# HAZARD FUNCTIONS OF EXPONENTIAL MIXTURES AND THEIR LINK WITH MIXED POISSON DISTRIBUTIONS

**Moses Wamalwa Wakoli**

A thesis submitted to the University of Nairobi
for the award of the degree of
Doctor of Philosophy in Mathematical Statistics

**School of Mathematics**
**UNIVERSITY OF NAIROBI**

December 2015

# DECLARATION

This thesis is my original work and has not been presented in part or whole for a degree in any other University.

Moses Wamalwa Wakoli
Reg. No. I80/9211/06
Signature: .............................................. Date .........

## Approval

I/We declare that this work has been under my/our supervision and has my/our approval for submission

Prof. J.A.M Ottieno
School of Mathematics
University of Nairobi
P. O. Box 30196, GPO-00100
**NAIROBI**
Signature: .............................................. Date .........

Prof. M. M. Manene
School of Mathematics
University of Nairobi
P. O. Box 30196, GPO-00100
**NAIROBI**
Signature: .............................................. Date .........

Prof. G. P. Pokhariyal
School of Mathematics
University of Nairobi
P. O. Box 30196, GPO-00100
**NAIROBI**
Signature: .............................................. Date .........

# DEDICATION

God Almighty for being gracious and merciful.

# Table of Contents

## Abbreviations

**CPD** compound Poisson distribution

**pdf** probability density function

**pmf** probability mass function

## List of symbols

$\boldsymbol{N_t}$ Number of changes or failures in the interval [0, t].

$\boldsymbol{P_n(t)}$ Probability that the number of failures in the interval [0, t] are n.

$\boldsymbol{g(\lambda)}$ The mixing distribution

$\boldsymbol{f(x|\lambda)}$ The conditional probability.

$\boldsymbol{f(x)}$ The exponential mixture.

$\boldsymbol{s(x)}$ The survival function of the mixture.

$\boldsymbol{h(x)}$ The hazard function or failure rate.

$\boldsymbol{m(x)}$ Mean Excess Loss.

$\boldsymbol{E(X^r)}$ The rth moment of the mixture.

$\boldsymbol{H(s,t)}$ Generating function of the mixed Poisson distribution. also the generating function of the compound Poisson distribution.

$\boldsymbol{g(x)}$ pmf of the iids.

$\boldsymbol{\theta'(t)}$ Hazard function of an exponential mixture.

$\boldsymbol{\theta(t)}$ Cumulative hazard function of an exponential mixture.

$\boldsymbol{M(f(x),s)}$ Mellin transform of the function f(x).

$\boldsymbol{\Psi(a,c;x)}$ Tricomi confluent hyper-geometric function.

$\boldsymbol{{}_1F_1(a,c;x)}$ Kummer's confluent hyper-geometric function.

$\boldsymbol{\gamma(a,x)}$ Incomplete gamma function.

$\boldsymbol{L_\Lambda(t)}$ Laplace transform of the mixing distribution.

# ACKNOWLEDGMENT

First and foremost, I would like to sincerely thank my lead supervisor, Professor J. A. M. Ottieno, for the dedication and sacrifice he had to make; my other supervisors, Professor M. M. Manene (You were available whenever I needed guidance) and G. P. Pokhariyal (You were a great inspiration and you made it look easy all the time).

Professor Weke, the Director of School, for continuously encouraging me and asking for the progress report. It is not possible to quantify the impact these scholars have had on me, more specifically as a doctoral student. I am also very grateful to my fellow doctoral candidates; Rachael and Ida for the encouragement and support they provided, given that our areas of study were related and we shared a common lead supervisor.

I also wish to thank the management at the Technical University of Kenya, especially the Vice-Chancellor, Professor F.W.O Aduol, for the immense support. Lastly, but not least, I am greatly indebted to my family, for the patience, love and encouragement throughout the period of study.

