frac{x(x-0.25)}{(0.5-0)(0.5-0.25)} = \frac{x^2 - 0.25x}{0.125}$$

(2) Quadratic basis functions for element two (e_2) are:

$$\varphi_0^{(2)}(x) = \frac{(x-0.75)(x-1)}{(0.5-0.75)(0.5-1)} = \frac{x^2-1.75x+0.75}{0.125}$$

$$\varphi_1^{(2)}(x) = \frac{(x-0.5)(x-1)}{(0.75-0.5)(0.75-1)} = \frac{x^2-1.5x+0.5}{-0.0625}$$

$$\varphi_2^{(2)}(x) = \frac{(x-0.5)(x-0.75)}{(1-0.5)(1-0.75)} = \frac{x^2-1.25x+0.375}{0.125}$$

To obtained the stiffness matrix S for the each element we solve the matrix,

$$S = \begin{bmatrix} \int_0^1(1-x^2)\varphi_0'(x)\varphi_0'(x)dx & \int_0^1(1-x^2)\varphi_0'(x)\varphi_1'(x)dx & \int_0^1(1-x^2)\varphi_0'(x)\varphi_2'(x)dx \\ \int_0^1(1-x^2)\varphi_1'(x)\varphi_0'(x)dx & \int_0^1(1-x^2)\varphi_1'(x)\varphi_1'(x)dx & \int_0^1(1-x^2)\varphi_1'(x)\varphi_2'(x)dx \\ \int_0^1(1-x^2)\varphi_2'(x)\varphi_0'(x)dx & \int_0^1(1-x^2)\varphi_2'(x)\varphi_1'(x)dx & \int_0^1(1-x^2)\varphi_2'(x)\varphi_2'(x)dx \end{bmatrix}$$

To determine the mass matrix for each element we solve the matrix shown below,

$$M = \begin{bmatrix} \int_0^1 \varphi_0(x)\varphi_0(x)dx & \int_0^1 \varphi_0(x)\varphi_1(x)dx & \int_0^1 \varphi_0(x)\varphi_2(x)dx \\ \int_0^1 \varphi_1(x)\varphi_0(x)dx & \int_0^1 \varphi_1(x)\varphi_1(x)dx & \int_0^1 \varphi_1(x)\varphi_2(x)dx \\ \int_0^1 \varphi_2(x)\varphi_0(x)dx & \int_0^1 \varphi_1(x)\varphi_1(x)dx & \int_0^1 \varphi_2(x)\varphi_2(x)dx \end{bmatrix}$$

4.2.2 Stiffness and mass matrix for element one (e_1)

Thus solutions for stiffness matrix is has follows,

$$\begin{aligned} 64\int_0^{0.5}(1-x^2)(2x-0.75)^2dx &= \tfrac{137}{30} \\ -128\int_0^{0.5}(1-x^2)(2x-0.75)(2x-0.5)dx &= -\tfrac{77}{15} \\ 64\int_0^{0.5}(1-x^2)(2x-0.75)(2x-0.25)dx &= \tfrac{17}{30} \\ 256\int_0^{0.5}(1-x^2)(2x-0.5)^2dx &= \tfrac{48}{5} \\ -128\int_0^{0.5}(1-x^2)(2x-0.5)(2x-0.25)dx &= -\tfrac{67}{15} \\ 64\int_0^{0.5}(1-x^2)(2x-0.25)^2dx &= \tfrac{39}{10} \end{aligned} \quad (4.20)$$

This matrix is given by

$$S^1_{ij} = \begin{bmatrix} 4.566666667 & -5.133333333 & 0.566666666 \\ -5.133333333 & 9.6 & -4.466666667 \\ 0.566666666 & -4.466666667 & 3.9 \end{bmatrix}$$

The solution for the mass matrix of element one is has follows;

$$\begin{aligned}
\tfrac{1.55}{0.85} * 64 \int_0^{0.5} (x^2 - 0.75x + 0.125)^2 dx &= 0.121568627 \\
\tfrac{1.55}{0.85} * 128 \int_0^{0.5} (x^2 - 0.75x + 0.125)(x^2 - 0.5x) dx &= 0.060784314 \\
\tfrac{1.55}{0.85} * 64 \int_0^{0.5} (x^2 - 0.75x + 0.125)(x^2 - 0.25x) dx &= -0.030392157 \\
\tfrac{1.55}{0.85} * 256 \int_0^{0.5} (x^2 - 0.5x)^2 dx &= 0.48627451 \\
\tfrac{1.55}{0.85} * 128 \int_0^{0.5} (x^2 - 0.5x)(x^2 - 0.25x) dx &= 0.060784314 \\
\tfrac{1.55}{0.85} * 64 \int_0^{0.5} (x^2 - 0.25x)^2 dx &= 0.121568627
\end{aligned} \tag{4.21}$$

This matrix is given below;

$$\frac{B}{K} M^1_{ij} = \begin{bmatrix} 0.121568627 & 0.060784314 & -0.030392157 \\ 0.060784314 & 0.48627451 & 0.060784314 \\ -0.030392157 & 0.060784314 & 0.121568627 \end{bmatrix}$$

To obtain $A^1_{ij} = S^1_{ij} + \frac{B}{K} M^1_{ij}$

$$A^1_{ij} = \begin{bmatrix} 4.688235294 & -5.072549019 & 0.536274509 \\ -5.072549019 & 10.08627451 & -4.405882353 \\ 0.536274509 & -4.405882353 & 4.021568627 \end{bmatrix}$$

4.2.3 Stiffness and mass matrix for element two (e_2)

Solutions obtained for the stiffness matrix is given has follows;

$$
\begin{aligned}
64 * \int_{0.5}^{1}(1-x^2)(2x-1.75)^2 dx &= 2.9 \\
-128 * \int_{0.5}^{1}(1-x^2)(2x-1.75)(2x-1.5) dx &= -3.133333333 \\
64 * \int_{0.5}^{1}(1-x^2)(2x-1.75)(2x-1.25) dx &= 0.233333333 \\
256 * \int_{0.5}^{1}(1-x^2)(2x-1.5)^2 dx &= 4.266666667 \\
-128 * \int_{0.5}^{1}(1-x^2)(2x-1.5)(2x-1.25) dx &= -1.133333333 \\
64 * \int_{0.5}^{1}(1-x^2)(2x-1.25)^2 dx &= 0.9
\end{aligned}
\tag{4.22}
$$

This matrix is given by;

$$
S_{ij}^2 = \begin{bmatrix} 2.9 & -3.133333333 & 0.233333333 \\ -3.133333333 & 4.266666666 & -1.13333333 \\ 0.233333333 & -1.13333333 & 0.9 \end{bmatrix}
$$

The solution for mass matrix is has follows;

$$
\begin{aligned}
\tfrac{1.55}{0.65} * 64 \int_{0.5}^{1}(x^2-1.75x+0.75)^2 dx &= 0.158974359 \\
\tfrac{1.55}{0.65} * 128 \int_{0.5}^{1}(x^2-1.75x+0.75)(x^2-1.5x+0.5) dx &= 0.079487179 \\
\tfrac{1.55}{0.65} * 64 \int_{0.5}^{1}(x^2-1.75x+0.75)(x^2-1.25x+0.375) dx &= -0.039743588 \\
\tfrac{1.55}{0.65} * 256 \int_{0.5}^{1}(x^2-1.5x+0.5)^2 dx &= 0.635897436 \\
\tfrac{1.55}{0.65} * 128 \int_{0.5}^{1}(x^2-1.5x+0.5)(x^2-1.25x+0.375) dx &= 0.079487179 \\
\tfrac{1.55}{0.65} * 64 \int_{0.5}^{1}(x^2-1.25x+0.375)^2 dx &= 0.158974359
\end{aligned}
\tag{4.23}
$$

This matrix is given by;

$$
\frac{B}{K} M_{ij}^2 = \begin{bmatrix} 0.158974359 & 0.079487179 & -0.039743588 \\ 0.079487179 & 0.635897436 & 0.079487179 \\ -0.039743588 & 0.079487179 & 0.158974359 \end{bmatrix}
$$

The value of $A^2_{ij} = S^2_{ij} + M^2_{ij}$

$$A^2_{ij} = \begin{bmatrix} 3.058974359 & -3.053846154 & 0.193589745 \\ -3.053846154 & 4.902564102 & -1.053846151 \\ 0.193589745 & -1.053846151 & 1.058974359 \end{bmatrix}$$

4.2.4 Matrix assemblying of the two elements

The matrix after assemblying is given below;

$$A = \begin{bmatrix} 4.688235294 & -5.072549019 & 0.536274509 & 0 & 0 \\ -5.072549019 & 10.08627451 & -4.405882353 & 0 & 0 \\ 0.536274509 & -4.405882353 & 7.080542986 & -3.053846154 & 0.193589745 \\ 0 & 0 & -3.053846154 & 4.902564102 & -1.053846151 \\ 0 & 0 & 0.193589745 & -1.053846151 & 1.058974359 \end{bmatrix}$$

4.2.5 Load vector for two elements

From Eq.(4.19) by evaluating integral of the equation on the right hand side we obtained solutions for each element as follows.
The integral solution for both elements are shown below;

$$\begin{aligned} 8 * \int_0^{0.5}(x^2 - 0.75x + 0.125)dx &= \tfrac{1}{12} \\ 16 * \int_0^{0.5}(x^2 - 0.5x)dx &= \tfrac{1}{3} \\ 8 * \int_0^{0.5}(x^2 - 0.2x)dx &= \tfrac{1}{12} \\ 8 * \int_{0.5}^{1}(x^2 - 1.75x + 0.75)dx &= \tfrac{1}{12} \\ 16 * \int_{0.5}^{1}(x^2 - 1.5x + 0.5)dx &= \tfrac{1}{3} \\ 8 * \int_{0.5}^{1}(x^2 - 1.25x + 0.375)dx &= \tfrac{1}{12} \end{aligned} \tag{4.24}$$

The value for albedo and S(x) is obtained by use of data from S.G Warren and S.T.Schneider in (1979) [36].These values are given in the table below.
The value for $Q = \frac{1}{4}Q = 342.5Wm^{-2}$

Table 4.1: **Values for K(x),Albedo,S(x) and QS(x)β(x,μ)-A per latitude belt for two elements**

x	S(x)	QS(x)	Albedo	S(x)β(x,μ)-A	K
0-0.5	1.176	402.78	0.258	87.6676	0.85
0.5-1	0.804	275.40425	0.443	-57.79983275	0.65

From Eq.(4.19) we obtained the values for $F(x)$ has shown in the table below;

Table 4.2: **Values of F(x) for two elements**

x	F(x)
0	8.594388235
0.25	34.37755294
0.5	1.184153267
0.75	-29.64093987
1	-7.410234968

From Eq.(4.16) the solution of temperature for five elements is obtained by solving the equation $A\vec{T} = \vec{F}$ by use of Gaussian elimination method.Here A is a 5 by 5 matrix, $\vec{F}$ is a 5 by 1 matrix and $\vec{T}$ are unknowns to be determine.The solution for temperature are shown in the table below and are round off to 3 significant figures.

