



**UNIVERSITY OF NAIROBI**  
**COLLEGE OF BIOLOGICAL AND PHYSICAL**  
**SCIENCES**

**SCHOOL OF MATHEMATICS**  
**NUMERICAL BIFURCATION ANALYSIS FOR**  
**ADVECTIVE-DIFFUSIVE EQUATIONS IN**  
**CLIMATE MODELLING**

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A project submitted in partial fulfillment for a degree of Master  
of Science in Applied Mathematics

August 4, 2016

# Declaration

I the undersigned declare that this project is my original work and to the best of my knowledge has not been presented for the award of a degree in any other university.

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## Declaration by the supervisor

This project has been submitted for examination with my approval as the supervisor.

Signature: .....

Name: Dr. Charles Nyandwi

Date:.....

Signature:.....

# Statement

This dissertation has been submitted in partial fulfillment of requirement for a Master of Science degree at the University of Nairobi and is deposited in the University library to be made available to the borrowers under the rules of the library.

# Dedication

To my family

# Acknowledgment

I would like to acknowledge and sincerely thank my sponsors, the University of Nairobi and the East African Universities Mathematics Programme (EAUMP).

I would also like to extend my gratitude to all those who have helped me to complete this work, both academically and personally.

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Last but not least, I thank the Almighty God for having enabled me finish this project successfully and for the many countless blessings.

# Abstract

The earth's climate system is a highly complex, interconnected system formed by the atmosphere, land surface, ocean and snow together with all living organisms and powered by the solar radiation. Mathematical models have been developed to model the complex processes within the climate system which include radiative, convective, advective and diffusive processes. These models range from simple models to complex models and they require tools to generate the relevant information needed to understand the phenomenon behind them. Therefore some of the tools used to study these model equations include linear stability analysis and other dynamical system methods like the numerical continuation method which we will use here to study bifurcation for the advective-diffusive models.

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# Chapter 1

## INTRODUCTION

### 1.1 Complexity of the Climate System

Earth's climate system is a complex system composed of the ocean, atmosphere, land surface and snow. All these components are powered by the solar radiation from the sun.

One of the greatest global challenges facing humankind is climate change and weather prediction. Some of the gases that contribute to this change are carbon and nitrogen cycles which influence the amount of carbon dioxide, nitrous acid and methane circulating in the atmosphere. As a result, the study of climate change and its consequences is of great importance for our future [1].

Our understanding of climate change draws on expertise from a variety of scientific disciplines but climate models are the best means we have on predicting likely future changes and these models rely wholly on advanced mathematical equations.

Generally, a climate model is a mathematical representation of the atmosphere, ocean and the earth based on the physical, chemical and biological principles. Because of the complexity of the Earth's climate system, building any model that can simulate the changing climate system to the satisfaction of policy makers is a challenge and it is an ongoing subject of research [22].

The equations used to build these models are partial differential equations

and ordinary differential equations which are resolved using numerical techniques over grids that span the entire globe thus providing solutions that are discrete in time and space and nontrivial. This time interval would vary depending on the problem at hand, capacity of the computer being used and on the choice of the numerical method used.

In this project we will describe the models used to represent the complex processes of the climate systems ranging from the simple models i.e Energy Balance Models (EBMs) to the most complex models like the General Circulation Models (GCMs). However, our focus will be on the convective/advection models which we will study using a tool called bifurcation theory.

## 1.2 Convection/Advection

The Navier-Stokes equation is at the heart of modeling dynamical flows in the atmosphere and ocean, which describe the flows of fluids such as water, air and gases [25].

Convection is the bulk flow of fluids i.e gases and liquids due to temperature difference. In the atmosphere for example it enables the transfer of thermal energy by conduction and convection and also evaporation of water from water storage sources like rivers and oceans etc.

Convection also plays a very important role in geophysical systems such as the earth's core because all these systems experience temperature difference [9].

One of the convective system which has been widely studied is the Rayleigh-Benard convection after Benard (1900) and Rayleigh (1916). This system has a simplified geometry and thermodynamic properties just like the one in geophysical systems and therefore is straightforward to study. The theory studied by Rayleigh in 1916 was the theory of linear stability which was to predict the value at which convection would begin called the Rayleigh number. However, further work was later done by many researchers who introduced nonlinearity within the system.

Building on the nonlinear, finite amplitude instability of Rayleigh-Benard convection below, Lorenz derived a system of three equations which model weather prediction in the atmosphere [1].

$$\begin{aligned}\frac{\partial}{\partial t} \vec{\nabla}^2 \psi - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \vec{\nabla}^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} \vec{\nabla}^2 \psi &= v \vec{\nabla}^4 \psi + g\alpha \frac{\partial \tilde{\theta}}{\partial y} \\ \frac{\partial \tilde{\theta}}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \tilde{\theta}}{\partial y} + \frac{\partial \psi}{\partial y} \frac{\partial \tilde{\theta}}{\partial z} &= \kappa \vec{\nabla}^2 \tilde{\theta} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial y}\end{aligned}$$

together with the boundary conditions below:

$$\begin{aligned}\psi &= 0 && \text{at the boundary} \\ \frac{\partial \tilde{\theta}}{\partial y} &= 0 && \text{given } y = 0 \text{ and } y = H/a, \\ \tilde{\theta} &= 0 && \text{given } z = 0 \text{ and } z = H\end{aligned}$$

Lorenz system of equations are obtained from the Galerkin truncation of the two dimensional Rayleigh-Benard convection. Lorenz results showed that these equations which are deterministic produced chaotic solutions for different values of the initial conditions and parameters. Therefore, this provided a motivation for research in Chaos theory which has long been used to solve problems from many disciplines [9].

Therefore in this project, we will use numerical bifurcation analysis to analyze the Lorenz system of equation below [1].

$$\begin{aligned}\frac{dX}{dt} &= -\sigma X + \sigma Y \\ \frac{dY}{dt} &= -XZ + rX - Y \\ \frac{dZ}{dt} &= XY - bZ\end{aligned}$$

With

$$\sigma = \frac{v}{\kappa}, \quad r = \frac{g\alpha H^3 \Delta T}{v\kappa} \frac{a^2}{\pi^4(1+a^2)^3}, \quad b = \frac{4}{1+a^2}$$

The following definitions will also be used in this project:

**Definition 1.1. Bifurcation:** It is the qualitative study of a dynamical system under variation of one or more parameters, say  $r$ .

$$\frac{dx}{dt} = f(x, r) \tag{1.1}$$

Here  $\frac{dx}{dt}$  is the unknown function,  $x$  the independent variable and  $r$  is the parameter to be varied.

## 1.3 Problem Statement

Bifurcation theory was originally developed for tackling flow problems in ordinary differential equations. However in the last few decades, a lot of research has been done on this theory and extended to transition problems in partial differential equations like the Navier-Stokes equation. For this project, we will use the method of numerical bifurcation to study partial differential equations in climate modeling. In particular, we will study the Rayleigh-Benard convection.

## 1.4 Objectives of the Study

### 1.4.1 Main Objective

To understand the complexity of the climate system using mathematical modeling and the theory of bifurcation.

### 1.4.2 Specific Objectives

- (1) Constructing the convective-diffusive equations in climate modeling.
- (2) To explain bifurcation theory for partial differential equations.
- (3) Derive the Lorenz equation from Rayleigh-Benard Convection and analyze it using numerical bifurcation method.

## 1.5 Project Outline

This project has the following structure. In chapter one we have introduction. Chapter 2 describes climate modeling, climate models and convection that occurs in the atmosphere. Bifurcation theory for partial differential equations is described in chapter 3. In chapter 4 we derive the Rayleigh-Benard equation and the Lorenz 63 model. Results for the Lorenz 63 model analysis using eigenvalue-eigenvector approach and bifurcation analysis is also

discussed in this chapter. In chapter 5 conclusions and recommendations are drawn.

## 1.6 Literature Review

In applied mathematics, the dynamics of Rayleigh-Benard convection has been studied in various setting and capacities i.e geophysical, astrophysical etc.

Edward Benard in 1900, was the first to study it experimentally and later in 1916, Lord Rayleigh did a theoretically study on the same experiments conducted by Edward in 1900 [27, 36].

However, Rayleigh analysis was based on the case of two free boundaries unlike Benard's where the lower boundary was fixed and the upper boundary set free. What Rayleigh studied was the linear theory of stability which was to predict the value at which convection would start for a given horizontal wave number.

Building on the work of Rayleigh (1916), in 1926, 1928 and 1940, Jeffreys, Pellow and Southwell [37, 38, 39] respectively derived the results for general boundary conditions. Moreover, in 1961 Chandrasekhar [40] combined and summarized all the work done by the previous scientists in one text.

However, the linear theory of Rayleigh-Benard convection predicted either exponential growth or decay and did not give further information on the behavior of the system as it became unstable. As a result, focus shifted to nonlinearity within the system so in 1958, Veronics and Malkus [41] did a nonlinear stability analysis of the system to finite amplitude perturbations and made predictions on the Nusselt number. In 1959, Veronics [42] further modified the work to include the effect of rotation.

Building on this nonlinear, finite amplitude instability, Lorenz in 1961, was doing a lot of research on fluid flow problem in the atmosphere and was looking for a system to describe this problem. Using the progress of Barry Saltzman who was then also modeling convection fluid motion, Lorenz found a solution to his problem [43].

Therefore in 1963, Lorenz derived a system of three deterministic ordinary differential equations which resulted into chaotic solutions for given initial conditions and parameters.

Lorenz aim was to come up with a simple model that would simplify the two dimensional Rayleigh-Benard equations that model convection in the atmosphere. Therefore his model was a good candidate because first, it summarized the thousands of variables and parameters involved into three variables and three parameters. Secondly, despite the simplifications, his model produced the same result as those of the origin complicated systems of equations i.e the solutions still showed unpredictable behavior in the system. [36].

Wouters J.E in 2013, did a brief analysis of the Lorenz'63 system [20].

In 2014, Benjamin J.Hepworth studied nonlinear two-dimensional Rayleigh Bernard convection in the geophysical and astrophysical setting [9]. Again in 2014, Mihailovic D.T, Mimic G and Arsenic wrote a paper on climate predictions to study chaos and complexity in climate models [14] . In same year, 2014, Erick S, Henk A and Michael G conducted a bifurcation analysis for partial differential equations and illustrated this by analyzing some atmospheric and oceanic models [3].