# ABSTRACT

In this study the mixed Poisson distribution has been defined in terms of the hazard function of an exponential mixture. This work, therefore, has shown that there is a link between exponential and Poisson mixtures, such that a hazard function of an exponential mixture characterizes an infinitely divisible mixed Poisson distribution, which is also a compound Poisson distribution.

It has been established that since a hazard function of an exponential mixture is completely monotone, then the mixing distribution is infinitely divisible through Laplace transform; and a Poisson mixture with an infinitely divisible mixing distribution is infinitely divisible too. Further, an infinitely divisible mixed Poisson distribution is a compound Poisson distribution. The compound Poisson distribution has been constructed recursively in terms of the probability mass function (pmf) of the independent and identically distributed (i.i.d.) random variables and the hazard function of the exponential mixture. It has also been shown that a sum of hazard functions of exponential mixtures gives rise to a convolution of infinitely divisible Poisson mixtures, hence a convolution of compound Poisson distributions. Given the importance of hazard functions of exponential mixtures in the development of these models, the hazard functions have been constructed using continuous mixing distributions through probability density functions and survival functions, and using Laplace transforms of probability density functions in continuous compound distributions.

From the literature reviewed, it was found that there are mixing distributions that have been used in the construction of mixed Poisson distributions that are not part of the exponential mixtures literature. Type I and type II exponential mixtures, that are in explicit form, in terms of modified Bessel function of the third kind and in terms of confluent hyper-geometric function, have been constructed using these mixing distributions. Whereas hazard functions of some of the exponential mixtures constructed are single hazard functions, others are sums of hazard functions. However, the Mellin transform technique that was used to obtain moments failed in some cases and this necessitated the use of an alternative method, the conditional expectation technique. The models developed were applied to a class of mixed Poisson distribution known as Hofmann distributions to show the link between exponential and Poisson mixtures.

The tools or methodologies in this study include; special functions, which have been used to construct some mixing distributions and exponential mixtures; transformations, which have been used to obtain moments of the mixtures that are in terms of the modified Bessel function of the third kind and confluent hyper-geometric function; generating functions, which have been used to determine the corresponding mixed Poisson distribution and the pmf of the iid random variables; conditional expectation, which has been used as an alternative technique in cases where the Mellin transform fails.

Although the Hofmann hazard function is a good illustration of the theory,

there is need to consider other classes of hazard functions, particularly those based on frailty models. There is room for further research to identify other families of hazard functions of exponential mixtures, which are not necessarily members of the family of Hofmann distributions, and whose sums of hazard functions give rise to convolutions of Poisson mixtures.

# Chapter 1

# GENERAL INTRODUCTION

## 1.1 Background Information

Currently, the most common applications of the exponential distribution is in the field of life-testing, where the lifetime can be usefully represented by an exponential random variable with a relatively simple associated theory. Occasionally, this representation has been found to be inadequate, and hence a modification of the exponential distribution has been essential. A good example in the life-testing context is when specimens tested differ in their quality and subsequently their lifetimes do not follow an exponential distribution with a constant failure rate. In other words, since populations are not homogeneous, the appropriate distributions to handle such populations are mixtures, which are modeled by considering the exponential distribution parameter as a random variable.

In many applications, especially those for biological organisms and mechanical systems that wear out over time, the hazard rate $h(t)$ is a rate of change function also known as failure rate function in reliability engineering and the force of mortality in life contingency theory. The rate of change is a constant when the time until the next change is exponentially distributed and it is a function of $t$ in the case of exponential mixtures. It is an event or failure rate at time $t$ conditional on survival until time $t$ or later (that is, $T \geq 0$) and it is an increasing function of $t$. In other words, the older the life in question (the larger the $t$), the higher chance of failure at the next instant.

The hazard rate function can provide information about the tail of a distribution. If the hazard rate function is decreasing, it is an indication that the distribution has a heavy tail, thus, the distribution significantly puts more probability on larger values. Conversely, if the hazard rate function is increasing, it is an indication of a lighter tail. In an insurance context, heavy tailed distributions (for example the Pareto distribution) are suitable candidates for modeling large insurance losses.