Table 4.3: **Temperature per latitudes in degrees celsius**

x	zone(°)	T(°c)
0	0	17.0
0.25	14	14.8
0.5	30	6.22
0.75	48.59	-5.01
1	90	-13.1

4.3 Evaluation of Stiffness matrix S,Mass matrix M and Load vector F for five elements.

From the following equation Eq.(4.17),Eq.(4.18) and Eq.(4.19) we evaluated values for these matrices by using quadratic basis function derived below.

4.3.1 Quadratic basis function for five elements.

We let $h_i = 0.1$

We partition the interval $I = (0, 1)$ into subinterval given below,

$0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$

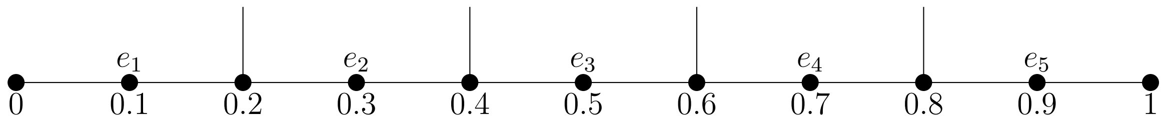


Figure 4.2: Quadratic finite elements for five elements

(i) Quadratic basis functions for element one (e_1) are;

$$\varphi_0^{(1)}(x) = \frac{(x-0.1)(x-0.2)}{(0-0.1)(0-0.2)} = \frac{x^2 - 0.3x + 0.02}{0.02}$$

$$\varphi_1^{(1)}(x) = \frac{(x)(x-0.2)}{(0.1-0)(0.1-0.2)} = \frac{x^2 - 0.2x}{-0.01}$$

$$\varphi_2^{(1)}(x) = \frac{(x)(x-0.1)}{(0.2-0)(0.2-0.1)} = \frac{x^2 - 0.1x}{0.02}$$

(ii) Quadratic basis functions for element two (e_2) are;

$$\varphi_0^{(2)}(x) = \frac{(x-0.3)(x-0.4)}{(0.2-0.3)(0.2-0.4)} = \frac{x^2-0.7x+0.12}{0.02}$$

$$\varphi_1^{(2)}(x) = \frac{(x-0.2)(x-0.4)}{(0.3-0.2)(0.3-0.4)} = \frac{x^2-0.6x+0.08}{-0.01}$$

$$\varphi_2^{(2)}(x) = \frac{(x-0.2)(x-0.3)}{(0.4-0.2)(0.4-0.3)} = \frac{x^2-0.5x+0.06}{0.02}$$

(iii) Quadratic basis functions for element three (e_3) are;

$$\varphi_0^{(3)}(x) = \frac{(x-0.5)(x-0.6)}{(0.4-0.5)(0.4-0.6)} = \frac{x^2-1.1x+0.3}{0.02}$$

$$\varphi_1^{(3)}(x) = \frac{(x-0.4)(x-0.6)}{(0.5-0.4)(0.5-0.6)} = \frac{x^2-x+0.24}{-0.01}$$

$$\varphi_2^{(3)}(x) = \frac{(x-0.4)(x-0.5)}{(0.6-0.4)(0.6-0.5)} = \frac{x^2-0.9x+0.2}{0.02}$$

(iv) Quadratic basis functions for element four (e_4) are;

$$\varphi_0^{(4)}(x) = \frac{(x-0.7)(x-0.8)}{(0.6-0.7)(0.6-0.8)} = \frac{x^2-1.5x+0.56}{0.02}$$

$$\varphi_1^{(4)}(x) = \frac{(x-0.6)(x-0.8)}{(0.7-0.6)(0.7-0.8)} = \frac{x^2-1.4x+0.48}{-0.01}$$

$$\varphi_2^{(4)}(x) = \frac{(x-0.6)(x-0.7)}{(0.8-0.6)(0.8-0.7)} = \frac{x^2-1.3x+0.42}{0.02}$$

(v) Quadratic basis functions for element five (e_5) are;

$$\varphi_0^{(5)}(x) = \frac{(x-0.9)(x-1)}{(0.8-0.9)(0.8-1)} = \frac{x^2-1.9x+0.9}{0.02}$$

$$\varphi_1^{(5)}(x) = \frac{(x-0.8)(x-1)}{(0.9-0.8)(0.9-1)} = \frac{x^2-1.8x+0.8}{-0.01}$$

$$\varphi_2^{(5)}(x) = \frac{(x-0.8)(x-0.9)}{(1-0.8)(1-0.9)} = \frac{x^2-1.7x+0.72}{0.02}$$

4.3.2 Stiffness and Mass matrix for element one (e_1)

From Eqs.(4.17) and (4.18) we obtained values for the stiffness and the mass matrix for element one. Solution obtained for stiffness matrix is has follows;

$$\begin{aligned}
2500 * \int_0^{0.2}(1-x^2)(2x-0.3)^2 dx &= \tfrac{872}{75} \\
-5000 * \int_0^{0.2}(2x-0.3)(2x-0.2) dx &= \tfrac{-994}{75} \\
2500 * \int_0^{0.2}(1-x^2)(2x-0.3)(2x-0.1) dx &= \tfrac{122}{75} \\
10000 * \int_0^{0.2}(1-x^2)(2x-0.2)^2 dx &= \tfrac{656}{25} \\
-5000 * \int_0^{0.2}(1-x^2)(2x-0.2)(2x-0.1) dx &= \tfrac{-974}{75} \\
2500 * \int_0^{0.2}(1-x^2)(2x-0.1)^2 dx &= \tfrac{284}{25}
\end{aligned} \tag{4.25}$$

This matrix is given by;

$$S_{ij}^{(1)} = \begin{bmatrix} 11.62666667 & -13.25333333 & 1.626666667 \\ -13.25333333 & 26.24 & -12.98666667 \\ 1.626666667 & -12.98666667 & 11.36 \end{bmatrix}$$

The solution of mass matrix is has follows;

$$\begin{aligned}
\tfrac{1.55}{0.90} * 2500 * \int_0^{0.2}(x^2-0.3x+0.02)^2 dx &= 0.94148148 \\
\tfrac{1.55}{0.90} * -5000 * \int_0^{0.2}(x^2-0.3x+0.02)(x^2-0.2x) dx &= 0.02296296 \\
\tfrac{1.55}{0.90} * 2500 * \int_0^{0.2}(x^2-0.3x+0.02)(x^2-0.1x) dx &= -0.011481481 \\
\tfrac{1.55}{0.90} * 10000 * \int_0^{0.2}(x^2-0.2x)^2 dx &= 0.1837037 \\
\tfrac{1.55}{0.90} * -5000 * \int_0^{0.2}(x^2-0.2x)(x^2-0.1x) dx &= 0.02296296 \\
\tfrac{1.55}{0.90} * 2500 * \int_0^{0.2}(x^2-0.1x)^2 dx &= 0.04592593
\end{aligned} \tag{4.26}$$

This matrix is given by;

$$\frac{B}{K} M_{ij}^{(1)} = \begin{bmatrix} 0.94148148 & 0.02296296 & -0.011481481 \\ 0.02296296 & 0.1837037 & 0.02296296 \\ -0.011481481 & 0.02296296 & 0.04592593 \end{bmatrix}$$

Then the matrix $A_{ij}^{(1)} = S_{ij}^{(1)} + \frac{B}{K} M_{ij}^{(1)}$

$$A_{ij}^{(1)} = \begin{bmatrix} 12.56814815 & -13.23037037 & 1.615185186 \\ -13.23037037 & 26.4237037 & -12.96370371 \\ 1.615185186 & -12.96370371 & 11.40592593 \end{bmatrix}$$

4.3.3 Stiffness and mass matrix for element two (e_2)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element two. The solution of stiffness matrix is has follows;

$$\begin{aligned} 2500 * \int_{0.2}^{0.4} (1-x^2)(2x-0.7)^2 dx &= \tfrac{274}{25} \\ -5000 \int_{0.2}^{0.4} (1-x^2)(2x-0.7)(2x-0.6) dx &= \tfrac{-934}{75} \\ 2500 * \int_{0.2}^{0.4} (1-x^2)(2x-0.7)(2x-0.5) dx &= \tfrac{112}{75} \\ 10000 * \int_{0.2}^{0.4} (1-x^2)(2x-0.6)^2 dx &= \tfrac{1808}{75} \\ -5000 * \int_{0.2}^{0.4} (1-x^2)(2x-0.6)(2x-0.5) dx &= \tfrac{-874}{75} \\ 2500 * \int_{0.2}^{0.4} (1-x^2)(2x-0.5)^2 dx &= \tfrac{254}{25} \end{aligned} \tag{4.27}$$

This matrix is given by;

$$S_{ij}^{2} = \begin{bmatrix} 10.96 & -12.45333333 & 1.493333333 \\ -12.45333333 & 24.10666667 & -11.65333333 \\ 1.493333333 & -11.65333333 & 10.16 \end{bmatrix}$$

The solution for the mass matrix is given by;

$$\begin{aligned}
\tfrac{1.55}{0.85} * 2500 * \textstyle\int_{0.2}^{0.4} (x^2 - 0.7x + 0.12)^2 dx &= 0.0462745 \\
\tfrac{1.55}{0.85} * -5000 * \textstyle\int_{0.2}^{0.4} (x^2 - 0.7x + 0.12)(x^2 - 0.6x + 0.08) dx &= 0.02431373 \\
\tfrac{1.55}{0.85} * 2500 * \textstyle\int_{0.2}^{0.4} (x^2 - 0.7x + 0.12)(x^2 - 0.5x + 0.06) dx &= -0.012156863 \\
\tfrac{1.55}{0.85} * 10000 * \textstyle\int_{0.2}^{0.4} (x^2 - 0.6x + 0.08)^2 dx &= 0.1945098 \\
\tfrac{1.55}{0.85} * -5000 * \textstyle\int_{0.2}^{0.4} (x^2 - 0.6x + 0.08)(x^2 - 0.5x + 0.06) dx &= 0.04548209 \\
\tfrac{1.55}{0.85} * 2500 * \textstyle\int_{0.2}^{0.4} (x^2 - 0.5x + 0.06)^2 dx &= 0.04862745
\end{aligned} \tag{4.28}$$