## Chapter 2

# MODELING THE CLIMATE SYSTEM

Earth's climate system is a complex system composed of the ocean, atmosphere, land surface and snow. All these components are powered by the solar radiation from the sun as illustrated by figure (2.1)

Pressure, humidity, temperature, velocity of wind in the atmosphere and of currents in the ocean, concentration of gaseous components e.t.c. are some of the distributed parameters involved [12].

Some of the processes that make the climate system complex and which take place among the various climate components include biological, physical and chemical processes.

One of the greatest global challenges facing humankind is climate change and weather prediction which are caused by natural and anthropogenic changes in the climate components. Some of the gases that contribute to this change are carbon and nitrogen cycles which influence the amount of carbon dioxide, nitrous acid and methane circulating in the atmosphere. [23].

As a result, the study of climate change and its consequences is of great importance for our future and our understanding of climate change draws an expertise from a variety of scientific disciplines.

Global climate models are the best means we have on predicting likely future changes and these models rely fully on advanced mathematical equations.

A climate model is a mathematical representation of the atmosphere, ocean

and the earth based on the physical, chemical and biological principles. The equations used to build these models are partial differential equations and ordinary differential equations solved using numerical techniques over grids that span the entire globe thus providing solutions that are discrete in time and space and nontrivial. The time interval could vary depending on the problem involved, capacity of the computer being used and on the choice of the numerical method used.

Therefore because of the complexity of the earth's climate system, building any model that can simulate the changing climate system to the satisfactory of policy makers is a challenge and its an ongoing subject of research [22].

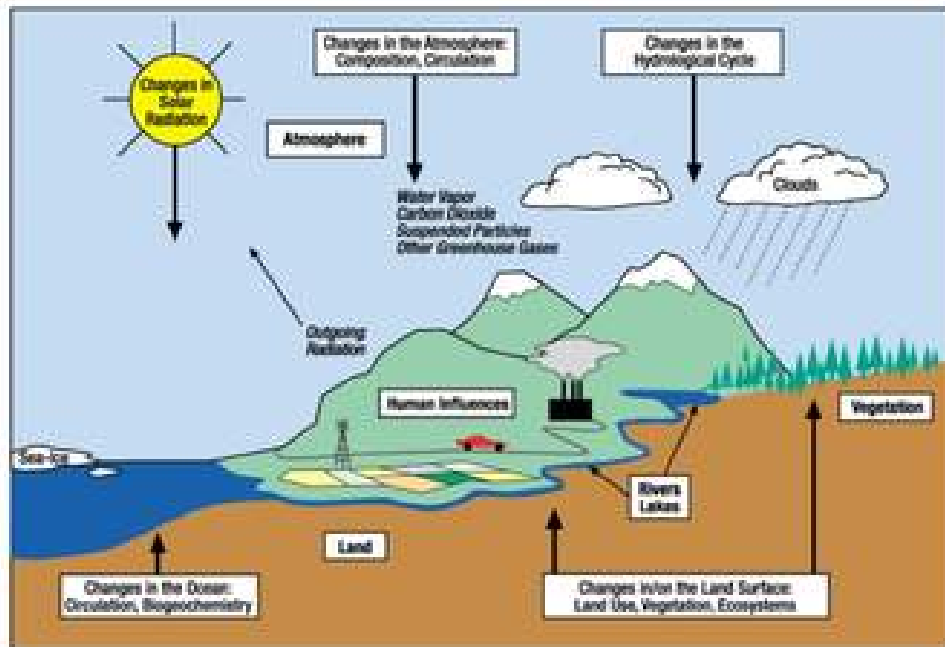


Figure 2.1: Components of the climate system, their interaction and processes. From IPCC (2007)

## 2.1 Earth's Energy Budget

The earth receives almost all of its energy directly from the sun through radiation. This solar radiation heats the earth and it is emitted back to space and therefore cooling the earth (Fig 2.2).

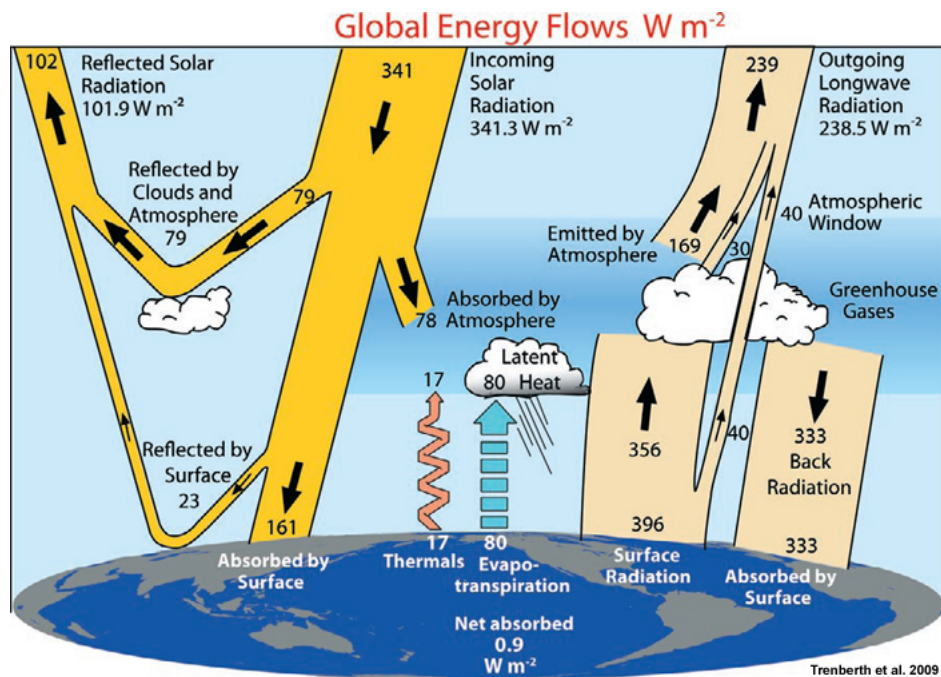


Figure 2.2: Energy flow in the earth. From Trenberth et al.(2009)

At the top of the earth-atmosphere, the total amount of energy that reaches the earth is called the solar constant, denoted by  $(S_0)$ . This solar constant depends on the distance between the earth and the sun and is given by  $1368 W/m^2$ . However, this value is an average and because of the variation of the earth's orbit, it is not a constant.

The interaction of the atmosphere and the solar radiation from the sun solely relies on the wavelength involved and the solar radiation flux intensity. This relationship between the energy flux and the wavelength is known as the spectrum. The incoming solar radiation is shortwave radiation, while the outgoing thermal radiation is long-wave radiation in the visible and infrared.

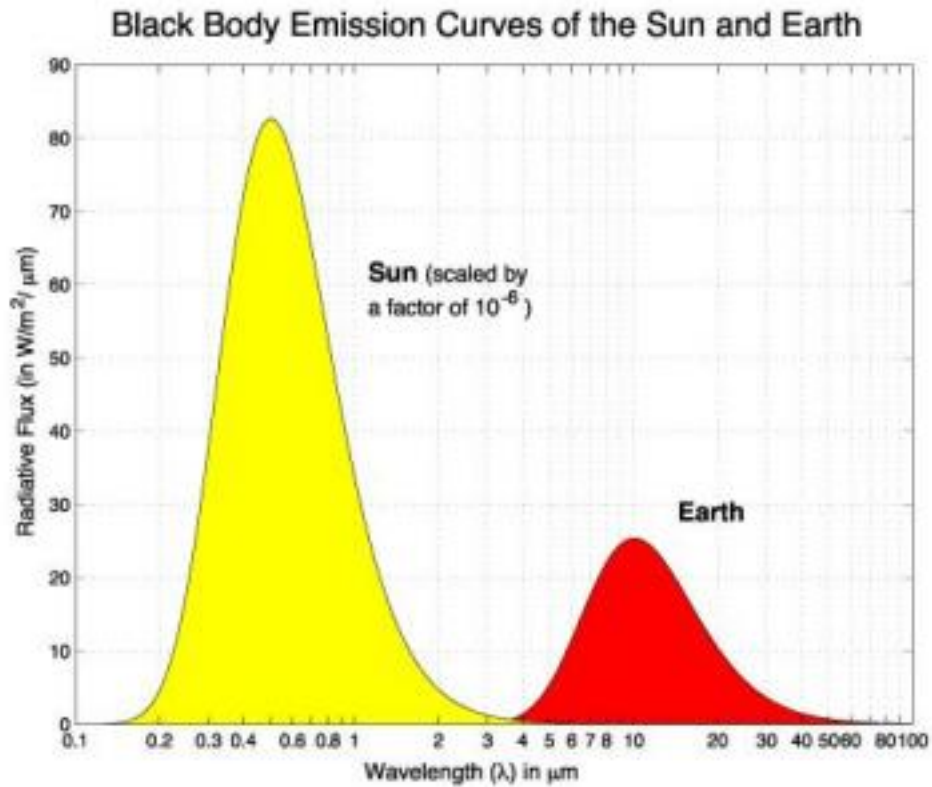


Figure 2.3: Black body radiation for the sun and the earth. From <http://www.Ideo.columbia.edu/~kushir/MPA-ENVP/Climate/lecture/energy>

part of the spectrum respectively . The measured radiation is approximated by a Planck curve or blackbody spectrum (Fig 2.3).

Water vapor, water droplets, carbon-dioxide and dust particles in the atmosphere absorbs some of the solar radiation coming from the sun but a greater portion of the radiation from the sun reach the earth's surface.

When this energy reaches the earth, some of it is absorbed while a fraction of it is reflected back to space by the clouds, light surfaces such as deserts and especially snow and ice on the ground. This ratio of reflected to incident solar radiation is called **planetary albedo**. For the earth  $\alpha = 0.3$

## 2.2 Climate Models

The complex processes within the climate system which include radiative, convective and diffusive processes are represented by climate models. This climate models facilitate in the interpretation of the phenomenon and mechanisms regarding climate change and weather prediction.