If $\mathbf{N_t}$ is the number of changes or failures in the time interval $[0, t]$ and $T = t$ is the time until the occurrence of the first change, then $p_n(t) = Prob\ (N_t = n)$ is an infinitely divisible mixed Poisson distribution, which is also a compound Poisson distribution.

Basically, there are 3 (three) types of exponential mixtures; the finite mixture, which is achieved by taking $k$ different distributions with mixing weights; discrete mixtures which are achieved by considering discrete mixing distributions and continuous mixtures, which are constructed using continuous mixing distributions.

The continuous exponential mixtures can be categorized into type I and type II exponential mixtures and the difference between type I and type II exponential mixtures is that, whereas the mean of type I is the reciprocal of the parameter, that of type II is the parameter itself. The survival function of the type I mixture is the Laplace transform while that of the type II mixture is not.

## 1.2 Notations, terminologies and formulations

### Mixtures

Mathematically speaking, a mixture arises when a probability density function $\boldsymbol{f(x\,|\lambda)}$ depends on a parameter $\boldsymbol{\lambda}$ that is uncertain and is itself a random variable with density $\boldsymbol{g(\lambda)}$. Then taking the weighted average of $\boldsymbol{f(x\,|\lambda)}$ with $\boldsymbol{g(\lambda)}$ as weight produces the mixture distribution.
The pdf of a continuous mixture is given by

$$\boldsymbol{f(x) = \int_0^\infty f(x\,|\lambda)\, g(\lambda)\, d\lambda}$$

where,

$$\boldsymbol{f(x\,|\lambda)} = \text{the conditional pdf}$$

and,

$$\boldsymbol{g(\lambda)} = \text{the continuous mixing distribution}$$

The survival function is,

$$\boldsymbol{S(x) = \int_0^\infty S(x\,|\lambda)\, g(\lambda)\, d\lambda}$$

and the hazard function is

$$\boldsymbol{h(x) = \frac{f(x)}{S(x)}}$$

### 1.2.1 Exponential Mixtures

In this study we have categorized exponential mixtures into type I and type II mixtures. The type I exponential mixtures are such that:

$$\boldsymbol{f(x\,|\lambda) = \lambda\, e^{-\lambda\, x}; \quad x > 0;, \quad \lambda > 0}$$

and,

$$\boldsymbol{f(x) = \int_0^\infty \lambda\, e^{-\lambda\, x}\, g(\lambda)\, d\lambda} \tag{1.1}$$

The type II exponential mixtures, on the other hand, are such that:

$$\boldsymbol{f\,(x\,|\,\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad x \geq 0} \quad \text{for} \quad \boldsymbol{\lambda > 0}$$

and,

$$\boldsymbol{f\,(x) = \int_0^\infty \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}\, g\,(\lambda)\, d\lambda} \tag{1.2}$$

## 1.3 Literature review

### 1.3.1 Introduction

Although mixed Poisson distributions and exponential mixtures have been studied extensively, these studies have been done separately and the link between the two distributions has not been explored. In this section the literature on type I and type II exponential mixtures has been separated from that on mixed Poisson distributions.

### 1.3.2 Type I exponential mixtures

Mc Nolty et al. (1980) determined that when the distribution function of the random time-to-failure for any component is exponential, the hazard rate is a constant. However, in a population of such components there may be an ubiquitous variation in the parameter-values because of small fluctuations in manufacturing tolerances. Thus, a component selected at random can be regarded as having a random hazard rate. They considered an exponential-gamma distribution and stated that the hazard rate is decreasing, since the derivative of the hazard function is less than or equal to zero.