This matrix is given by;

$$\frac{B}{K_2} M_{ij}^{(2)} = \begin{bmatrix} 0.04862745 & 0.02431373 & -0.012156863 \\ 0.02431373 & 0.1945098 & 0.04548209 \\ -0.012156863 & 0.04548209 & 0.04862745 \end{bmatrix}$$

Then the matrix $A_{ij}^{(2)} = S_{ij}^{(2)} + \frac{B}{K_2} M_{ij}^{(2)}$

$$A_{ij}^{(2)} = \begin{bmatrix} 11.00862745 & -12.4290196 & 1.48117647 \\ -12.4290196 & 24.30117647 & -11.60785124 \\ 1.48117647 & -11.60785124 & 10.20862745 \end{bmatrix}$$

4.3.4 Stiffness and mass matrix for element three (e_3)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element three. Solution of the stiffness matrix is has follows;

$$\begin{aligned}
2500 * \int_{0.4}^{0.6}(1-x^2)(2x-1.1)^2 dx &= \tfrac{234}{25} \\
-5000 * \int_{0.4}^{0.6}(1-x^2)(2x-1.1)(2x-1) dx &= \tfrac{-794}{75} \\
2500 * \int_{0.4}^{0.6}(1-x^2)(2x-0.9)(2x-1.1) dx &= \tfrac{92}{75} \\
10000 * \int_{0.4}^{0.6}(1-x^2)(2x-1)^2 dx &= \tfrac{496}{25} \\
-5000 * \int_{0.4}^{0.6}(1-x^2)(2x-1)(2x-0.9) dx &= \tfrac{-694}{75} \\
2500 * \int_{0.4}^{0.6}(1-x^2)(2x-0.9)^2 dx &= \tfrac{602}{75}
\end{aligned} \tag{4.29}$$

This matrix is given below;

$$S_{ij}^{(3)} = \begin{bmatrix} 9.36 & -10.58666667 & 1.226666667 \\ 10.58666667 & 19.84 & -9.253333333 \\ 1.226666667 & -9.253333333 & 8.026666667 \end{bmatrix}$$

The solution for the mass matrix is has follows;

$$\begin{aligned}
\tfrac{1.55}{0.80} * 2500 * \int_{0.4}^{0.6}(x^2-1.1x+0.3)^2 dx &= 0.051666667 \\
\tfrac{1.55}{0.80} * -5000 * \int_{0.4}^{0.6}(x^2-1.1x+0.3)(x^2-x+0.24) dx &= 0.025833333 \\
\tfrac{1.55}{0.80} * 2500 * \int_{0.4}^{0.6}(x^2-1.1x+0.3)(x^2-0.9x+0.2) dx &= -0.012916667 \\
\tfrac{1.55}{0.80} * 10000 * \int_{0.4}^{0.6}(x^2-x+0.24)^2 dx &= 0.20666667 \\
\tfrac{1.55}{0.80} * -5000 * \int_{0.4}^{0.6}(x^2-x+0.24)(x^2-0.9x+0.2) dx &= 0.025833333 \\
\tfrac{1.55}{0.80} * 2500 * \int_{0.4}^{0.6}(x^2-0.9x+0.2)^2 dx &= 0.051666667
\end{aligned} \tag{4.30}$$

This matrix is given by;

$$\frac{B}{K_3} M_{ij}^{(3)} = \begin{bmatrix} 0.051666667 & 0.025833333 & -0.012916667 \\ 0.025833333 & 0.20666667 & 0.025833333 \\ -0.012916667 & 0.025833333 & 0.051666667 \end{bmatrix}$$

Then the matrix $A_{ij}^{(3)} = S_{ij}^{(3)} + \frac{B}{K_{(3)}} M_{ij}^{(3)}$

$$A_{ij}^{(3)} = \begin{bmatrix} 9.411666667 & -10.56083334 & 1.21375 \\ -10.56083334 & 20.04666667 & -9.2275 \\ 1.21375 & -9.2275 & 8.078333334 \end{bmatrix}$$

4.3.5 Stiffness and mass matrix for element four (e_4)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element four. Solution of stiffness matrix is has follows;

$$\begin{aligned} 2500 * \int_{0.6}^{0.8} (1-x^2)(2x-1.5)^2 dx &= 6.826666667 \\ -5000 * \int_{0.6}^{0.8} (1-x^2)(2x-1.5)(2x-1.4) dx &= -7.653333333 \\ 2500 * \int_{0.6}^{0.8} (1-x^2)(2x-1.5)(2x-1.3) dx &= 0.826666666 \\ 10000 * \int_{0.6}^{0.8} (1-x^2)(2x-1.4)^2 dx &= 13.44 \\ -5000 * \int_{0.6}^{0.8} (1-x^2)(2x-1.4)(2x-1.3) dx &= -5.786666667 \\ 2500 * \int_{0.6}^{0.8} (1-x^2)(2x-1.3)^2 dx &= 4.96 \end{aligned} \tag{4.31}$$

This matrix is given by;

$$S_{ij}^{(4)} = \begin{bmatrix} 6.826666667 & -7.653333333 & 0.826666666 \\ 0.826666666 & 13.44 & -5.786666667 \\ 0.826666666 & -5.786666667 & 4.96 \end{bmatrix}$$

The solution of mass matrix is has follows;

$$\begin{aligned}
\tfrac{1.55}{0.75} * 2500 * \textstyle\int_{0.6}^{0.8} (x^2 - 1.5x + 0.56)^2 dx &= 0.055111111 \\
\tfrac{1.55}{0.75} * -5000 * \textstyle\int_{0.6}^{0.8} (x^2 - 1.5x + 0.56)(x^2 - 1.4x + 0.48) dx &= 0.027555555 \\
\tfrac{1.55}{0.75} * 2500 * \textstyle\int_{0.6}^{0.8} (x^2 - 1.5x + 0.56)(x^2 - 1.3x + 0.42) dx &= 0.013777778 \\
\tfrac{1.55}{0.75} * 10000 * \textstyle\int_{0.6}^{0.8} (x^2 - 1.4x + 0.48)^2 dx &= 0.22044444 \\
\tfrac{1.55}{0.75} * -5000 * \textstyle\int_{0.6}^{0.8} (x^2 - 1.4x + 0.48)(x^2 - 1.3x + 0.42) dx &= 0.027555555 \\
\tfrac{1.55}{0.75} * 2500 * \textstyle\int_{0.6}^{0.8} (x^2 - 1.3x + 0.42)^2 dx &= 0.055111111
\end{aligned} \tag{4.32}$$

This matrix is given by;

$$\frac{B}{K_4} M_{ij}^{(4)} = \begin{bmatrix} 0.055111111 & 0.027555555 & -0.013777778 \\ 0.027555555 & 0.22044444 & 0.027555555 \\ -0.013777778 & 0.027555555 & 0.055111111 \end{bmatrix}$$

Then the matrix $A_{ij}^{(4)} = S_{ij}^{(4)} + \frac{B}{K_{(4)}} M_{ij}^{(4)}$

$$A_{ij}^{(4)} = \begin{bmatrix} 6.881777778 & -7.625777778 & 0.812888888 \\ -7.625777778 & 13.66044444 & -5.759111112 \\ 0.812888888 & -5.759111112 & 5.015111111 \end{bmatrix}$$

4.3.6 Stiffness and Mass matrix for element five (e_5)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element five. Solution of the stiffness matrix is has follows;

$$\begin{aligned}
2500 * \int_{0.8}^{1}(1-x^2)(2x-1.9)^2dx &= 3.36\\
-5000 * \int_{0.8}^{1}(1-x^2)(2x-1.9)(2x-1.8)dx &= -3.653333333\\
2500 * \int_{0.8}^{1}(1-x^2)(2x-1.9)(2x-1.7)dx &= 0.293333333\\
10000 * \int_{0.8}^{1}(1-x^2)(2x-1.8)^2dx &= 4.906666667\\
-5000 * \int_{0.8}^{1}(1-x^2)(2x-1.8)(2x-1.7)dx &= -1.253333333\\
2500 * \int_{0.8}^{1}(1-x^2)(2x-1.7)^2dx &= 0.96
\end{aligned} \tag{4.33}$$

This matrix is given by;

$$S_{ij}^{5} = \begin{bmatrix} 3.36 & -3.653333333 & 0.293333333 \\ -3.653333333 & 4.906666667 & -1.253333333 \\ 0.293333333 & -1.253333333 & 0.96 \end{bmatrix}$$

The solution for mass matrix is given has follows;

$$\begin{aligned}
\tfrac{1.55}{0.60} * 2500 * \int_{0.8}^{1}(x^2-1.9x+0.9)^2dx &= 0.068888889\\
\tfrac{1.55}{0.60} * -5000 * \int_{0.8}^{1}(x^2-1.9x+0.9)(x^2-1.8x+0.8)dx &= 0.034444444\\
\tfrac{1.55}{0.60} * 2500 * \int_{0.8}^{1}(x^2-1.9x+0.9)(x^2-1.7x+0.72)dx &= -0.017222223\\
\tfrac{1.55}{0.60} * 10000 * \int_{0.8}^{1}(x^2-1.8x+0.8)^2dx &= 0.275555556\\
\tfrac{1.55}{0.60} * -5000 * \int_{0.8}^{1}(x^2-1.8x+0.8)(x^2-1.7x+0.72)dx &= 0.034444444\\
\tfrac{1.55}{0.60} * 2500 * \int_{0.8}^{1}(x^2-1.7x+0.72)^2dx &= 0.068888889
\end{aligned} \tag{4.34}$$

This matrix is given by;

$$\frac{B}{K_5}M_{ij}^{(5)} = \begin{bmatrix} 0.069421117 & 0.034710559 & -0.01735528 \\ 0.034710559 & 0.27768447 & 0.034710559 \\ -0.01735528 & 0.034710559 & 0.069421117 \end{bmatrix}$$

Then the matrix $A_{ij}^{(5)} = S_{ij}^{(5)} + \frac{B}{K_5} M_{ij}^{(5)}$

$$A_{ij}^{(5)} = \begin{bmatrix} 3.429421117 & -3.618888889 & 0.27611111 \\ -3.618888889 & 5.182222223 & -1.218888889 \\ 0.27611111 & -1.218888889 & 1.028888889 \end{bmatrix}$$