Generally, a climate model is a mathematical representation of the atmosphere, ocean and the earth based on the physical, chemical and biological principles being studied. Because of the complexity of the Earth's climate system, building any model that can simulate the changing climate system to the satisfactory of policy makers is a challenge and its an ongoing subject of research [22].

However, different disciplines i.e physics, chemistry and biology use climate models depending on the phenomenon being investigated. For example, in order to model roughly all the climate components, information about the radius of the earth rotation, the amount of solar radiation from the sun, the period of earth's etc has to be provided.

On the other if the focus is only to model a few components of the climate system like the dynamics of the atmosphere, ocean etc, then just few relevant information like boundary conditions should be provided.

In order to understand in details the phenomenon of the complex processes interactions within the climate components and to achieve the goals set for a particular problem, climate models should be very simple. This is because of the thousands of variables and parameter involved in the complex processes being modeled and the time scale which could range from seconds to millions and billions of years.

When constructing climate models, modelers have to make a choice on variables to include and the ones to leave out as well as the ones to consider as constants. However, the physical behavior of the phenomenon and the component being modeled have to be included in order to get the right information needed.

Development of models started in the 1960s to deal with the problem of climate change at that time and to predict the future trends i.e. to understand the the effect of  $CO_2$  and other greenhouse gases on climate change [1].

Climate models are grouped into classes depending on the complexity of the

processes being modeled, objective and problem being studied represented by hierarchy of model shown in figure (2.4).

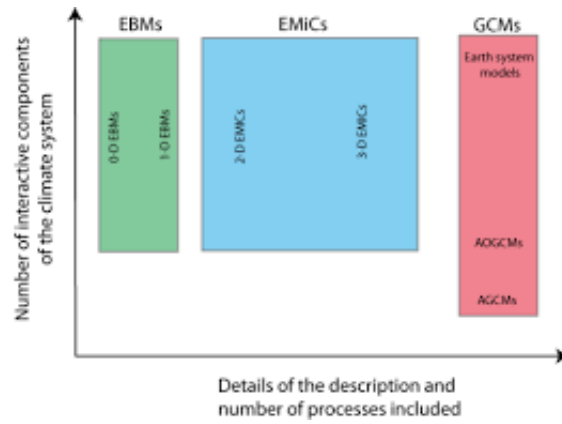


Figure 2.4: Hierarchy of models. From ([1])

## 2.2.1 Energy Balance Models (EBMs)

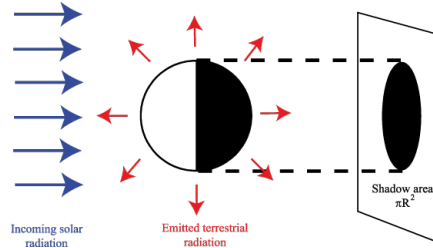


Figure 2.5: Heat absorbed and emitted by the Earth. From Goose et al.

As the name suggests, energy balance models are used to simulate, estimate and predict climate dynamics based on the earth's energy budget. They focus on major climate components such as planetary albedo and incoming solar radiation and their interactions.

Based on their simplicity, EBMs play an important role in advancing our understanding of and ability to predict climate. Simple climate models, for example enables the study of climate sensitivity to change in forcing as different parameters are varied [8].

Budyko (1969) and Sellar (1969) were the first scholars to introduce the simplest forms of the energy balance models referred as the zero-dimensional energy balance models.

The zero dimensional models solves for the equality between incoming and outgoing radiation on the surface of the earth's. In order to construct this conceptual models and acquire this equality alot of assumptions have to be made.

The first assumption made is that the solar radiation from the sun and absorbed by the earth is estimated by

$$A \downarrow = (1 - \alpha_p) \frac{S_o}{4}$$

In this equation,  $S_o$  is roughly  $(1370 W m^{-2})$  and  $\alpha_p$  is the planetary albedo accounting for the reflection of clouds, atmosphere and the reflective surfaces of the earth like ice (roughly we take it to be **0.32**).

The other assumption made is that the earth is a black body which absorbs all the radiation that fall on it and the Stefan-Boltzman law for black body

radiation states that a body emits radiation proportional to the 4th power of its temperature, i.e emission from the earth's surface is given by:

$$\mathbf{A} \uparrow = \epsilon \cdot \sigma \cdot \mathbf{T}_s^4$$

where  $\sigma$  is the Stefan-Boltzman constant and its approximate value is  $5.67 \times 10^{-8} (\text{Wm}^{-2}\text{K}^{-4})$ .

$\epsilon$  is the emissivity of the object which is a measure of how 'good' the black body object is over the range of wavelength in which it is emitting radiation and  $\mathbf{T}_s$  is the surface temperature.

We will approximate the surface temperature as representing the average of an earth covered by 70% ocean and approximate the thermodynamic effect of the mixed layer ocean in terms of effective heat capacity of the earths (land+earth) surface,  $\mathbf{C}_E = 2.08 \times 10^8 \text{JK}^{-1}\text{m}^{-2}$ .

The condition of energy balance can then be described in terms of thermodynamics, which states that any change in the internal energy per unit area per unit time must balance the rate of net heating, which in this case is the difference between shortwave radiation and the longwave radiation.

Mathematically we write:

$$\mathbf{C}_E \frac{\partial \mathbf{T}_s}{\partial t} = (\mathbf{A} \downarrow - \mathbf{A} \uparrow) \quad (2.1)$$

At equilibrium,  $\mathbf{C}_E \frac{\partial \mathbf{T}_s}{\partial t}$  is zero and we must acquire equality between the incoming and the outgoing radiation i.e between the two terms in the right hand equation (2.1). Thus we obtain

$$\mathbf{A} \uparrow = \mathbf{A} \downarrow$$

i.e

$$\epsilon \cdot \sigma \cdot T_s^4 = \frac{S(1 - \alpha)}{4} \quad (2.2)$$

The factor 1/4 comes from the fact that the earth emits radiation over its entire spherical surface area but at a given time only receiving incoming solar radiation over its cross-sectional area.

Linearizing equation (2.2) gives,

$$\epsilon \cdot \sigma \cdot T_s^4 = A + BT$$



where  $A$  and  $B$  are constant determined by satellite. Approximate values are ( $A = 315Wm^{-2}$  and  $B = 4.6Wm^{-2}K^{-1}$ ). Using this approximation  $T_s$  can be easily solved as

$$T_s = \left[ \frac{S(1 - \alpha)}{4} \right] / B \quad (2.3)$$

Latitudinal and longitudinal distribution of temperature can be included in the zero-dimensional EBMs to obtain one or two dimensional EBMs . The incoming solar radiation is symmetric with respect to longitude but varies drastically with latitude so the latitude degrees of freedom can be resolved. This leads to the 1-dimensional energy balance model, where the earth is explicitly divided into latitude bands and treated as uniform with respect to longitude (Fig 2.6). By this, processes like ice feedback which have a strong latitudinal component can be realistically represented.

From the linearized zero dimensional energy balance model, a similar radiation and energy balance equation for each latitude band  $i$  can be computed:

$$C_p \frac{dT_i}{dt} = (1 - \alpha_i)S_i - A - BT_i \quad (2.4)$$

where  $i$  represent each latitude band.

In this equation,  $T_i$ , the temperature at latitude band  $i$ ,  $\alpha_i$  which is albedo and  $T_s$ , solar radiation from the sun are latitude dependent variables enabling the representation of incoming solar radiation between equator and pole. The global temperature  $T_s$  is computed by averaging  $T_i$ 's.

The meridional heat transport caused by the atmospheric circulation and ocean currents in the different latitudes has to be accounted for in order to get realistic results. Therefore the heat advective process is represented by,

$$F(T_i - T_s)$$

for a constant  $F$ .

This advective term is responsible for regional warming and cooling. Combining this expression and equation (2.4) the final form of the one-dimensional EBM is obtained which can be solved numerically.

$$C_p \frac{dT_i}{dt} + F(T_i - T_s) = (1 - \alpha_i)S_i - A - BT_i \quad (2.5)$$

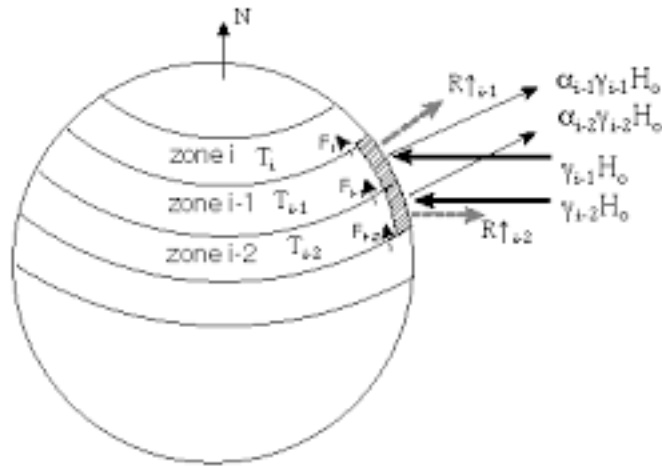


Figure 2.6: Representation of a one-dimensional EBM for which the temperature  $T_i$  is averaged over a band of longitude. From Goose et al.

### 2.2.2 Intermediate Complex Models (EMICs)

Intermediate Complex Models are in between simple models and complex models. They model the dynamics of the oceans and atmosphere that contains also some components of the earth systems [1].

However, the parameter involved are not easily adjustable as in the simpler models since this can easily give wrong results. Finally, the time scale is also very wide and they requires powerful computers to resolve [12].

### 2.2.3 General Circulation Models (GCMs)

This are sophisticated, 3-dimensional models which attempts to simulate and resolve all relevant components of the climate system on a wide scale.

GCMs takes into account and integrate almost all chemical, biological and physical equation derived from the basic physical phenomena.

Unlike simpler models, GCMs divide the atmosphere or earth into grids of discrete cells which represents computational units. [2]

General circulation models are divided into either Atmospheric General Circulation Models or Ocean General Circulation Models depending on the com-

ponent being modeled.

Using interactive atmospheric and oceanic components and coupling the two equations we get Atmospheric Oceanic General Circulation Model (AOGCM) and Coupled General Circulation Models (CGCMs).