The exponential-gamma (Pareto) distribution was obtained in terms of survival function, $\boldsymbol{S(t)}$; probability density function, $\boldsymbol{f(t)}$ and hazard function, h(t), using the gamma mixing distribution. The mixed exponential functions, $\boldsymbol{S(t)}$, $\boldsymbol{f(t)}$ and $\boldsymbol{h(t)}$ were further expressed in terms of Laplace transform, Fourier transform, characteristic function and special functions.

*Hesselager et al. (1998)* studied exponential mixtures of type I with single hazard functions and those with a sum of two hazard functions and determined that whereas Pareto (exponential-gamma) and exponential-inverse Gaussian distributions have single hazard functions of exponential mixtures, exponential-shifted gamma is a mixture whose hazard function is the sum of a constant (which is a hazard function of an exponential distribution) and the hazard function of a Pareto. Benktander II distribution, on the other hand, is an exponential mixture whose hazard function is the sum of a hazard function of a Pareto and that of exponential-Hougaard distribution in which case, Benktander II distribution is based on mean excess loss function. Gampertz-Makeham distribution is an exponential mixture whose hazard function is a sum of a constant and a hazard function of Gompertz distribution which is an exponential mixture with Poisson mixing distribution. Hesselager also determined that exponential-Hougaard mixture is based on the Laplace transform of Hougaard distribution which is a Laplace transform of a compound Poisson distribution with gamma independent and identically distributed random variables.

*Drozdenko and Yadrenko (2012)* constructed type I exponential mixtures in terms of probability density functions, survival functions and hazard functions. Whereas the functions are in explicit form for gamma mixing distribution, for the inverse gamma and generalized inverse Gaussian (GIG) mixing distribution, the functions are in terms of the modified Bessel function of the third kind. The first method is based on Cauchy inequality, while the second is based on Block-Savich inequality. They further defined average residual time of life which is mean excess loss function and gave a formula for this mean excess loss function for an exponential mixture. Using Cauchy inequality they proved that the mean excess loss function for an exponential mixture is non-decreasing. Using the gamma distribution with two parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ they obtained the mean excess loss function for the exponential-gamma distribution.

### 1.3.3 Type II exponential mixtures

*Bhattacharya and Holla (1965)* constructed type II exponential distributions when the mixing distribution is distributed uniformly over a range $[\boldsymbol{\theta - \delta, \theta + \delta}]$ with $\boldsymbol{\theta > \delta > 0}$ and with beta II as the mixing distribution. They expressed the mixtures in terms of Tricomi confluent hypogeometric function.
They obtained the $\boldsymbol{r^{th}}$ moment of the mixtures directly using Mellin transform of Tricomi confluent hypogeometric function and also obtained the first four moments of the mixtures.
The model was used to show that if a set of individuals, exposed to an accident risk, are under observation in two successive periods of given length, and if the accident liabilities vary from individual to individual, then a selection of the individuals free from the accidents in the first period results in a decrease in the mean number of accidents in the second period.

*Bhattacharya (1966)* constructed type II exponential mixtures when the exponential distribution is the mixing distribution. The mixture was expressed in terms of modified Bessel function of the third kind. Using Mellin transform of modified Bessel function of third kind, Bhattacharya directly obtained the $\boldsymbol{r^{th}}$ moment of the exponential-exponential mixture. For the scaled Beta I mixing distribution, he expressed the pdf of the type II exponential mixture in terms of Tricomi confluent hypergeometric function through use of Riemann-Lienville integral and Whittaker function. They obtained the $\boldsymbol{r^{th}}$ moment of the mixture using the Laplace transform of a Tricomi confluent function.

They also used scaled Beta I as a mixing distribution and expressed the corresponding pdf in terms of Wittaker model and confluent hypergeometric function. The type II exponential mixture was obtained with an exponential mixing distribution and the mixture was expressed in terms of modified Bessel function of the third kind.

*Frangos and Vrontos (2001)* obtained Pareto II (Lomax) distribution by considering type II exponential mixture with inverse gamma mixing distribution and used the model in insurance claim to obtain severity component.