4.3.7 Matrix assemblying for five elements

To 9 decimal place, we assembly all the stiffness and mass matrix has shown below;

$$A = \begin{bmatrix} A_{ij}^{(1)} & 0 & 0 & \dots\dots & 0 \\ 0 & A_{ij}^{(2)} & \dots\dots\dots & 0 & 0 \\ 0 & \dots & A_{ij}^{(3)} & \dots & 0 \\ 0 & 0 & \dots & A_{ij}^{(4)} & 0 \\ 0 & 0 & \dots & 0 & A_{ij}^{(5)} \end{bmatrix}$$

This matrix is of order 11 and is symmetric

4.3.8 Load vector for five elements

From Eq.(4.19) by evaluating the integral of the equation on right hand side we obtained solutions for each element as follows.
The integral solution for element one (e_1) are;

$$\begin{aligned} 50 * \int_0^{0.2} (x^2 - 0.3x + 0.02)dx &= \tfrac{1}{30} \\ -100 * \int_0^{0.2} (x^2 - 0.2x) &= \tfrac{2}{15} \\ 50 * \int_0^{0.2} (x^2 - 0.1x) &= \tfrac{1}{30} \end{aligned} \tag{4.35}$$

The integral solution for element two (e_2) are;

$$\begin{aligned} 50 * \int_{0.2}^{0.4} (x^2 - 0.7x + 0.12)dx &= \tfrac{1}{30} \\ -100 * \int_{0.2}^{0.4} (x^2 - 0.6x + 0.08)dx &= \tfrac{2}{15} \\ 50 * \int_{0.2}^{0.4} (x^2 - 0.5x + 0.06)dx &= \tfrac{1}{30} \end{aligned} \tag{4.36}$$

The integral solution for element three (e_3) are;

$$\begin{aligned} 50 * \int_{0.4}^{0.6}(x^2 - 1.1x + 0.3)dx &= \tfrac{1}{30} \\ -100 * \int_{0.4}^{0.6}(x^2 - x + 0.24)dx &= \tfrac{2}{15} \\ 50 * \int_{0.4}^{0.6}(x^2 - 0.9x + 0.2)dx &= \tfrac{1}{30} \end{aligned} \tag{4.37}$$

The integral solution for element four (e_4) are;

$$\begin{aligned} 50 * \int_{0.6}^{0.8}(x^2 - 1.5x + 0.56)dx &= \tfrac{1}{30} \\ -100 * \int_{0.6}^{0.8}(x^2 - 1.4x + 0.48)dx &= \tfrac{2}{15} \\ 50 * \int_{0.6}^{0.8}(x^2 - 1.3x + 0.42)dx &= \tfrac{1}{30} \end{aligned} \tag{4.38}$$

The integral solution for element five (e_5) are;

$$\begin{aligned} 50 * \int_{0.8}^{1}(x^2 - 1.9x + 0.9)dx &= \tfrac{1}{30} \\ -100 * \int_{0.8}^{1}(x^2 - 1.8x + 0.8)dx &= \tfrac{2}{15} \\ 50 * \int_{0.8}^{1}(x^2 - 1.7x + 0.72)dx &= \tfrac{1}{30} \end{aligned} \tag{4.39}$$

Similarly we can find the values for albedo and S(x) for five elements by using data from S.G Warren and S.T.Schneider in (1979) [36].These values are shown in the table below.

Table 4.4: **Values for K(x),Albedo,S(x) and QS(x)β(x,μ)-A for five elements**

x	S(x)	K(x)	QS(x)	albedo	QS(x)β(x,μ)-A
0-0.2	1.204	0.90	412.37	0.251	87.898617
0.2-0.4	1.1545	0.85	395.41625	0.26	69.19682125
0.4-0.6	1.0705	0.80	366.64625	0.2905	39.1484115
0.6-0.8	0.8943	0.75	306.29775	0.358	-14.5568445
0.8-1	0.6063	0.60	207.65775	0.498	-106.9558095

From Eq.(4.19) values for $F(x)$ per latitude is given in the table below;

Table 4.5: **Values of F(x) for five elements**

x	F(x)
0	3.617227037
0.1	14.4690815
0.2	6.809698606
0.3	12.76988627
0.4	5.231451353
0.5	8.155919063
0.6	1.3920089
0.7	-2.587883467
0.8	-6.588960283
0.9	-23.76795767
1	-5.941989417

From Eq.(4.16) the solution of temperature for five elements is obtained by solving the equation $A\vec{T} = \vec{F}$ by use of Gaussian elimination method.Here A is a 11 by 11 matrix, $\vec{F}$ is a 11 by 1 matrix and $\vec{T}$ are unknowns to be determine.The solution for temperature are shown in the table below and are round off to 3 significant figures.

Table 4.6: **Temperature in degree celsius for five elements**

x	zones(°)	T(°c)
0	0	15.0
0.1	6	15.9
0.2	12	16.1
0.3	17	15.4
0.4	24	14.0
0.5	30	11.7
0.6	37	8.66
0.7	44	4.82
0.8	53	0.12
0.9	64	-8.14
1	90	-15.4

4.4 Evaluation of Stiffness matrix S,Mass matrix M and Load vector F for ten elements.

From the following equation Eq.(4.17),Eq.(4.18) and Eq.(4.19) we evaluated values of this matrices by using quadratic basis function derived below.

4.4.1 Quadratic basis function for five elements.

We let $h_i = 0.05$

We partition the interval $I = (0, 1)$ into subintervals given below;

$0 < 0.1 < 0.2 < 0.3 < 0.4 < 0.5 < 0.6 < 0.7 < 0.8 < 0.9 < 1$

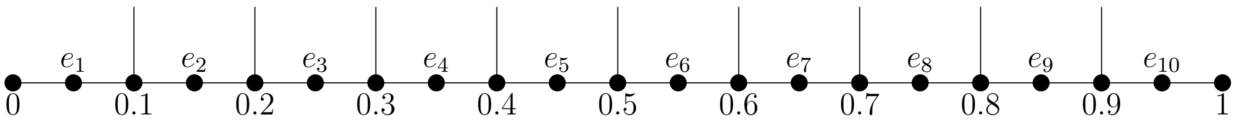


Figure 4.3: Quadratic finite elements for ten elements

(i) Quadratic basis function for element one (e_1)

$$\varphi_0^{(1)}(x) = \frac{(x-0.05)(x-0.1)}{(0-0.05)(0-0.1)} = \frac{x^2 - 0.15x + 0.005}{0,005}$$

$$\varphi_1^{(1)}(x) = \frac{(x)(x-0.1)}{(0.05-0)(0.05-0.1)} = \frac{x^2 - 0.1x}{-0.0025}$$

$$\varphi_2^{(2)}(x) = \frac{(x)(x-0.05)}{(0.1-0)(0.1-0.05)} = \frac{x^2 - 0.05x}{0.005}$$

(ii) Quadratic basis function for element two (e_2)

$$\varphi_0^{(2)}(x) = \frac{(x-0.15)(x-0.2)}{(0.1-0.15)(0.1-0.2)} = \frac{x^2-0.25x+0.03}{0.005}$$

$$\varphi_1^{(2)}(x) = \frac{(x-0.1)(x-0.2)}{(0.15-0.1)(0.15-0.2)} = \frac{x^2-0.3x+0.02}{-0.0025}$$

$$\varphi_2^{(2)}(x) = \frac{(x-0.1)(x-0.15)}{(0.2-0.1)(0.2-0.15)} = \frac{x^2-0.25x+0.015}{0.005}$$

(iii) Quadratic basis function for element three (e_3)

$$\varphi_0^{(3)}(x) = \frac{(x-0.25)(x-0.3)}{(0.2-0.25)(0.2-0.3)} = \frac{x^2-0.55x+0.075}{0.005}$$

$$\varphi_1^{(3)}(x) = \frac{(x-0.2)(x-0.3)}{(0.25-0.2)(0.25-0.3)} = \frac{x^2-0.5x+0.06}{-0.0025}$$

$$\varphi_2^{(3)}(x) = \frac{(x-0.2)(x-0.25)}{(0.3-0.2)(0.3-0.25)} = \frac{x^2-0.45x+0.05}{0.005}$$

(iv) Quadratic basis function for element four (e_4)

$$\varphi_0^{(4)}(x) = \frac{(x-0.35)(x-0.4)}{(0.3-0.35)(0.3-0.4)} = \frac{x^2-0.75x+0.14}{0.005}$$

$$\varphi_1^{(4)}(x) = \frac{(x-0.3)(x-0.4)}{(0.35-0.3)(0.35-0.4)} = \frac{x^2-0.7x+0.12}{-0.0025}$$

$$\varphi_2^{(4)}(x) = \frac{(x-0.3)(x-0.35)}{(0.4-0.3)(0.4-0.35)} = \frac{x^2-0.65x+0.105}{0.005}$$

(v) Quadratic basis function for element five (e_5)

$$\varphi_0^{(5)}(x) = \frac{(x-0.45)(x-0.5)}{(0.4-0.45)(0.4-0.5)} = \frac{x^2-0.95x+0.225}{0.005}$$

$$\varphi_1^{(5)}(x) = \frac{(x-0.4)(x-0.5)}{(0.45-0.4)(0.45-0.5)} = \frac{x^2-0.9x+0.2}{-0.0025}$$

$$\varphi_2^{(5)}(x) = \frac{(x-0.4)(x-0.45)}{(0.5-0.4)(0.5-0.45)} = \frac{x^2-0.85x+0.18}{0.005}$$

(vi) Quadratic basis function for element six (e_6)

$$\varphi_0^{(6)}(x) = \frac{(x-0.55)(x-0.6)}{(0.5-0.55)(0.5-0.6)} = \frac{x^2-1.15x+0.33}{0.005}$$

$$\varphi_1^{(6)}(x) = \frac{(x-0.5)(x-0.6)}{(0.55-0.5)(0.55-0.6)} = \frac{x^2-1.1x+0.3}{-0.0025}$$

$$\varphi_2^{(6)}(x) = \frac{(x-0.5)(x-0.55)}{(0.6-0.5)(0.6-0.55)} = \frac{x^2-1.05x+0.275}{0.005}$$