The commonly used set of equations to construct these models are introduced below and are known as the primitive equations [12]:

(i) Newton's second law [12]

$$\frac{d\vec{v}}{dt} = \frac{1}{\rho} \vec{\nabla} p - \vec{g} + \vec{F}_{fric} - 2\vec{\Omega} \times \vec{v} \quad (2.6)$$

Where,  $\frac{d}{dt}$  is the total derivative which includes the transport term given by,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$$

where  $\vec{g}$  is the vector representing gravity,  $\vec{F}_{fric}$  is the frictional force and  $\vec{\Omega}$  is the earth's angular velocity.

(ii) Equation of mass conservation.

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v}) \quad (2.7)$$

(iii) Equation of mass conservation for water vapor

$$\frac{\partial \rho q}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v} q) + \rho(E - C) \quad (2.8)$$

Here  $E$  represent evaporation and  $C$  represent condensation.

(iv) The law of conservation of energy

$$Q = C_p \frac{dT}{dt} - \frac{1}{\rho} \frac{dp}{dt} \quad (2.9)$$

Here  $Q$  is the rate of heating per unit mass and  $C_p$  is the specific heat capacity.

(v) State Equation

$$p = \rho R_g T \quad (2.10)$$

Many assumptions and approximation have to be made in the use of these primitive equations include considering the quasi-Boussinesq approximation which is related to the continuity equation [12].

## 2.3 Convection/Advection in the atmosphere

The Navier-Stokes equations is at the heart of modeling dynamical flows in the atmosphere and ocean which describe the flows of fluids such as water, oil, air and gases [25].

This equations are non-linear due to the convective term which is as a result of mathematical reasoning and cannot be bypassed through changing the physical model.

Convection is the bulk flow of fluids i.e gases and liquids due to temperature difference. In the atmosphere for example it enables the transfer of thermal energy by conduction and convection and also evaporation of water from water storage sources like rivers and oceans etc.

For convection to start, the ratio of thermal diffusion time to displacement time. This ration is what is called Rayleigh number after Lord Rayleigh.

The basic physical process behind thermal convection is that of a layer of fluid with two regions. It is then heated in one region and the fluids there expand and lowers the density compared to the other region. This lighter fluid then rises up and is replaced by the cold, heavier fluid from the other region. This patterns continues as the fiuid is heated further.

Convection also plays an important role in many geophysical systems because all these systems experience a temperature difference when heated.[9].

This mechanism described by convection is a simple process but very instrumental in fields where behaviors like fluid turbulence and instability leads to interesting mathematical analysis.

Some of the atmospheric models that are driven by convection include quasi-geostrophic three-level T21QG model, Lorenz 63 model and the Rosenzweig-MacArthur system which is driven by theLorenz 84 atmospheric equations.

The convective system and the one we will study in details in this project is the Rayleigh-Benard convection after Benard (1900) and Rayleigh (1916).

## 2.4 Rayleigh-Benard Convection

This is a natural convection used mostly in the study of pattern forming non-linear systems because of the ease to to conduct experiments on it and analyze it analytically.

In the experiment, a layer of fluid between two parallel planes at different heights is considered e.g water between two parallel planes is considered. Then the fluid is heated from the bottom plane which causes heat to flow in the fluid. Temperature difference across the fluid results into a density difference across the fluid i.e the regions near the bottom are warmed, the fluid expand and the density decreases compared to regions near the top. The difference in density results to buoyancy force which makes the warmer,,lighter fluid at the bottom to move up and replace the colder, heavier fluid at the top.

For small temperature difference, no fluid motion takes place and heat is transferred across the fluid through conduction but beyond a certain critical value of the temperature difference, heat transfer through the reorganization of the hot and cold fluid become more vigorous and this set the fluid into motion.

Henri Benard was the first to conduct the experiment on convention motion in 1900-1901 while Rayleigh in 1916 [27] provided a theoretical explanation on the motion caused due to buoyancy force [28].

This theory of linear stability introduced by Rayleigh could only predict either exponential growth or decay. However, further work was later done by many other researchers introducing nonlinearity within the system. The set of equations governing the Rayleigh-Benard convection (2.11) derived in chapter four is from the Boussinesq approximation which is an approximation from the Navier-Stokes equation and is given by [1]:

$$\begin{aligned} \frac{\partial}{\partial t} \vec{\nabla}^2 \psi - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \vec{\nabla}^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} \vec{\nabla}^2 \psi &= v \vec{\nabla}^4 \psi + g \alpha \frac{\partial \tilde{\theta}}{\partial y} \\ \frac{\partial \tilde{\theta}}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \tilde{\theta}}{\partial y} + \frac{\partial \psi}{\partial y} \frac{\partial \tilde{\theta}}{\partial z} &= \kappa \vec{\nabla}^2 \tilde{\theta} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial y} \end{aligned} \tag{2.11}$$

together with the following boundary conditions

$$\begin{aligned} \psi &= 0 && \text{at the boundary} \\ \frac{\partial \tilde{\theta}}{\partial y} &= 0 && \text{given } y = 0 \text{ and } y = H/a, \end{aligned}$$

$$\tilde{\theta} = 0 \quad \text{given } z = 0 \text{ and } z = H$$

Building on the nonlinear, finite amplitude instability of Rayleigh-Benard convection below, Lorenz derived a system of three equations which model weather prediction in the atmosphere called Lorenz'63 [1].

This Lorenz-63 model is what we will study in details in chapter four using bifurcation theory discussed in chapter three.

# Chapter 3

## BIFURCATION THEORY

**Definition 3.1.** Bifurcation analysis is the qualitative study of a dynamical system under the variation of one or more parameter, say  $r$ .

$$\frac{dx}{dt} = f(x, r) \tag{3.1}$$

Here  $\frac{dx}{dt}$  is the unknown function,  $x$  the independent variable and  $r$  is the parameter to be varied.

The results from this theory of bifurcation is used to follow up on the behavior of climate change which are modeled using climate models. This model solutions can be periodic, turbulent, chaotic etc [3].

In this project, focus is drawn on local bifurcation where change in the solution of the dynamical system occur near the fixed point. Furthermore, the nature of the eigenvalues and eigenvectors of the linearized system is used to determine the type of bifurcation and the dynamics of the system as a whole.

In climate models though, mostly global bifurcations such us homoclinic and heteroclinic bifurcations occur and they are a bit complex to analyze [32, 33]. As a result we will just give examples and diagram illustration without going into details.

## 3.1 Bifurcation Analysis

Let us consider the system below. This is an ordinary differential equation which is autonomous and depends on the parameter  $r$ ,

$$\frac{dx}{dt} = f(x, r) \quad (3.2)$$

In this equation,  $x$  is the solution of a general partial differential equation model discretized using any of the following methods: finite element, finite difference or spectral methods.  $r$  is the parameter to be varied and if in a model with many parameters only one of the parameters is varied then that is called co-dimensional-1 bifurcation.

It is possible that under variation of  $r$ , nothing interesting happens to the system. When we do not experience any qualitative behavior/sifted equilibria then it is said to be structurally stable but when it changes qualitatively then bifurcation will have occurred. We will indicate the solution at time  $t$  with initial condition  $x_o$  as  $\Phi_t(x_o)$ .

To find an equilibrium state i.e when  $x(t) = \bar{x}$  when  $t$  is any value then,

$$f(\bar{x}, \bar{r}) = 0 \quad (3.3)$$

and solve the resultant equation.

### 3.1.1 Linear Stability

Studying the Linear stability of a fixed point requires to first identify the steady state by letting

$$f(\bar{x}, \bar{r}) = 0 \quad (3.4)$$

Next is to study whether the system is stable around  $\bar{x}$ , so infinitesimally small perturbation is applied to the current situation and the interest is to observe whether the system returns back to the original steady state or deviates off.

$$x = \bar{x} + \epsilon \tilde{x}(t) \quad (3.5)$$



If eq. (3.5) is substituted into eq.(3.2), expanded by Taylor's series and the linear terms retained then a linear autonomous system of ordinary differential equation is obtained plus a classic first order algebraic system.

$$\frac{d\tilde{x}}{dt} = J\tilde{x} \quad (3.6)$$

$J$  is the Jacobian matrix in this eq.(3.6) and is obtained as follows

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \quad (3.7)$$

The linearized system has solution of the form  $\tilde{x} = U_o e^{rt}$  where  $U_o$  is a small perturbation.

Substituting this solution into (3.6) and expanding leads to an eigenvalue problem

$$J(\bar{x}, \bar{r})U = \mu U \quad (3.8)$$

for the complex growth factor

$$\mu = \mu_r + i\mu_c \quad (3.9)$$

where  $i^2 = -1$

When  $\mu_r > 0$ , then the equilibrium points are unstable because the disturbances are growing exponentially while when  $\mu_r < 0$ , the equilibrium points are stable since the disturbances converges to zero.

Eigenspaces associated with eigenvalues  $\mu_r > 0$  will be denoted by  $E^u$ , eigenspaces associated with eigenvalues  $\mu_c < 0$  will be denoted by  $E^s$  whereas those eigenspaces associated with the eigenvalues  $\mu = 0$  will be denoted by  $E^c$  [3].

This analysis leads us to stating and proving the following two theorems that will be very useful in the study of bifurcations.

**Theorem 3.1. Hartman-Grobman Theorem:** *This theorem states that the solutions of the system  $\frac{dx}{dt} = f(x, r)$  around a hyperbolic equilibrium point is homeomorphic to the solutions of the linear system  $\frac{d\tilde{x}}{dt} = J\tilde{x}$ .*

In simple terms, the two systems stated in the theorem are locally equivalent and the local structure of the system around the equilibrium point is stable. So for instance if nonlinear terms are added onto the linear system at a hyperbolic equilibrium point, the phase-space portrait of the system does not change.

However when this theorem is tested and does not work for a given system then bifurcation occurs.

Also, in the neighborhood of an hyperbolic equilibrium, stable and unstable manifolds exist :

$$W^s(\bar{x}) = x \quad \text{if} \quad \lim_{t \rightarrow +\infty} \phi_t(x) = \bar{x}$$

$$W^u(\bar{x}) = x \quad \text{if} \quad \lim_{t \rightarrow -\infty} \phi_t(x) = \bar{x}$$

This manifolds corresponds to the eigenspaces  $E^u$  and  $E^s$  for the linearized system at  $\bar{x}$ .