*Frangos and Karlis (2004)* derived a type II exponential mixture using a Generalized Inverse Gaussian (GIG) distribution as the mixing distribution and expressed the mixture in terms of modified Bessel function of the third kind.

Although they obtained the $r^{th}$ moment of the mixing distribution, thus the GIG, in terms of a ratio of modified Bessel functions of the third kind, they did not obtain the $r^{th}$ moment of the mixture. However, they fitted the model to data from a large Greek insurance company that was concerned with the size of car accident claims and provided an EM type algorithm to facilitate the estimation procedure.

*Nadarajah and Kotz (2006)* used sixteen mixing distributions to obtain the pdfs of type II exponential mixtures which were expressed in terms of the following special functions: Incomplete gamma function, Appell function of the first kind, modified Bessel function of the third kind, Tricomi confluent hypergeometric function, generalized hypergeometric function and Kummer's confluent hypergeometric function. The mixing distributions used are:- exponential, gamma, half logistic, inverse Gaussian, Weibull, Stacy, half-normal, Fréchet, Pareto, two sided power, Beta, inverted Beta, Lomax, generalized Pareto, Burr III, and Burr XII. They however did not obtain the $r^{th}$ moments for the mixtures.

Bhattacharya and his colleagues made a major breakthrough in studying type II exponential mixtures as early as in 1960's. Though their success has been attributed to their knowledge of special functions, they confined themselves to exponential, scaled beta I, uniform and Beta II mixing distributions.

### 1.3.4 Mixed Poisson distributions

*Walhin and Paris (1999)* defined a class of mixed Poisson distributions known as Hofmann distributions in terms of Laplace transform and determined that this class encompasses Poisson, negative binomial, Poisson-Inverse Gaussian, Polya-Aeppli and Neyman Type A distributions.
These distributions are infinitely divisible and hence are compound Poisson distributions. Parameterizations of this class have been obtained by Klugman et al. in 2008 and it is known as extended negative binomial distribution whereas Hougaard et al. called it Poisson-Hougaard distribution.

*Walhin and Paris (2002)* assumed that the cost for repairing a fault is given by a random variable X that is discrete and distributed along equidistant mass points. They also assumed that the realizations of X are independent and independent of N (t).
Their interest was in the probability distribution of the total cost $S(t) = X_1 + \cdots + X_{N(t)}$, where $N(t)$ is the number of flaws in a roll of length $t$. They concluded that the Hofmann process, convoluted with a Poisson process, may be of interest in order to analyze industrial problems where defects are counted on varying element sizes. The model they developed had interesting properties that could be given a physical interpretation in the case of data like that of Bissell (1972a). Moreover, it is tractable due to the recursions giving the probability function.

### 1.3.5 Summary of literature review

Exponential mixtures have been expressed in terms of survivor times, namely; $\boldsymbol{f(x)}$ (pdf), $\boldsymbol{s(x)}$ (survival functions) and $\boldsymbol{h(x)}$ (hazard function). The functions are explicit, in terms of the modified Bessel function of the third kind and in terms of confluent hyper-geometric function. Moments of the mixtures have been obtained using Mellin transform technique. Hazard functions of exponential mixtures have been constructed through Laplace transform and mean excess loss. Some of the hazard functions are single while others are sums of individual hazard functions. Mixed Poisson distributions have been expressed in terms of Laplace transforms.

Some mixing distributions have been obtained by considering sums of independent random variables. In one case the number of the random variables is fixed and in another case it is also a random variable. In other words, Laplace transforms have been obtained through compound Poisson distributions and through convolutions of random variables.

The derivative of a function has been used in the definition of the mixed Poisson distribution, and this seems to be the link between exponential and Poisson mixtures that needs to be explored. Mixed Poisson distributions and exponential mixtures have been studied extensively but separate from each other. Nonetheless, there are a number of mixing distributions that do not form part of the exponential mixtures literature but they have been used in the construction of mixed Poisson distributions .