(vii) Quadratic basis function for element seven (e_7)

$$\varphi_0^{(7)}(x) = \frac{(x-0.65)(x-0.7)}{(0.6-0.65)(0.6-0.7)} = \frac{x^2-1.35x+0.455}{0.005}$$

$$\varphi_1^{(7)}(x) = \frac{(x-0.6)(x-0.7)}{(0.65-0.6)(0.65-0.7)} = \frac{x^2-1.3x+0.42}{-0.0025}$$

$$\varphi_2^{(7)}(x) = \frac{(x-0.6)(x-0.65)}{(0.7-0.6)(0.7-0.65)} = \frac{x^2-1.25x+0.39}{0.005}$$

(viii) Quadratic basis function for element eight (e_8)

$$\varphi_0^{(8)}(x) = \frac{(x-0.75)(x-0.8)}{(0.7-0.75)(0.7-0.8)} = \frac{x^2-1.55x+0.6}{0.005}$$

$$\varphi_1^{(8)}(x) = \frac{(x-0.7)(x-0.8)}{(0.75-0.7)(0.75-0.8)} = \frac{x^2-1.5x+0.56}{-0.0025}$$

$$\varphi_2^{(8)}(x) = \frac{(x-0.7)(x-0.75)}{(0.8-0.7)(0.8-0.75)} = \frac{x^2-1.45x+0.525}{0.005}$$

(ix) Quadratic basis function for element nine (e_9)

$$\varphi_0^{(9)}(x) = \frac{(x-0.85)(x-0.9)}{(0.8-0.85)(0.8-0.9)} = \frac{x^2-1.75x+0.765}{0.005}$$

$$\varphi_1^{(9)}(x) = \frac{(x-0.8)(x-0.9)}{(0.85-0.8)(0.85-0.9)} = \frac{x^2-1.7x+0.72}{-0.0025}$$

$$\varphi_2^{(9)}(x) = \frac{(x-0.8)(x-0.85)}{(0.9-0.8)(0.9-0.85)} = \frac{x^2-1.65x+0.68}{0.005}$$

(x) Quadratic basis function for element ten (e_{10})

$$\varphi_0^{(10)}(x) = \frac{(x-0.95)(x-1)}{(0.9-0.95)(0.9-1)} = \frac{x^2-1.95x+0.95}{0.005}$$

$$\varphi_1^{(10)}(x) = \frac{(x-0.9)(x-1)}{(0.95-0.9)(0.95-1)} = \frac{x^2-1.9x+0.9}{-0.0025}$$

$$\varphi_2^{(10)}(x) = \frac{(x-0.9)(x-0.95)}{(1-0.9)(1-0.95)} = \frac{x^2-1.85x+0.855}{0.005}$$

4.4.2 Stiffness and Mass matrix for element one (e_1)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element one. Solution of stiffness matrix is given by;

$$\begin{aligned}
40000 * \int_0^{0.1}(1-x^2)(2x-0.15)^2dx &= 23.31333333 \\
-80000 * \int_0^{0.1}(1-x^2)(2x-0.15)(2x-0.1)dx &= -26.49333333 \\
40000 * \int_0^{0.1}(1-x^2)(2x-0.15)(2x-0.05)dx &= 3.313333333 \\
160000 * \int_0^{0.1}(1-x^2)(2x-0.1)^2dx &= 53.12 \\
-80000 * \int_0^{0.1}(1-x^2)(2x-0.1)(2x-0.05)dx &= -26.49333333 \\
40000 * \int_0^{0.1}(1-x^2)(2x-0.05)^2dx &= 23.28
\end{aligned} \tag{4.40}$$

This matrix is given by;

$$S_{ij}^{(1)} = \begin{bmatrix} 23.31333333 & -26.49333333 & 3.313333333 \\ -26.49333333 & 53.12 & -26.49333333 \\ 3.313333333 & -26.49333333 & 23.28 \end{bmatrix}$$

The solution for mass matrix is has follows;

$$
\begin{aligned}
\tfrac{1.55}{0.95} * 40000 * \textstyle\int_0^{0.1} (x^2 - 0.15x + 0.005)^2 dx &= 1.97964912 \\
\tfrac{1.55}{0.95} * -80000 * \textstyle\int_0^{0.1} (x^2 - 0.15x + 0.005)(x^2 - 0.1x) dx &= 0.01026558 \\
\tfrac{1.55}{0.95} * 40000 * \textstyle\int_0^{0.1} (x^2 - 0.15x + 0.005)(x^2 - 0.05x) dx &= -0.005438596 \\
\tfrac{1.55}{0.95} * 160000 * \textstyle\int_0^{0.1} (x^2 - 0.1x)^2 dx &= 0.08701754 \\
\tfrac{1.55}{0.95} * -80000 * \textstyle\int_0^{0.1} (x^2 - 0.1x)(x^2 - 0.05x) dx &= 0.01087719 \\
\tfrac{1.55}{0.95} * 40000 * \textstyle\int_0^{0.1} (x^2 - 0.05x)^2 dx &= 0.02175439
\end{aligned} \tag{4.41}
$$

This matrix is given has follows;

$$
\frac{B}{K_1} M_{ij}^{(1)} = \begin{bmatrix} 1.97964912 & 0.01026558 & -0.005438596 \\ 0.01026558 & 0.08701754 & 0.01087719 \\ -0.005438596 & 0.01087719 & 0.02175439 \end{bmatrix}
$$

Then the matrix $A_{ij}^{(1)} = S_{ij}^{(1)} + \frac{B}{K_1} M_{ij}^{(1)}$ is given by;

$$
A_{ij}^{1} = \begin{bmatrix} 25.29298245 & -26.48245614 & 3.307894737 \\ -26.48245614 & 53.20701754 & -26.48245614 \\ 3.307894737 & -26.48245614 & 23.30175439 \end{bmatrix}
$$

4.4.3 Stiffness and Mass matrix for element two (e_2)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element two. Solution of stiffness matrix is given by;

$$\begin{aligned}
40000*\int_{0.1}^{0.2}(1-x^2)(2x-0.35)^2dx &= 22.98\\
-80000*\int_{0.1}^{0.2}(1-x^2)(2x-0.35)(2x-0.3)dx &= -26.22666667\\
40000*\int_{0.1}^{0.2}(1-x^2)(2x-0.35)(2x-0.25)dx &= 3.246666667\\
160000*\int_{0.1}^{0.2}(1-x^2)(2x-0.3)^2dx &= 52.05333333\\
-80000*\int_{0.1}^{0.2}(1-x^2)(2x-0.3)(2x-0.25)dx &= -25.82666667\\
40000*\int_{0.1}^{0.2}(1-x^2)(2x-0.25)^2dx &= 22.58
\end{aligned} \tag{4.42}$$

This matrix is shown below;

$$S_{ij}^{(2)} = \begin{bmatrix} 22.98 & -26.22666667 & 3.246666667 \\ -26.22666667 & 52.05333333 & -25.82666667 \\ 3.246666667 & -25.82666667 & 22.58 \end{bmatrix}$$

The solution for mass matrix is given by solving the following;

$$\begin{aligned}
\tfrac{1.55}{0.92}*40000*\int_{0.1}^{0.2}(x^2-0.35x+0.03)^2dx &= 0.03481884\\
\tfrac{1.55}{0.92}*-80000*\int_{0.1}^{0.2}(x^2-0.35x+0.03)(x^2-0.3x+0.02)dx &= 0.01123188\\
\tfrac{1.55}{0.92}*40000*\int_{0.1}^{0.2}(x^2-0.35x+0.03)(x^2-0.25x+0.015)dx &= -0.005615942\\
\tfrac{1.55}{0.92}*160000*\int_{0.1}^{0.2}(x^2-0.3x+0.02)^2dx &= 0.08985507\\
\tfrac{1.55}{0.92}*-80000*\int_{0.1}^{0.2}(x^2-0.3x+0.02)(x^2-0.25x+0.015)dx &= 0.01123188\\
\tfrac{1.55}{0.92}*40000*\int_{0.1}^{0.2}(x^2-0.25x+0.015)^2dx &= 0.02246377
\end{aligned} \tag{4.43}$$

This matrix is given has follows;

$$\frac{B}{K_2}M_{ij}^{(2)} = \begin{bmatrix} 0.3481884 & 0.01123188 & -0.005615942 \\ 0.01123188 & 0.08985507 & 0.01123188 \\ -0.005615942 & 0.01123188 & 0.02246377 \end{bmatrix}$$

Then the matrix$A_{ij}^{(2)} = S_{ij}^{(2)} + \frac{1}{K_2}M_{ij}^{(2)}$ is given below;

$$A_{ij}^{(2)} = \begin{bmatrix} 23.01481884 & -26.21543479 & 3.241050725 \\ -26.21543479 & 52.1431884 & -25.81543479 \\ 3.241050725 & -25.81543479 & 22.60431373 \end{bmatrix}$$

4.4.4 Stiffness and Mass matrix of element three (e_3)

From Eqs.(4.17) and (4.18) we obtained values of the stiffness and the mass matrix for element three. Solution of the stiffness matrix is given has follows;

$$\begin{aligned} 40000 * \textstyle\int_{0.2}^{0.3}(1-x^2)(2x-0.55)^2dx &= 22.18 \\ -80000 * \textstyle\int_{0.2}^{0.3}(1-x^2)(2x-0.55)(2x-0.5)dx &= -24.89333333 \\ 40000 * \textstyle\int_{0.2}^{0.3}(1-x^2)(2x-0.55)(2x-0.45)dx &= 3.113333333 \\ 160000 * \textstyle\int_{0.2}^{0.3}(1-x^2)(2x-0.5)^2dx &= 49.92 \\ -80000 * \textstyle\int_{0.2}^{0.3}(1-x^2)(2x-0.5)(2x-0.45)dx &= -24.62666667 \\ 40000 * \textstyle\int_{0.2}^{0.3}(1-x^2)(2x-0.45)^2dx &= 21.51333333 \end{aligned} \tag{4.44}$$

This matrix is shown below;

$$S_{ij}^{(3)} = \begin{bmatrix} 22.18 & -24.89333333 & 3.113333333 \\ -24.89333333 & 49.92 & -24.62666667 \\ 3.113333333 & -24.62666667 & 21.51333333 \end{bmatrix}$$