The system will have a bifurcation equilibrium point if for the eigenvalues of the Jacobian matrix, at least one of them is zero or the real part for the complex ones is zero. This leads us to another useful theorem called the center manifold theorem which is very instrumental in finding the bifurcation equilibrium points.

**Theorem 3.2. Center Manifold Theorem:** *This theorem states that at a given equilibrium point  $\bar{x}$ , there can exist three manifolds namely  $W^s$  which is unique, stable and tangent to  $E^s$ ,  $W^u$  which is also unique, unstable and tangent to  $E^u$  and a non unique manifold  $W^c$  tangent to  $E^c$  called the center manifold. However, all the three manifolds are independent on the flow  $\phi_t$ .*

The study of the dynamics of the center manifold together with the systems parameters lead to the concept of bifurcation.

To illustrate this let us take  $\bar{x} = 0$  then

$$\frac{du}{dt} = L_o u + N(u, r) \tag{3.10}$$

where  $N(u, r)$  has a Taylor's expansion starting with at least a quadratic term,  $u$  belongs to  $\mathbb{R}^{n_o}$  and  $L_o$  has  $n_o$  eigenvalue with zero real part. We can make system (3.9) as simple as possible by finding a change of coordinates and the resulting vector field is what is known as the normal form. The normal form is however defined for only local bifurcation.

## 3.2 Local Bifurcation of Steady States

This is change in the solution of the dynamical system near a fixed point. We have three cases for the single zero eigenvalue which include:

- (i) **Saddle-node bifurcation** which occur when a fixed point in the system disappears.
- (ii) **Transcritical bifurcation** which occur fixed points in the system collide.
- (iii) **Pitchfork bifurcation** which occur when new points emerge in the system.

### 3.2.1 Saddle-node Bifurcation

This kind of bifurcation occur in the case when after system (3.9) has been reduced its normal we get

$$\dot{x} = r \pm x^2 \tag{3.11}$$

which corresponds to the supercritical saddle node bifurcation and the subcritical saddle node bifurcation respectively .

Therefore to find the equilibrium points for this bifurcations i.e the subcritical case;

$$r + x^2 = 0 \tag{3.12}$$

and we get that

$$x_* = \pm\sqrt{-r} \tag{3.13}$$

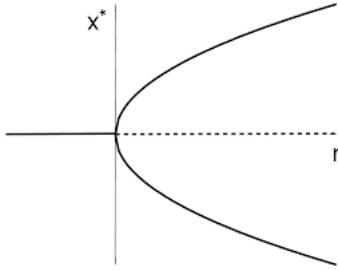


Figure 3.1: Saddle-node bifurcation. From [45]

which exists only if  $r < 0$ .

Next is to find the stability of the system by getting the derivative of

$$f(x) = r + x^2$$

and evaluating at the equilibrium point i.e

$$f'(x) = 2x \tag{3.14}$$

$$f'(\pm\sqrt{-r}) = 2\sqrt{-r}$$

Thus

$x_* = -\sqrt{-r}$  becomes stable because of the negative sign and  $x_{**} = \sqrt{-r}$  become unstable because of the positive sign.

### 3.2.2 Transcritical Bifurcation

In the normal form this bifurcation is given by

$$\dot{x} = rx \pm x^2 \quad (3.15)$$

for the subcritical and supercritical case. Bifurcation occur at  $r = 0$  which is one of the fixed point together with  $x = r$ . This is because in the case when ( $r < 0$ ),  $x = 0$  become stable and  $x = r$  becomes unstable. On the other hand when  $r > 0$ ,  $x > 0$  becomes unstable and  $x = r$  stable.

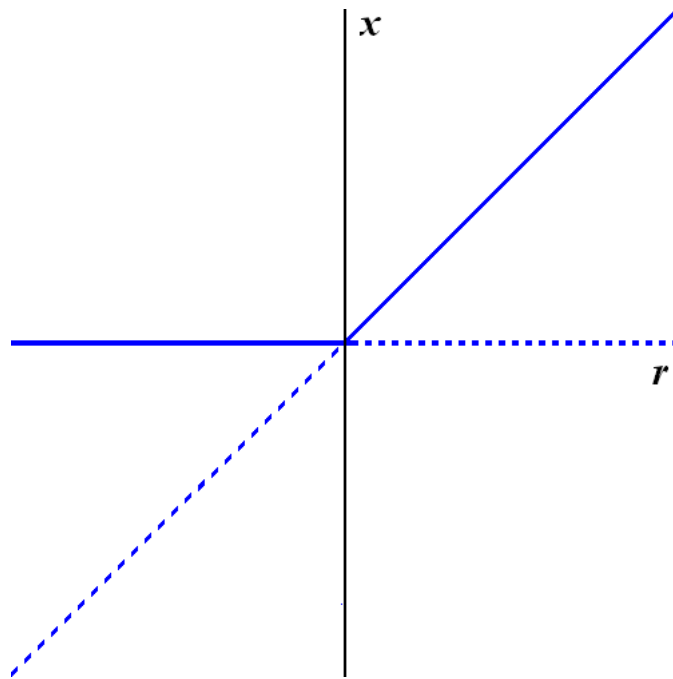


Figure 3.2: Transcritical bifurcation for subcritical and supercritical cases.

### 3.2.3 Pitchfork Bifurcation

The normal form is given by

$$\dot{x} = rx \pm x^3 \quad (3.16)$$

for the supercritical and subcritical cases.

To illustrate what happens we consider the supercritical case,

$$\dot{x} = rx - x^3$$

Here the bifurcation point is still at  $r = 0$  since for the fixed point  $x = 0$ , when  $r < 0$  we have one stable equilibrium point while for  $r > 0$  we have a unstable equilibrium point. We also have two stable equilibria at  $x = \pm\sqrt{r}$ . In this case this system does not go far from the neighborhood of equilibrium so the transition is soft and non catastrophic. This is however opposite for the subcritical case. [3]

This type of bifurcation illustrates the situation where there is something special about the formation of the problem i.e it is constrained by reflection and symmetry as we shall see in the analysis of the Lorenz equation [3].

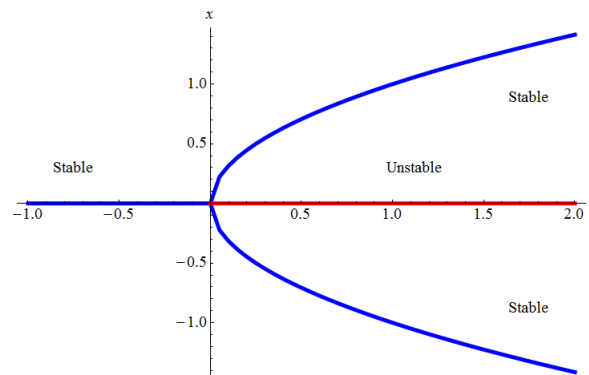


Figure 3.3: Pitchfork bifurcation.

### 3.3 Hopf Bifurcation

Hopf bifurcation occur in the case where, when the fixed points becomes unstable when it was stable or stable when it was unstable as the parameter in question is varied. As a result either a periodic solution or a limit cycle is emerges with the stability properties of the equilibrium point before the disturbance [34].

In this case the eigenvalues of the Jacobian matrix must be a purely complex pair, that is, the real part of the complex eigenvalues is zero.

The normal form is expressed in polar coordinates

$$\begin{aligned} \dot{r} &= \mu r \pm r^3 \\ \dot{\theta} &= w \end{aligned} \tag{3.17}$$

The super-critical and the sub-critical is illustrated by figure (3.4) below.

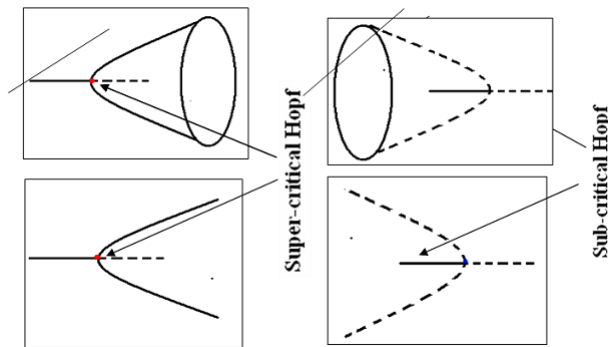


Figure 3.4: Super-critical and sub-critical Hopf bifurcation.

### 3.4 Global Bifurcation

This bifurcation are experienced in large invariant sets where for example when a periodic orbit collides with the equilibrium points, a complete transformation in the topology of the flow of phase-space of the system is experienced [3].

This global bifurcations are generally responsible for the emergence of chaos

and strange attractors. Here will only mention one example of global bifurcation which is homoclinic bifurcation which is also one of the bifurcations we will identify in the Lorenz system. Other types of global bifurcation include torus bifurcation, heteroclinic bifurcation etc.



## Chapter 4

# BIFURCATION ANALYSIS FOR THE LORENZ 63 MODEL OF ATMOSPHERIC CIRCULATION

### 4.1 Lorenz-Saltzman Model

The basic physical process behind Lorenz-63 model is that of a layer of incompressible fluid contained in a cell . It is then heated from below and the fluids there expand and lowers the density compared to the top. This lighter fluid from the bottom then rises up and is replaced by the cold, heavier fluid from the top. This patterns continues as the fluid is heated further.

Barry Saltzman (1931-2001) derived an approximation consisting of a non-linear system of ordinary differential equations from the governing equations of a viscous, stably stratified flow for the thermally driven flows. [43].

This system was describing circulation in the atmosphere and was the first system to describe chaotic systems thus discovering chaos theory [1].

This system is deterministic and is represented by several coupled ordinary differential equations of first order in time. It is named after the first scientist who solved and interpreted it completely called Edward Lorenz [36].

### 4.1.1 Derivation of the Model

We obtain our derivation from the work of Stoker in the book, Introduction to Climate Modelling [1]. He considered a plane in the (y,z) frame. This plane was also fixed and not rotating and solutions were assumed to uniform in the x-axis. Another assumption was that the heated fluid was moving in the clockwise direction [1].