### 1.3.6 The framework

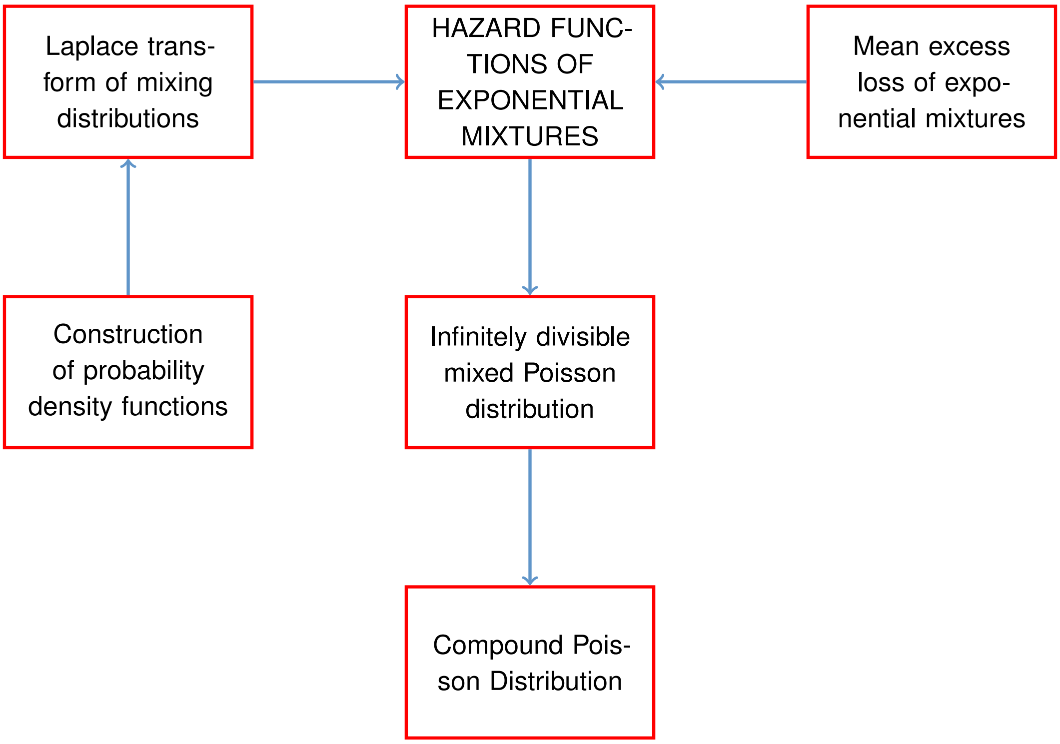


Figure 1.1: The link between exponential and Poisson mixtures

## 1.4 The Research problem

Walhin and Paris (1999) defined a class of mixed Poisson distribution in terms of the derivative of some function. While re-examining this definition, we realized that this derivative is a hazard function of type 1 exponential mixture. Hence there is a link between mixed Poisson distributions and exponential mixtures. Although both mixed Poisson distributions and exponential mixtures have been studied extensively, this link has not been explored.

There are many mixing distributions that have been used in the study of mixed Poisson distributions that are not part of the exponential mixtures literature. For this link to be explored effectively, there is need to construct exponential mixtures using these mixing distributions.

Hazard functions of exponential mixtures through Laplace transform and mean excess loss function can be single or sums of individual hazard functions of exponential mixtures. The link between these hazard functions and mixed Poisson distributions has also not been studied. From literature, moments of exponential mixtures have been obtained using Mellin transform technique, however the technique fails in some cases. There is need for a more robust method of obtaining moments of exponential mixtures that would serve an alternative technique to the Mellin transform.

Laplace transform of a compound Poisson distribution for gamma independent and identically distributed random variables was considered by Aallen (1992). However, the corresponding mixed Poisson distribution has not been explored and hence the need to study compound Poisson distributions with continuous iid random variables.