The solution for mass matrix is given has follows;

$$
\begin{aligned}
\tfrac{1.55}{0.90} * 40000 * \textstyle\int_{0.2}^{0.3} (x^2 - 0.55x + 0.075)^2 dx &= 0.02296296 \\
\tfrac{1.55}{0.90} * -80000 * \textstyle\int_{0.2}^{0.3} (x^2 - 0.55x + 0.075)(x^2 - 0.5x + 0.06) dx &= 0.01148148 \\
\tfrac{1.55}{0.90} * 40000 * \textstyle\int_{0.2}^{0.3} (x^2 - 0.55x + 0.075)(x^2 - 0.45x + 0.05) dx &= -0.005740741 \\
\tfrac{1.55}{0.90} * 160000 * \textstyle\int_{0.2}^{0.3} (x^2 - 0.5x + 0.06)^2 dx &= 0.09185185 \\
\tfrac{1.55}{0.90} * -80000 * \textstyle\int_{0.2}^{0.3} (x^2 - 0.5x + 0.06)(x^2 - 0.45x + 0.05) dx &= 0.01148148 \\
\tfrac{1.55}{0.90} * 40000 * \textstyle\int_{0.2}^{0.3} (x^2 - 0.45x + 0.05)^2 dx &= 0.02296296
\end{aligned} \tag{4.45}
$$

This matrix is shown below;

$$
\frac{B}{K_3} M_{ij}^3 = \begin{bmatrix} 0.02296296 & 0.01148148 & -0.005740741 \\ 0.01148148 & 0.09185185 & 0.01148148 \\ -0.005740741 & 0.01148148 & 0.02296296 \end{bmatrix}
$$

Then the matrix $A_{ij}^{(3)} = S_{ij}^{(3)} + \frac{B}{K_3} M_{ij}^{(3)}$ is given has follows;

$$
A_{ij}^{(3)} = \begin{bmatrix} 22.20296296 & -24.88185185 & 3.107592592 \\ -24.88185185 & 50.01185185 & -24.61518519 \\ 3.107592592 & -24.61518519 & 21.53629629 \end{bmatrix}
$$

4.4.5 Stiffness and mass matrix for element four (e_4)

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element four. As in the case of element three we did the same for element four and we obtain the matrix, $A_{ij}^{(4)} = S_{ij}^{(4)} + \frac{B}{K_4} M_{ij}^{(4)}$ given below;

$$
A_{ij}^{(4)} = \begin{bmatrix} 20.93470085 & -23.81450981 & 2.907254902 \\ -23.81450981 & 46.8172549 & -22.88117647 \\ 2.907254902 & -22.88117647 & 20.00431373 \end{bmatrix}
$$

4.4.6 Stiffness and Mass matrix for element five (e_5)

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element five. As in the case of element three we did the same for element five and we obtain the matrix, $A_{ij}^{(5)} = S_{ij}^{(5)} + \frac{B}{K_5} M_{ij}^{(5)}$ given below;

$$A_{ij}^{(5)} = \begin{bmatrix} 19.20583333 & -21.81375 & 2.640208334 \\ -21.81375 & 42.55666666 & -20.61375 \\ 2.640208334 & -20.61375 & 18.00583333 \end{bmatrix}$$

4.4.7 Stiffness and Mass matrix for element six (e_6)

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element six. As in the case of element three we did the same for element six and we obtain the matrix, $A_{ij}^{(6)} = S_{ij}^{(6)} + \frac{B}{K_5} M_{ij}^{(6)}$ given below;

$$A_{ij}^{(6)} = \begin{bmatrix} 16.94755556 & -19.27955555 & 2.306444444 \\ -19.27955555 & 37.23022222 & -17.81288889 \\ 2.306444444 & -17.81288889 & 15.54088889 \end{bmatrix}$$

4.4.8 Stiffness and mass matrix for element seven (e_7)

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element seven. As in the case of element three we did the same for element six and we obtain the matrix, $A_{ij}^{(7)} = S_{ij}^{(7)} + \frac{B}{K_7} M_{ij}^{(7)}$ given below;

$$A_{ij}^{(7)} = \begin{bmatrix} 14.34285714 & -16.21190477 & 1.905952381 \\ -16.21190477 & 30.83809524 & -14.47857143 \\ 1.905952381 & -14.47857143 & 12.60952381 \end{bmatrix}$$

4.4.9 Stiffness and mass matrix for element eight (e_8)

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element eight. As in the case of element three we did the same for element eight and we obtain the matrix $A_{ij}^{(8)} = S_{ij}^{(8)} + \frac{B}{K_8} M_{ij}^{(8)}$ given below;

$$A_{ij}^{(8)} = \begin{bmatrix} 11.21179487 & -12.61076923 & 1.838717949 \\ -12.61076923 & 23.38051282 & -10.61076923 \\ 1.8387717949 & -10.61076923 & 9.211794872 \end{bmatrix}$$

4.4.10 Stiffness and mass matrix for element nine (e_9)

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element nine. As in the case of element three we did the same for element nine and we obtain the matrix, $A_{ij}^{(9)} = S_{ij}^{(9)} + \frac{B}{K_9} M_{ij}^{(9)}$ given below;

$$A_{ij}^{(9)} = \begin{bmatrix} 7.614444444 & -8.476111111 & 0.904722221 \\ -8.476111111 & 14.85777778 & -6.209444445 \\ 0.904722221 & -6.209444445 & 5.347777777 \end{bmatrix}$$

4.4.11 Stiffness and mass matrix for element ten (e_{10})

From Eqs.(4.17) and (4.18) we obtained stiffness and mass matrix for element ten. As in the case of element three we did the same for element ten and we obtain the matrix, $A_{ij}^{(10)} = S_{ij}^{(10)} + \frac{B}{K_{10}} M_{ij}^{(10)}$given below;

$$A_{ij}^{(10)} = \begin{bmatrix} 3.550909091 & -3.807878788 & 0.303939393 \\ -3.807878788 & 5.27030303 & -1.274545454 \\ 0.303939393 & -1.274545454 & 1.017575758 \end{bmatrix}$$

4.4.12 Matrix assemblying for ten elements

Similarly with the same case as in two element we assembly all the stiffness and mass matrix for ten element as shown below:

$$A = \begin{bmatrix} A_{ij}^{(1)} & 0 & \dots\dots\dots\dots & & 0 \\ 0 & A_{ij}^{(2)} & \dots\dots & & 0 \\ 0 & \dots\dots & A_{ij}^{(3)} & & \dots\dots 0 \\ 0 & \dots\dots\dots\dots & \dots\dots & & 0 \\ 0 & & \ddots & & 0 \\ 0 & & \ddots & & \\ 0 & 0 & \dots\dots\dots\dots\dots\dots\dots\dots & & A_{ij}^{(10)} \end{bmatrix}$$

This is a matrix of order 21 and is symmetric.

4.4.13 Load vector for ten elements

From Eq.(4.19) by evaluating the integral of the equation on right hand side we obtained solutions for each element as follows.
The integral solution for element one (e_1) are;

$$\begin{aligned} 200 * \int_{0}^{0.1}(x^2 - 0.15x + 0.005)dx &= \tfrac{1}{60} \\ 400 * \int_{0}^{0.1}(x^2 - 0.1x)dx &= \tfrac{1}{15} \\ 200 * \int_{0}^{0.1}(x^2 - 0.05x)dx &= \tfrac{1}{60} \end{aligned} \tag{4.46}$$

The integral solution for element two (e_2) are;

$$\begin{aligned} 200 * \int_{0.1}^{0.2}(x^2 - 0.35x + 0.03)dx &= \tfrac{1}{60} \\ -400 * \int_{0.1}^{0.2}(x^2 - 0.3x + 0.02)dx &= \tfrac{1}{15} \\ 200 * \int_{0.1}^{0.2}(x^2 - 0.25x + 0.015)dx &= \tfrac{1}{60} \end{aligned} \tag{4.47}$$

The integral solution for element three (e_3) are;

$$\begin{aligned} 200 * \int_{0.2}^{0.3}(x^2 - 0.55x + 0.075)dx &= \tfrac{1}{60} \\ -400 * \int_{0.2}^{0.3}(x^2 - 0.5x + 0.06)dx &= \tfrac{1}{15} \\ 200 * \int_{0.2}^{0.3}(x^2 - 0.45x + 0.05)dx &= \tfrac{1}{60} \end{aligned} \tag{4.48}$$

The integral solution for element four (e_4) are;

$$\begin{aligned} 200 * \int_{0.3}^{0.4}(x^2 - 0.75x + 0.14)dx &= \tfrac{1}{60} \\ -400 * \int_{0.3}^{0.4}(x^2 - 0.7x + 0.12)dx &= \tfrac{1}{15} \\ 200 * \int_{0.3}^{0.4}(x^2 - 0.65x + 0.105)dx &= \tfrac{1}{60} \end{aligned} \tag{4.49}$$

The integral solution for element five (e_5) are;

$$\begin{aligned} 200 * \int_{0.4}^{0.5}(x^2 - 0.95x + 0.225)dx &= \tfrac{1}{60} \\ -400 * \int_{0.4}^{0.5}(x^2 - 0.9x + 0.2)dx &= \tfrac{1}{15} \\ 200 * \int_{0.4}^{0.5}(x^2 - 0.85x + 0.18)dx &= \tfrac{1}{60} \end{aligned} \tag{4.50}$$

The integral solution for element six (e_6) are;

$$\begin{aligned} 200 * \int_{0.5}^{0.6}(x^2 - 1.15x + 0.33)dx &= \tfrac{1}{60} \\ -400 * \int_{0.5}^{0.6}(x^2 - 1.1x + 0.3)dx &= \tfrac{1}{15} \\ 200 * \int_{0.5}^{0.6}(x^2 - 1.05x + 0.275)dx &= \tfrac{1}{60} \end{aligned} \tag{4.51}$$

The integral solution for element seven (e_7) are;

$$\begin{aligned} 200 * \int_{0.6}^{0.7}(x^2 - 1.35x + 0.455)dx &= \tfrac{1}{60} \\ -400 * \int_{0.6}^{0.7}(x^2 - 1.3x + 0.42)dx &= \tfrac{1}{15} \\ 200 * \int_{0.6}^{0.7}(x^2 - 1.25x + 0.39)dx &= \tfrac{1}{60} \end{aligned} \tag{4.52}$$

The integral solution for element eight (e_8) are;

$$\begin{aligned} 200 * \int_{0.7}^{0.8}(x^2 - 1.55x + 0.6)dx &= \tfrac{1}{60} \\ -400 * \int_{0.7}^{0.8}(x^2 - 1.5x + 0.56)dx &= \tfrac{1}{15} \\ 200 * \int_{0.7}^{0.8}(x^2 - 1.45x + 0.525)dx &= \tfrac{1}{60} \end{aligned} \tag{4.53}$$

The integral solution for element nine (e_9) are;

$$\begin{aligned} 200 * \int_{0.8}^{0.9}(x^2 - 1.75x + 0.765)dx &= \tfrac{1}{60} \\ -400 * \int_{0.8}^{0.9}(x^2 - 1.7x + 0.72)dx &= \tfrac{1}{15} \\ 200 * \int_{0.8}^{0.9}(x^2 - 1.65x + 0.68)dx &= \tfrac{1}{60} \end{aligned} \tag{4.54}$$

The integral solution for element ten (e_{10}) are;

$$\begin{aligned} 200 * \int_{0.9}^{1}(x^2 - 1.95x + 0.95)dx &= \tfrac{1}{60} \\ -400 * \int_{0.9}^{1}(x^2 - 1.9x + 0.9)dx &= \tfrac{1}{15} \\ 200 * \int_{0.9}^{1}(x^2 - 1.85x + 0.855)dx &= \tfrac{1}{60} \end{aligned} \tag{4.55}$$

Similarly we can obtain the values for albedo and S(x) by using the data from S.G.Warren and S.T.Schneider in (1979) [36].Finally we then solve them to get results shown in the table below.