We consider a incompressible fluid and bring in the continuity equation

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4.1)$$

From the mass conservation equation, a stream function is defined

$$v = \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial y} \quad (4.2)$$

As a result of the stream function, the relative vorticity in the meridional y-z-plane is given by

$$\zeta = \vec{\nabla}^2 \psi \quad (4.3)$$

For us to derive the vorticity equation, the momentum equations have to be brought fourth.

$$\frac{Dv}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial y} + v \vec{\nabla}^2 v \quad (4.4)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial z} + v \vec{\nabla}^2 w - \frac{g}{\rho_o} \tilde{\rho} \quad (4.5)$$

Here  $v$  is the kinematic velocity and the last term describe acceleration caused by buoyancy. Equations (5.5), (4.4) and (4.1) gives

$$\frac{D\zeta}{Dt} = v \vec{\nabla}^2 \zeta - \frac{g}{\rho_o} \frac{\partial \tilde{\rho}}{\partial y} \quad (4.6)$$

This equation can further be expressed differently by applying the volume coefficient equation.

$$\frac{D\zeta}{Dt} = v \vec{\nabla}^2 \zeta + g\alpha \frac{\partial \theta}{\partial y} \quad (4.7)$$

then

$$\frac{\partial \zeta}{\partial t} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} = v \vec{\nabla}^2 \zeta + g\alpha \frac{\partial \theta}{\partial y} \quad (4.8)$$

Considering to use the temperature distribution equation and the heat equation we get

$$\frac{\partial \tilde{\theta}}{\partial t} + v \frac{\partial \tilde{\theta}}{\partial y} - w \frac{\Delta T}{H} + w \frac{\partial \tilde{\theta}}{\partial z} = \kappa \frac{\partial^2 \tilde{\theta}}{\partial y^2} + \kappa \frac{\partial^2 \tilde{\theta}}{\partial z^2} \quad (4.9)$$

Here  $\kappa$  is the thermal diffusivity. merging (4.2),(4.3), (4.7) and (4.9) results gives the required system of equation [1].

$$\begin{aligned} \frac{\partial}{\partial t} \vec{\nabla}^2 \psi - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \vec{\nabla}^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} \vec{\nabla}^2 \psi &= v \vec{\nabla}^4 \psi + g\alpha \frac{\partial \tilde{\theta}}{\partial y} \\ \frac{\partial \tilde{\theta}}{\partial t} - \frac{\partial \psi}{\partial z} \frac{\partial \tilde{\theta}}{\partial y} + \frac{\partial \psi}{\partial y} \frac{\partial \tilde{\theta}}{\partial z} &= \kappa \vec{\nabla}^2 \tilde{\theta} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial y} \end{aligned} \quad (4.10)$$

This system (4.10) is only complete when conditions for boundary are set. In this case the conditions are, no transport across the boundaries and no heat flow across the meridional boundaries. Temperatures are fixed at the top and bottom , hence equation (4.10) is considered together with the following boundary conditions.

$$\begin{aligned} \psi &= 0 && \text{at the boundary} \\ \frac{\partial \tilde{\theta}}{\partial y} &= 0 && \text{if } y = 0 \text{ and } y = H/a, \\ \tilde{\theta} &= 0 && \text{if } z = 0 \text{ and } z = H \end{aligned}$$

## Truncated Galerkin Expansion

When this set of equations (4.10) are subjected to a three mode spectral truncation approximation we obtain three systems of ordinary differential equations. We approximate the stream function  $\psi$  and temperature  $\tilde{\theta}$  by

$$\psi(y, z, t) = X(t) \sin\left(\frac{\pi a y}{H}\right) \sin\left(\frac{\pi z}{H}\right) + \dots \quad (4.11)$$

$$\tilde{\theta}(y, z, t) = Y(t) \cos\left(\frac{\pi a y}{H}\right) \sin\left(\frac{\pi z}{H}\right) - Z(t) \sin\left(\frac{2\pi z}{H}\right) + \dots \quad (4.12)$$

Expanding (4.11) and (4.12) gives:

$$\frac{\partial \psi}{\partial y}(y, z, t) = \frac{\pi a}{H} X(t) \sin\left(\frac{\pi z}{H}\right) \cos\left(\frac{\pi a y}{H}\right) + \dots \quad (4.13)$$

$$\frac{\partial \psi}{\partial z}(y, z, t) = \left(\frac{\pi}{H}\right)X(t) \cos\left(\frac{\pi z}{H}\right) \sin\left(\frac{\pi ay}{H}\right) + \dots \quad (4.14)$$

$$\frac{\partial \theta}{\partial y}(y, z, t) = -\left(\frac{\pi a}{H}\right)Y(t) \sin\left(\frac{\pi z}{H}\right) \sin\left(\frac{\pi ay}{H}\right) + \dots \quad (4.15)$$

$$\frac{\partial \theta}{\partial z}(y, z, t) = \left(\frac{\pi}{H}\right)Y(t) \cos\left(\frac{\pi z}{H}\right) \cos\left(\frac{\pi ay}{H}\right) + \left(\frac{2\pi z}{H}\right) \cos\left(\frac{2\pi z}{H}\right) + \dots \quad (4.16)$$

When this equations are substituted back to equation (4.10) and approximated to the first order then eliminating the common factor  $\sin\left(\frac{\pi ay}{H}\right) \sin\left(\frac{\pi z}{H}\right)$  we find

$$\left(\frac{\pi}{H}\right)^2(1+a^2)\frac{dX}{dt} = -v\left(\frac{\pi}{H}\right)^4(1+a^2)^2X + g\alpha\frac{\pi a}{H}Y \quad (4.17)$$

and

$$\begin{aligned} & \cos\left(\frac{\pi ay}{H}\right) \sin\left(\frac{\pi z}{H}\right) \left\{ \frac{dY}{dt} - \frac{\pi a}{H} \frac{2\pi}{H} XZ \cos\left(\frac{2\pi z}{H}\right) \right. \\ & \quad \left. + \kappa\left(\frac{\pi}{H}\right)^2(1+a^2)Y - \frac{\Delta T}{H} \frac{\pi a}{H} X \right\} \\ & = \sin\left(\frac{2\pi z}{H}\right) \left\{ \frac{dZ}{dt} - \frac{1}{2} \frac{\pi a}{H} \frac{\pi}{H} XY + \kappa\left(\frac{2\pi}{H}\right)^2 Z \right\} \end{aligned} \quad (4.18)$$

i.e. using  $\sin\left(\frac{2\pi z}{H}\right) = 2 \sin\left(\frac{\pi}{H}\right) \cos\left(\frac{\pi z}{H}\right)$  in (4.22) gives,

$$\begin{aligned} & \cos\left(\frac{\pi ay}{H}\right) \left\{ \frac{dY}{dt} - \frac{\pi a}{H} \frac{2\pi}{H} XZ \cos\left(\frac{2\pi z}{H}\right) \right. \\ & \quad \left. + \kappa\left(\frac{\pi}{H}\right)^2(1+a^2)Y - \frac{\Delta T}{H} \frac{\pi a}{H} X \right\} \\ & = 2 \cos\left(\frac{2\pi z}{H}\right) \left\{ \frac{dZ}{dt} - \frac{1}{2} \frac{\pi a}{H} \frac{\pi}{H} XY + \kappa\left(\frac{2\pi}{H}\right)^2 Z \right\} \end{aligned} \quad (4.19)$$

This equation is valid for all values  $0 \leq y \leq H/a$  and  $0 \leq z \leq H$  and so we eliminate the sum in the curly brackets. Further assumptions are made on the vertical range to be considered. Therefore,  $\cos(2\pi z/H) \approx -1$  and we obtain from [1].

$$\begin{aligned} \frac{dX}{dt} & = -cX + dY \\ \frac{dY}{dt} & = -eXZ + fX - gY \\ \frac{dZ}{dt} & = hXY - kZ \end{aligned} \quad (4.20)$$

with the seven constants,

$$\begin{aligned}
c &= v\left(\frac{\pi}{H}\right)^2(1+a^2) \\
d &= \frac{g\alpha a H}{\pi(1+a^2)} \\
e &= \frac{2\pi^2 a}{H^2} \\
f &= \frac{\Delta T \pi a}{H^2} \\
g &= \kappa\left(\frac{\pi}{H}\right)^2(1+a^2) \\
h &= \frac{\pi^2}{2H^2} \\
k &= 4\kappa\left(\frac{\pi}{H}\right)^2
\end{aligned} \tag{4.21}$$

The equations are rescaled as follows,

$$\begin{aligned}
t &= \left(\frac{\pi}{H}\right)^2(1+a^2)\kappa t \\
X &= \frac{a}{\kappa(1+a^2)}X \\
Y &= \frac{a}{\kappa(1+a^2)}\frac{g\alpha a H^3}{\pi^3(1+a^2)^2 v}Y \\
Z &= 2\frac{a}{\kappa(1+a^2)}\frac{g\alpha a H^3}{\pi^3(1+a^2)^2 v}Z
\end{aligned} \tag{4.22}$$

Thus we get the standard form of Lorenz equation [1]:

$$\begin{aligned}
\frac{dX}{dt} &= -\sigma X + \sigma Y \\
\frac{dY}{dt} &= -XZ + rX - Y \\
\frac{dZ}{dt} &= XY - bZ
\end{aligned} \tag{4.23}$$

With

$$\sigma = \frac{v}{\kappa}, \quad r = \frac{g\alpha H^3 \Delta T}{v\kappa} \frac{a^2}{\pi^4(1+a^2)^3}, \quad b = \frac{4}{1+a^2}$$

We focus on  $r$  as the parameter to be varied because varying  $r$  is equivalent to the changes in the meridional temperature gradient between the equator and the pole.

The model parameters,  $\kappa$ ,  $\mathbf{r}$  and  $\mathbf{b}$  are the (Prandtl number), the (Rayleigh number) and the wavenumber respectively. The state space variables  $X$ ,  $Y$  and  $Z$  estimates convective motion intensity, the horizontal and vertical temperature gradient respectively.[26].

## 4.2 Bifurcation Analysis

In equation (4.23), the parameters  $\kappa, r$  and  $b$  are taken to be positive . In the analysis, we let  $\kappa = 10.0$  and  $b = 8/3$  which are standard values [36]. We then vary  $r$  over a wide range say 0 to 100 and study the behavior of the solution.