Table 4.7: **The values for K(x),S(x),Albedo and QS(x)β(x,μ)-A for ten elements**

x	S(x)	K(x)	QS(x)	Albedo	QS(x)β(x,μ)-A
0-0.1	1.219	0.95	417.5075	0.254	100.260595
0.1-0.2	1.204	0.92	412.37	0.251	97.66513
0.2-0.3	1.189	0.90	407.2325	0.248	95.03884
0.3-0.4	1.1545	0.85	395.41625	0.26	81.408025
0.4-0.5	1.120	0.80	383.6	0.272	68.0608
0.5-0.6	1.021	0.75	349.6925	0.309	30.4375175
0.6-0.7	0.9565	0.70	327.60125	0.333	7.31003375
0.7-0.8	0.8310	0.65	284.6175	0.382	-35.306385
0.8-0.9	0.697	0.60	238.7225	0.4295	-75.00881375
0.9-1	0.5517	0.55	188.95725	0.5283	-122.0688652

From Eq.(4.19) values for F(x) is shown on the table below;

Table 4.8: **values for F(x) of ten elements**

x	F(x)
0	1.758957807
0.05	7.035831228
0.1	3.52825364
0.15	7.077183333
0.2	3.529274352
0.25	7.039914074
0.3	3.356214303
0.35	6.385943137
0.4	3.014169117
0.45	5.671733333
0.5	2.09432261
0.55	2.705557111
0.6	0.850437699
0.65	0.69619369
0.7	-0.731243501
0.75	-3.621167692
0.8	-2.988870083
0.85	-8.334312639
0.9	-5.78263468
0.95	-14.79622608
1	-3.69905652

From Eq.(4.16) the solution for temperature of ten elements is obtained by solving the equation $A\vec{T} = \vec{F}$ by use of Gaussian elimination method.Here A is a 21 by 21 matrix, $\vec{F}$ is a 21 by 1 matrix and $\vec{T}$ are unknowns to be determine.The solution for temperature are shown in the table below and are round off to 3 significant figures.

Table 4.9: **Temperature in degrees celcius for ten elements**

x	zones(°)	T(°c)
0	0	6.73
0.05	3	7.33
0.1	6	7.73
0.15	9	7.95
0.2	12	7.93
0.25	15	7.85
0.3	17.5	7.65
0.35	20	7.22
0.4	24	6.53
0.45	27	5.57
0.5	30	4.32
0.55	33	2.77
0.6	37	0.96
0.65	41	-1.12
0.7	44	-3.51
0.75	49	-6.42
0.8	53	-8.77
0.85	58	-12.4
0.9	64	-16.4
0.95	72	-20.6
1	90	-24.5

Chapter 5

RESULTS,CONCLUSION AND RECOMMENDATION

5.1 Results

The results in table (5.1) to (5.3) shows the approximate values of temperature ($^\circ c$) at different latitude zone for two elements ,five elements and ten elements.

Table 5.1: **Results for two elements using quadratic basis function with step length h=0.25**

zone(°)	T(°c)
0	17.0
14	14.7
30	6.21
49	-4.99
90	-13.1

Table 5.2: **Results for five elements using quadratic basis function with step length h = 0.1**

zone(°)	T(°c)
0	15.0
6	15.9
12	16.1
17	15.4
26	14.0
30	11.7
37	8.66
44	4.83
53	0.12
64	-8.14
90	-15.4

Table 5.3: **Results for ten elements using quadratic basis function with step length h = 0.05**

zone(°)	T(°c)
0	6.73
3	7.33
6	7.73
9	7.95
12	7.93
14	7.85
17	7.65
20	7.22
24	6.53
27	5.57
30	4.32
33	2.77
37	0.96
41	-1.12
44	-3.51
49	-6.42
53	-8.77
58	-12.4
64	-16.4
72	-20.6
90	-24.5

In 1979, Warren and Schneider [36] considered the Budyko climate model and used the spectral method to obtain the results which correspond to some reality.We decided to use the finite element method hoping to improve the results or at least to see if we obtain the same.The starting point was, as it is suggested by the application of the finite element method,to use $2, 5$ and 10 elements in order to show how the results can be progressively improved.We also decided to use immediately the quadratic interpolation polynomials in the approximation which are more precise compared to the linear ones.

The results obtained above confirm,with some differences,the results of Warren and Schneider for only two and five elements.For 10 elements,we have observed a very large discrepancy and this may be due to the following factors,which we intend to research on in our future work.

- Budyko climate model used here deals mainly with the local energy balance and neglected possible interaction with other region at lower latitudes.Therefore the fluxes between the element (zones of $3°c$) may not be well represented.
- This model also ignored heat transport by both stationary waves and ocean currents thus this assumption made must have limited the accuracy of the model.
- The assumed values of meridional exchange coefficient might have lead to this difference,since we have neglected the exchange in the Southern hemisphere which naturally should influence the boundary conditions used at the equator.
- Also we assume that albedo varies with temperature and therefore the results changes considerably.The importance of albedo which should depend also on the components of the atmosphere above each zone(element) has not according to us ,was not properly parameterized.The numerous documents available on the determination of albedo take also into account paleo-climate data,which with time are changing considerably.

- The infrared radiation is to be analyzed in details due to its variation relative to land,vegetation,clouds and ocean covering element(zone) considered.We have assumed that infrared radiation from Eq.(2.4) that is;

$$I(x) = A + BT(x) \tag{5.1}$$

here we have taken A=211.2WM^{-2} and B=1.55WM^{-2} which holds in each latitude belt and this might be quite risky to change other variables without accounting for changes in this parameters.

- Another fundamental component of finite element method is the generation of a tridiagonal matrix which may not be very well conditioned and stable due to errors which may be part of the weak formulation obtained from the (initial)boundary value problem representing the climate model.As we all know,all the boundary conditions(Dirichlet,Neumann or Robin boundary conditions)must be taken into consideration in the derivation of weak formulation of the model.Since some boundary conditions keep changing due to climate change,we may not succeed in obtaining better results if we consider smaller elements i.e reducing the width of the zones since the corresponding boundary conditions for each smaller zone may not be known adequately.

- Finally,because of time we couldn't submit the obtained matrix of order 21 to normal tests of stability of the matrix using the techniques of numerical linear algebra as they appear for example in [38].

5.2 Conclusion

The simplest climate model used here is the one developed by Budyko [1] in 1969.It considers only the diffusive form of the heat transport.In our objectives,we wanted to study the advection(convection)-diffusion type of climate model but while in progress,especially considering the use finite element method ,we found that it will be a big challenge to consider simultaneously the diffusive and the advection term.

Romuald Szymkiewicz [22],describes how it is easier to consider one term at once and because the diffusive term is more dominant,when the altitude is not involved,we decided to use the Budyko climate model for our study,hoping that a more complete account of these two processes(diffusion and advection) will be considered in a future work.

As said above,our initial focus in this project was to solve by finite element method the advection-diffusion equation as it appears in the climate model developed by Seller in (1969),which incorporate both atmospheric and oceanic fluid dynamics [2]. Climate is regulated by complex interaction among components represented by the earth climate system.This influence the interaction which involves the sun,ocean,atmosphere,clouds,ice and land.Infact,this project is about the heat transfer with all its forms and the fluid dynamics theory.Concerning the fluid dynamics theory in particular,we should bear in mind the observations made by G.K.Batchelor in [39] namely that:

- The density of air in the atmosphere varies with height as a consequence of its compressibility.Therefore we regard the atmosphere and ocean as layer of incompressible fluid with uniform density.
- The upper boundary of the layer of air or water is a free surface and it should be spherical owing to the relatively strong action of the gravity.
- Vertical currents do occur in the atmosphere and the horizontal wind speed varies with height.

In all this we should consider the rotation of the earth. Motion of large horizontal extent in atmosphere or ocean,say with linear dimensions of atleast 100km is evident that coriolis force is very important.Coriolis force is refer to an inertia force which acts on objects that are in constant motion relative to a given rotating frame.It's motion is in the direction which is perpendicular to the axis of rotating and also depend on the velocity of the given body.

Another force we should consider is called centrifugal force which acts outwards toward the radial direction which is proportional to the distance of the body from the axis of a force rotating.The presences of this two forces can allows one to apply Newton's laws on a given rotating system.We shall therefore take earth as our rotating reference frame where the coriolis effect is due to its rotation.Thus because the earth completes only one rotation in per day coriolis force is smaller. Therefore it's effect is notice for motion that occur over a large distance for a long period of time.

Therefore these motion always takes place on the surface of the Earth in which they are deflected to the right in the Northern hemisphere while in the Southern hemisphere they are deflected to the left.This deflection is greater near the poles and becomes small at the equator since the rate at which the diameter changes for each circle of latitude increases as one move from North or South of the equator.Wind and currents then to flow toward the right of the North hemisphere and to the left in the south hemisphere.The climate modelling should try in a way or another use the observations above.

Budyko model neglect vertical heat transport by atmospheric motion.This type of one dimensional climate model use latitude as the spatial dimension and estimates the equilibrium surface temperature in terms of latitude.This model takes into account the first order vertical structure of the atmosphere which shows how the atmosphere can absorb longwave radiation leading to a warmer surface temperature that cause a decrease of temperature in the atmosphere with height.This warm surface temperature leads to the instability of convection processes in the atmosphere because air which is in contact with the ground becomes more lighter than the air above this could leads to a vertical motion of air.

George Hadley in (1735) was able to explained that strong solar heating in the tropics cause air to rise on the surface which flows toward the equator and the air above it toward the poles. Hadley observed that the extend of this circulating cell is limited to the tropics about $30N$ and $30S$ of the equator.Hadley circulation has important implication of hydrological cycles.Rising motion near the surface of the equator constitute to a strong convection process,with release of latent heat and precipitation [4].

Dry and cold air always flows toward poles in the upper troposphere and is subsidence in the subtropics leading to dry conditions near the earth surface.The flow near the surface of the equator picks up water vapour from evaporation at the surface and moves it into the intertropics convergence zone.Thus there is divergence of meridional water vapour transport in the subtropical temperature and convergence in the tropics.Therefore, if vertical energy transported in terms of convection is included in our model both the surface and the upper troposphere temperature will give good results compared with what we have obtained.

With respect to model improvement we offer the following suggestions.First, we have to consider Seller's model which incorporated both transport in the atmosphere and ocean.Taking into consideration of energy balance equation for the earth-atmosphere surface, with boundary condition of no meridional energy transport across the poles Seller's model considers each of the transporting mechanisms separately.

This model neglect heat storage in the oceans,land and atmosphere and therefore our energy balance climate model equation for the atmospheric system becomes;

$$R_s = L\nabla c + \nabla C + \nabla F \tag{5.2}$$

where R_s refers to the radiation balance for a given latitude belt,L is refers to the latent heat of condensation,∇c is represents net flux of water vapour by the atmospheric currents,∇C is refers to the net flux of sensible heat by the atmosphere currents and ∇F is refers to the net flux of sensible heat by ocean currents.Here we have the equations;

$$c = (vq - k_w \frac{\triangle q}{\triangle y})\frac{\triangle p}{g} \tag{5.3}$$

$$C = (vT_o - K_\kappa \frac{\Delta T}{\Delta y})\frac{c}{g}\Delta p, \tag{5.4}$$

$$F = -K_o \Delta z \frac{L'}{L_1}\frac{\Delta T}{\Delta y} \tag{5.5}$$

Where for each latitude circle we have;
v as the mean meridional wind speed.
q is the mean saturation of the specific humidity at sea level.
K_w is the eddy diffusivity for water vapour in air.
$\Delta y = 1.11 * 10^8 cm$
Δp is the pressure depth of the troposphere.

Δz is the ocean depth.
L' is the length of ocean-covered portion of L_1.
T_o is the average temperature of the surface.
g is the gravity of the earth surface.
K_o is the eddy diffusivity for the ocean currents.
C_p is the specific air capacity given a constant pressure.
P is the pressure of air currents.

Eqs.(5.3) and (5.4) assumed that the poleward transport of water vapour and sensible heat in the atmosphere consist of two parts,that is one in terms of the mean meridional motion and the other one involves large-scale eddies circulation or cyclones and anti cyclones movement of wind. [2].The condensation process provides most of energy transferred poleward in the atmosphere as sensible heat.The major source region for both atmospheric and oceanic sensible heat lies in the tropics between 20N and 20S.Latent heat, on the other hand originates primarily in the subtropics $15-35$N and $15-35$S [2].

If we consider both hemisphere transport is poleward and the bulk of required transport occurs in the atmosphere is in the form of sensible heat then we shall be able to get good approximate results.Latent heat or water vapour flux as a results of strong Hadley circulation is directed equatorward at 20N and 20S. The albedo is higher at the south pole than north pole and all this shows an interaction between the two hemisphere.Thus considering all this transport component in the earth-atmosphere, we shall obtained good approximate results of temperature per latitude belts.

The next major improvement concerns the use of mixed boundary value problem rather than Neumann boundary condition.Finally,more important than the above two improvement is the inclusion of the amount of cloud content in the atmosphere and its altitude feedback.In a future work,we intend to include the results obtained by David Goluskin,in his Ph.D thesis,defended in (2013) in the Graduate School of arts and science of Columbia University,about the zonal flow being driven by convection processes and convection which is driven by internal heating in the context of climate modelling [40].

Bibliography

[1] Budyko,M.I.(1969). The effect of solar radiation variations on the climate of the earth.Tellus, 21: $611-619$.

[2] Seller's,W.D.(1969). A global climate model based on the energy balance of the earth atmosphere system.J.Appl.Met.8 : $392-400$

[3] Simon, F.(2012). Energy balance models with three phases of water feedback.

[4] Andreas,S.(2015).Climate modelling, from concepts to codes. lecture notes for climate modelling Ats 421/521 course taught at Oregon State University.

[5] Yutian, W.(2009). Equilibria of diffusive moist static energy balance model.

[6] North,G.R.(1975). Theory of energy balance climate model,J.Atmospheric Sci, $32, pp.2033$

[7] Gerald,R.N.,Robert F.C and Jame A.C.(1981). Energy balance climate models,Review of Geophysics19(1) : $91-121$

[8] Bermejo, R.,Carpio, J.and Diaz J.I.(2009). Mathematical and numerical analysis of a nonlinear diffusive climate energy balance model.pure appl.geophysics.168.pp:$1025-1047$.

[9] Prof.T.Stocker.(2014). Introduction to Climate Modelling.Springer,179 pp

[10] Mohammad, A.(2012). An Introduction to the Finite Element Method (FEM) for Differential Equation.

[11] Applied Mathematics Scientific Research,(2011)

[12] Muhammad, A.(2008). Linear and Quadratic Finite Elements for a moving Mesh Method.

[13] Gordon, C.E.(2010). Numerical solution of Partial Differential Equations

[14] Paul, T. and Jason, A. FEM in 1D and 2D Quadratic shape functions

[15] Vassilios, A.D.(2013). Finite Element Methods for the Numerical solution of PDES.

[16] GHR .(2007). Derivation of energy balance model.

[17] Gerald, R.N.(1975). Analytical Solution to a Simple Climate Model with Diffusive Heat Transport.J.Atmos.Sci,32(7):1301 − 1307.

[18] Thorsten, W.B. and Boris, J.P.K.(2016). Numerical Modelling of Earth Systems.Lecture notes for USC Geol557,vol. 1.14 pp:45 − 177.

[19] Samuel and Gerald N.(1999). A simple proof of the slope stability theorem for energy balance climate model.Canadian Applied maths.Quaterly vol. 7. No2.

[20] James, W. and Richard, M.(2013). Modelling Climate Dynamically.college mathematical journal Vol.44, No.5, pp: 350 − 363.

[21] Mazaheri,M.,Samani,J.M.V. and Samani,H.M.V. (2013). Analytic solution to one dimensional advection diffusion equation with several point sources through arbitrary Time dependent Emission rates patterns.Journal of Agricultural Science and Technology,vol. 15:1231 − 1245.

[22] Romuald, S.(2010). Numerical modelling in open channel hydraulics water science.Water science and Technology.vol. 83,pp:219 − 261.

[23] Goosse,H.,Barriat,P.Y.,Lefebrre,W.,Loutre,M.F and Zunz,V.(2010). An introduction to climate dynamics and climate modelling.Cambridge University press.

[24] McGuffie,K.and Henderson,S.(2005). A Climate modelling primer,3rd.John Wiley and sons,Ltd. pp:1 − 116.

[25] Perez,J.S. G.,Pimentel,L.C.G.,Skaggs,T.H.and van,M.Th.G.(2009). Analytical solution of the advection-diffusion transport equation using a change-of-variable and integral transform technique.International Journal for Heat and Mass Transfer vol. 52:3297 − 3304.

[26] Boztosun,I.and Charafi,A.(2002). An analysis of linear advection-diffusion equation using mesh free and mesh dependent method.Eng.analy.with boundary elements vol. 26,pp:889 − 895.

[27] Obed, R.(2015). Study on the one dimensional linear advection-diffusion equation with finite differences and linear finite elements method.

[28] Guido, V.(2001). Fundamental of Physics and Chemistry of the atmosphere.Springer Science,ISBN: 978 − 3 − 662 − 04540 − 4.

[29] Zhenbu, Z.,Kwende,T. A. and Reddy S.(2016).An improved method for estimating ice line for zonal energy balance climate models.Electronic Journal of Differential Equation ,conference 23, pp.211 − 219

[30] Roland, W.L.,Perumal N. and Kankanhally, N.S.(2004). Fundamentals of the finite element method for heat and fluid flow.John Wiley and Sons,ltd.pp:38 − 91.

[31] Dr.Eleni,C.(2015). The Finite Element Method for analysis of non-linear and dynamic system.Institute of Structural Engineering.Lecture notes

[32] Cheuk, L.(2014). Finite Element Method (FEM)

[33] Ern,G. (2004). Theory and practise of Finite Element,Applied Mathematics Sciences,Vol.159,Springer.

[34] Tung,K.K.(2007). Topics in Mathematical modelling ,Princeton,University.Press,Princeton,NJ.

[35] Esther, R.W.(2013). Dynamics of the Budyko Energy balance model SIAM.I.Applied dynamical systems vol.12.No.pp 2068 − 2092

[36] Warren,S.G. and Schneider,S.H.(1979). Seasonal simulation as a test for uncertainties in the parameterizations of Budyko-Seller zonal climate model,J.Atmos.Sci −36 : 1377 − 1391.

[37] Kiehl,J.T. and Trenberth,k.E. (1997). Earth's annual global mean energy budget,AM.Met.Soc.78, 197 − 208.

[38] Lloyd,N.T. and David,B.(1997).Numerical Linear Algebra.Society for industrial and applied mathematics.ISBN: 0 − 89871 − 361 − 7. PP:87 − 163

[39] Batchelor,G.K. (2007).An introduction to fluid dynamics.Cambridge university press India Pvt.Ltd.ISBN 13 : 978 − 81 − 85618 − 24 − 1

[40] David,G.(2013).Zonal flow driven by convection and convection driven by internal heating.Columbia University.

[41] Gerald,R.N.,Louis H.,David,P. and Bruce,W. (1979).Variational formulation of Budyko-Seller models.National center for atmospheric research,Co 80307.

[42] Briggs,Smithson and Fall.(1989). Fundamental of Physical Geography,Toronto:Copp Clarke and Pitman.Canadian Edition.Copp Clark Pitman Ltd.

[43] Rahmaat, R.,Ariwahjoedi and Shaharin ,A.S.(2011).On the numerical exploration of Zero-dimensional greenhouse model using Newton-Raphson and Steepest Descent Method International journal of environmental science and development.Vol.2.No.3.