### 4.2.1 Equilibria/ Fixed points

At equilibria the system (4.23) does not change with time so we have:

$$\begin{aligned} X - Y &= 0 \\ rX - Y - XZ &= 0 \\ XY - bZ &= 0 \end{aligned} \tag{4.24}$$

From (4.24), one of the solution is the origin,  $(X, Y, Z) = (0, 0, 0)$  . Also from the first equation of (4.23),  $X - Y = 0 \Rightarrow X = Y$ .

From th third equation,  $XY - bZ = 0 \Rightarrow X^2 = bZ$  and  $X = \pm\sqrt{bZ}$

Finally from the second equation,  $rX - Y - XZ = 0 \Rightarrow XZ = rX - Y$  and therefore dividing both sides by  $X$  we get  $Z = r - 1$ .

So we have three equilibrium points abbreviated by  $C_o, C_1$  and  $C_2$  and given by :

$$\begin{aligned} C_o &= (0, 0, 0), \quad C_1 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \text{ and} \\ C_2 &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \end{aligned}$$

Out of the three equilibrium points,  $C_o$  persists for any values of the  $r$ .  $C_1$  ,  $C_2$  are real only if  $r \geq 1$ .

## 4.2.2 Bifurcations with varying $r$

We now study the behaviour of the solution of the system as the parameter  $r$  is varied and using matlab and octave draw the varies bifurcation diagrams.

(1)  $0 < r < 1$

We linearize the system of equation (4.23) by finding the Jacobian of the system i.e

$$J(X, Y, Z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z & -1 & -X \\ Y & X & -b \end{pmatrix} \quad (4.25)$$

and so at the origin,  $C_o = (0, 0, 0)$ :

$$J(0, 0, 0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

Finding the determinant of the Jacobian:

$$\begin{vmatrix} -(\sigma + \lambda) & \sigma & 0 \\ r & -(1 + \lambda) & 0 \\ 0 & 0 & -(b + \lambda) \end{vmatrix} = 0$$

When we expand the determinant we get:

$$-(\sigma + \lambda)[(1 + \lambda)(b + \lambda)] - r[-\sigma(b + \lambda)] = 0$$

$$-(\sigma + \lambda)[\lambda^2 + \lambda b + \lambda + b] + r\lambda\sigma + r\sigma b = 0$$

$$\lambda^3 + \lambda^2(1 + b + \sigma) + \lambda(\sigma + b + \sigma b - r\sigma) - r\sigma b + \sigma b = 0$$

solving this quadratic expression in  $\lambda$  we obtain three eigenvalues.

$$\lambda_1 = -b, \quad \lambda_2 = \frac{1}{2}(-1 - \sigma - \sqrt{(1 - \sigma)^2 + 4r\sigma}),$$

$$\lambda_3 = \frac{1}{2}(-1 - \sigma + \sqrt{(1 - \sigma)^2 + 4r\sigma})$$

$\lambda_1$  and  $\lambda_2$  are negative then in order for  $\lambda_3$  to be negative  $(1 - \sigma)^2 + 4r\sigma < (1 + \sigma)^2$  which happens if  $r < 1$ .



We take a few values of  $r$  in the interval  $0 < r < 1$ , compute the eigenvalues to verify our observations above.

Table 4.1: **Table of eigenvalues for different values of  $r$  in the interval  $0 < r < 1$**

$r$	$\lambda_1$	$\lambda_2$	$\lambda_3$
0.1	-10.1098	-0.8902	-2.6667
0.3	-10.3218	-0.6782	-2.6667
0.5	-10.5249	-0.4751	-2.6667
0.7	-10.7202	-0.2798	-2.6667
0.9	-10.9083	-0.0917	-2.6667

From the table (4.1) all the eigenvalues in the interval are negative and thus the origin equilibrium point is stable because all trajectories converge to the origin.

However, from the trends in the table we notice that  $\lambda_2$  is moving closer and closer to zero as we approach  $r = 1$ .

(2) **Super Critical Pitchfork Bifurcation at  $r = 1$**

At  $r = 1$  all the three equilibrium points converge to the origin. This is a pitchfork bifurcation since some of the eigenvalues tend to zero as shown below.

$$C_o = C_1 = C_2 = (-11.000, 0, -2.6667)$$

In the same way we carry out stability analysis for  $C_1$  and  $C_2$ . We calculate Jacobian matrix of the linearized system (4.23) and get,

$$J(X, Y, Z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z & -1 & -X \\ Y & X & -b \end{pmatrix} \quad (4.26)$$

so at  $C_1$  we have:

$$J(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{pmatrix}$$

And at  $C_2$  we have:

$$J(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$$

(3)  $1 < r < 1.345$

Table 4.2: **Table of eigenvalues for different values of r in the interval  $1 < r < 1.345$**

r	$C_o$			$C_1$ and $C_2$		
	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_1$	$\lambda_2$	$\lambda_3$
1.1	-11.0902	0.0902	-2.6667	-11.0260	-0.1980	-2.4427
1.2	-11.1789	0.1789	-2.667	-11.0515	-0.4447	-2.1705
1.3	-11.2663	0.2663	-2.6667	-11.0766	-0.8127	-1.7774
1.345	-11.3052	0.3052	-2.6667	-11.0878	-1.2345	-1.3444

In this range  $C_o$  is becomes unstable while  $C_1$  and  $C_2$  are stable. This is a **saddle-node** bifurcation. The two new equilibrium points that emerge are stable nodes thus at  $r = 1$  we have a supercritical pitchfork bifurcation.

(4)  $r=1.346$

At this point the first set of complex eigenvalues for  $C_1$  and  $C_2$  emerge so these equilibria changes from being nodes to **spirals**. Oscillatory behavior then first appear in the system as  $r$  increases. However  $C_o$  remains unstable.

$$C_o = (-11.3060, 0.3060, -2.6667)$$

$$C_1 = C_2 = (-11.0881, -1.2893 + 0.0442i, -1.2893 - 0.0442i)$$

(5)  $1.347 < r < 13.925$

Table 4.3: Table of eigenvalues for different values of  $r$  in the interval  $1.347 < r < 13.925$

r	$C_o$			$C_1$ and $C_2$		
	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_1$	$\lambda_2$	$\lambda_3$
1.347	-11.3069	0.3069	-2.6667	-11.0882	-1.2892+0.0760 $i$	-1.2892-0.0760 $i$
2.0	-11.8443	0.8443	-2.6667	-11.2422	-1.2122+1.8096 $i$	-1.2122-1.8096 $i$
5	-13.8815	2.8815	-2.6667	-11.8092	-0.9287+4.1476 $i$	-0.9287-4.1476 $i$
10	-16.4659	5.4659	-2.6667	-12.4757	-0.5955+6.1742 $i$	-0.5955-6.1742 $i$
12	-17.3427	6.3427	-2.6667	-12.6872	-0.4897+6.7824 $i$	-0.4897-6.7824 $i$
13	-17.7577	6.7577	-2.6667	-12.7849	-0.4409+7.0616 $i$	-0.4409-7.0616 $i$
13.925	-18.1293	7.1293	-2.6667	-12.8709	-0.3979+7.3074 $i$	-0.3978-7.3074 $i$

In this range the oscillatory behavior persists in the system as illustrated by the complex eigenvalues and as  $r$  increases the oscillations continue to grow becomes stronger and stronger. We also illustrated the behavior of the solutions leaving the origin by figure (4.1).

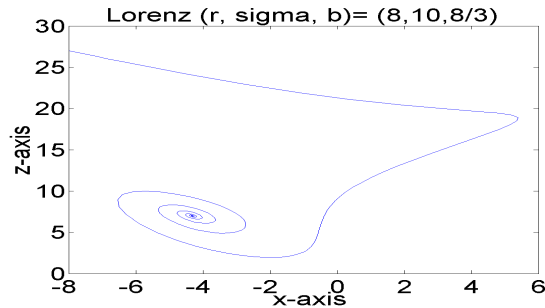


Figure 4.1: Bifurcation diagram for Lorenz equations at  $r=8$

(6)  $13.926 < r \leq 24.74$

In this interval the complex values for the eigenvalues persists and so

Table 4.4: **Table of eigenvalues for different values of r in the range  $13.926 < r < 24.74$**

r	$C_o$			$C_1$ and $C_2$		
	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_1$	$\lambda_2$	$\lambda_3$
13.926	-18.1297	7.1297	-2.6667	-12.8710	$-0.3978+7.3078 i$	$-0.3978-7.3078 i$
16	-18.9257	7.9257	-2.6667	-13.0510	$-0.3078+7.8233 i$	$-0.3078-7.8233 i$
20	-20.3408	9.3408	-2.6667	-13.3571	$-0.1548+8.7087i$	$-0.1548-8.7087i$
24	-21.6323	10.6323	-2.6667	-13.6216	$-0.0225+9.4896i$	$-0.0225-9.4896i$
24.05	-21.6478	10.6478	-2.6667	-13.6247	$-0.0210+9.4989i$	$-0.0210-9.499 i$
24.74	21.86	10.86	-2.6667	-13.6668	$0.0001+9.6251 i$	$0.0001-9.6251 i$

the oscillatory structure of the solution also persist.

For larger values of  $r$ , solutions of  $C_1$  and  $C_2$  go back and forth many times then finally settles into one of them. We call this type of solution **pre-chaotic transients** as illustrated by figure (4.2). Also in this range the limit cycles shrink around  $C_1$  and  $C_2$  as they head for a subcritical Hopf bifurcation which takes place at 24.74...

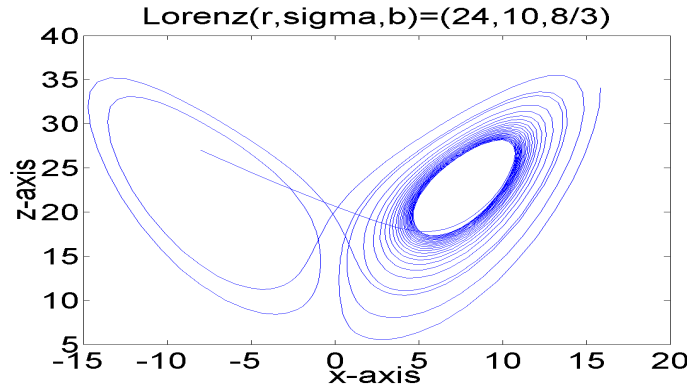


Figure 4.2: Bifurcation diagram for Lorenz equations at  $r=24$

(7) **Hopf bifurcation at  $r = 24.74$**

Reaching the bifurcation point  $r = 24.74$ , for  $C_1$  and  $C_2$ ,

$$\lambda_1 = -(b + \sigma + 1), \quad \lambda_{2,3} = \pm \sqrt{b(\sigma + r)}i$$

as shown in table (4.4) and so we have a **Hopf bifurcation** which is subcritical because the initially stable equilibrium points,  $C_1$  and  $C_2$  becomes unstable above this value of  $r$ .

(8)  **$24.74 < r \leq 28$**

Table 4.5: **Table of eigenvalues for different values of  $r$  in the range  $24.74 < r \leq 28$**

r	$C_o$			$C_1$ and $C_2$		
	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_1$	$\lambda_2$	$\lambda_3$
25	-21.9393	10.9393	-2.6667	-13.6825	0.0079+9.6721 $i$	0.0079-9.6721 $i$
27	-22.5367	11.5367	-2.6667	-13.7993	0.0663+10.0254 $i$	0.0663-10.0254 $i$
28	-22.8277	11.8277	-2.6667	-13.8546	0.0940+10.1945 $i$	0.0940-10.1945 $i$

In this range there are no stable critical points any more.  $C_o$ ,  $C_1$  and  $C_2$  all become unstable.

The trajectories of the solutions are bounded around  $C_1$  and  $C_2$ . At  $r = 28$ , the motion becomes aperiodic and highly dependent on the initial conditions.

We plot this in matlab and octave and a beautiful butterfly structure emerges as illustrated by figure (4.3).

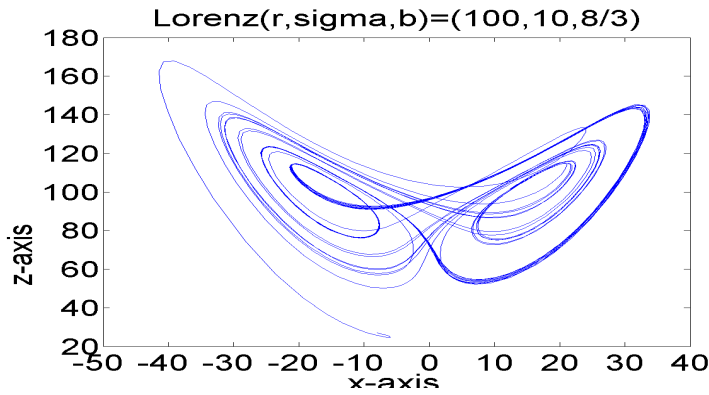


Figure 4.3: Bifurcation diagram for Lorenz equations at  $r=28$

(9)  $r > 28$

When we consider  $r = 100$ , we have the following eigenvalues

$$C_o = (-37.4414, 26.4416, -2.6667)$$

and

$$C_1 = C_2 = (-15.9829, 1.1581 + 18.1387i, 1.1581 - 18.1381i)$$

Therefore beyond  $r > 28$  all the equilibrium points remains unstable as shown by figure (4.4) and the motion remains aperiodic.

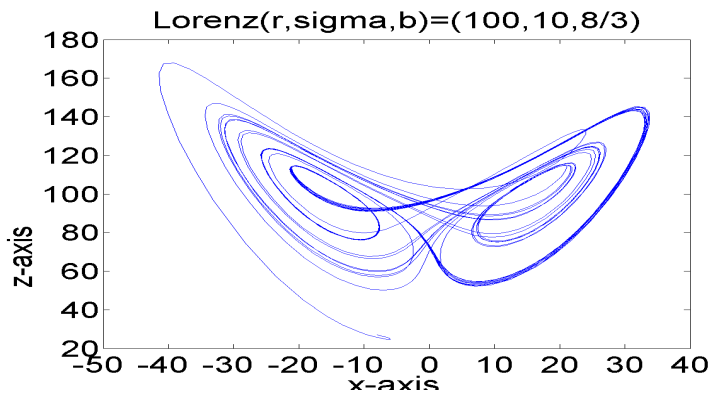


Figure 4.4: Bifurcation diagram for Lorenz equations at  $r=100$

The different bifurcations that take place as  $r$  increases from 0 to 28 and beyond is illustrated by figure (4.5)

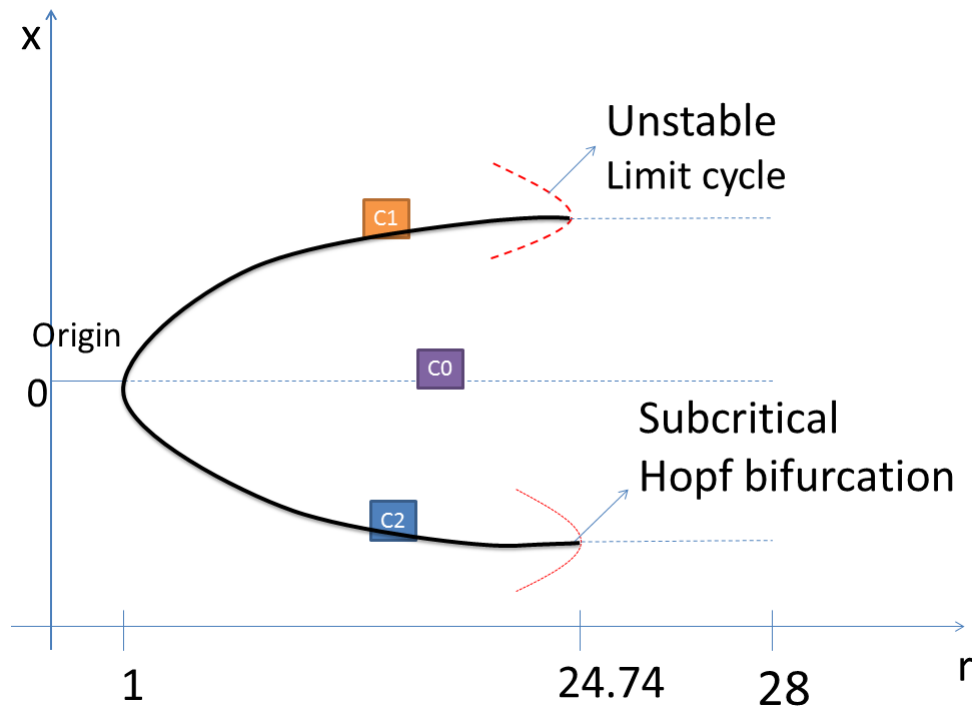


Figure 4.5: Illustration of different bifurcations as  $r$  increases

# Chapter 5

## CONCLUSION AND FURTHER STUDY

### 5.1 Conclusion

In this project our main objective was to understand the complexity of the climate modeling problem. In particular we wanted to apply the bifurcation theory to partial differential equations as represented by convective-diffusive models.

The highly complex climate system is really difficult to model as have been concluded by many researchers in this field. However several different classes of models have been developed over time by different scientists.

We present some of these model classes in order to underline the convective, advective, diffusive and radiative processes which form the core of boundary value problems which represent climate system.

Our focus here is the study of convective-radiative models for the earth-atmosphere system. In particular, we have looked at convection and its effect on climate change. The model used to this effect is the nonlinear Rayleigh-Benard convection after Benard (1900) and Rayleigh (1916).

They considered a fluid between two parallel plane: the height smaller compared to the horizontal distance and the fluid is heated from the bottom. The temperature difference between the two layers beyond a certain critical number called the Rayleigh number causes density difference and convection



onset.

This model becomes difficult to discretized by finite element method as was our first approach so we decided to use the approach used by Lorenz in [36]. By using double Fourier series method and a Galerkin truncation method to represent the Rayleigh-Benard equation, we obtain three system of ordinary differential equations after some useful simplification. This system obtained is what is known as the Lorenz system.

We then used the theory of bifurcation to study the dynamical changes in the Lorenz system of equations. This system of equations contains three parameters:  $r$  which represent the Rayleigh number,  $\kappa$  which represent Prandtl number and  $b$  which represent the wavenumber. While  $\kappa$  and  $b$  are constants, we consider different variation of  $r$ .

We have analyzed this system numerically using the eigenvalue-eigenvector approach. We first calculated the equilibrium points of the system then from the linearized system we were able to detect the stability of the different equilibrium points by computing their eigenvalues. We were also able to identify the different bifurcations as we varied  $r$ .

The bifurcations that we identified were super critical pitchfork bifurcation at the point  $r = 1$ , and subcritical hopf bifurcation at 24.74.

At  $r = 28$  a strange behavior of the system was noticed. The solution settled into an irregular oscillation that persisted as  $r$  increased but never repeated exactly. The solution was also very sensitive to initial conditions. This is what Lorenz called strange attractor.

In climate modeling,  $r$  represent the rate at which in the temperature increases between the equator and the poles which causes an increase in the heat flux represented by  $Z$  so the results of varying  $r$  is equivalent to the results of changes in the meridional temperature gradient between the equator and the pole.

The results obtained above shows that abrupt change in the dynamical system and the transitions in the systems are spontaneously caused by the dynamics of the system itself and not external disturbances.

This model provide an insight for numerical weather and climate prediction. Actually the abrupt change in the dynamical system shows that the weather conditions also changes abruptly from time to time and from zone to zone and therefore weather prediction is just limited to a few days.

In the objectives of this project, the last objective was to use numerical con-

tinuation to identify bifurcation points and branches of solutions. This is done by using softwares like AUTO and MATCONT package in Matlab as illustrated by [43, 44]. We tried to get any of this softwares but we did not succeed so therefore this objective was not achieved. However this has been left for further work to extend the work we already did.

## 5.2 Future research

In writing this project, I read extensively on this topic of Rayleigh-Benard convection. The main documents included the PHD thesis of Benjamin James Hepworth on Non-linear two dimensional Rayleigh-Benard convection presented in 2014 in the department of Applied Mathematics, University of Leeds. Benjamin considered a two dimensional non-linear Rayleigh-Benard convection in geophysical setting so for further work I intend to extend this work in the climate modeling context.

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