## 1.5 Objectives

The main objective of this work is to explore the derivative of the function in the definition of mixed Poisson distribution by Walhin and Paris.

### 1.5.1 Specific objectives

The specific objectives are to:

1. construct and derive moments of exponential mixtures.

2. show that single hazard functions of type I exponential mixtures characterize infinitely divisible mixed Poisson distributions which are also compound Poisson distributions.

3. show that sums of hazard functions of exponential mixtures characterize convolutions of infinitely divisible mixed Poisson distributions which are also convolutions of compound Poisson distributions.

4. obtain sums of hazard functions of exponential mixtures using Laplace transforms of sums of independent continuous random variables and associated mixed Poisson distributions.

5. obtain hazard functions of type I exponential mixtures using Laplace transforms of convolution of independent non-identical chi-squared random variables and associated mixed Poisson distributions.

6. obtain sums of hazard functions of exponential mixtures based on mean excess loss function of Benktander II (exponential-Hougaard) distribution and associated mixed Poisson distributions.

## 1.6 Research Methodologies

### 1.6.1 Introduction

Various mathematical tools have been used to achieve the above objectives and they include: generating functions, special functions, transforms, powers series expansions, inequalities and recurrence relations.

### 1.6.2 Special functions

The Beta I function

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx; \quad a>0,\ b>0 \tag{1.3}$$

The Beta II function

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b-1}}\, dt \tag{1.4}$$

$$B(a,b+1) = \frac{b}{a+b}\, B(a,b)$$

The Gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\, e^{-t}\, dt, \quad \alpha>0 \tag{1.5}$$

$$\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha)$$

$$\int_0^\infty e^{-\beta\, t}\, t^{\alpha-1}\, dt = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

$$B(a,b) = \frac{\Gamma(a)\, \Gamma b}{\Gamma(a+b)}$$

Modified Bessel function of the third kind

$$K_v(\omega) = \frac{1}{2}\int_0^\infty x^{v-1}\, e^{-\frac{\omega}{2}(x+\frac{1}{x})} dx \tag{1.6}$$

Some properties are:-

$$K_v(\omega) = K_{-v}\,(\omega)$$

$$K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}\, e^{-\omega} \tag{1.7}$$

$$K_{\frac{3}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}\, e^{-\omega}\, [1 + \frac{1}{\omega}] \tag{1.8}$$

**Confluent hyper-geometric function** (1.9)

1. Kummer's confluent Hyper-geometric function

$$ {}_1F_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{(c-a)-1}\, e^{ux}\, du \quad c > a > 0 \tag{1.10}$$

2. Tricomi confluent hyper-geometric function

$$\Psi(a;c;x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{(c-a)-1}\, e^{-xt}\, dt \quad a, b, > 0 \tag{1.11}$$

The following relations hold

$$\Psi(a,c;x) = x^{1-c}\Psi(1+a-c, 2-c;x) \tag{1.12}$$

$$\Psi(a;c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1(a;c;x) + \frac{\Gamma(c-1)}{\Gamma(a)}\, x^{1-c}\, {}_1F_1(a-c+1; 2-c; x) \tag{1.13}$$

The incomplete gamma function is defined by:

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \tag{1.14}$$

$$= a^{-1}x^a\, e^{-x}\, {}_1F_1(a; a+1; x) \tag{1.15}$$

Special functions have been useful in the construction of mixing distributions.

### 1.6.3 Transforms

1. Probability generating functions: have been used to determine the corresponding mixed Poisson distributions

2. Laplace transform: has provided the link between exponential mixtures and mixed Poisson distributions

3. Mellin transform: has been used to obtain moments of exponential mixtures that are based on special functions.

Generating functions

The pgf of a random variable $\boldsymbol{X}$ is given by

$$\begin{aligned} G(s) &= \sum_{k=0}^{\infty} Prob\,(X = k)\; s^k \\ &= E(S^k) \end{aligned} \tag{1.16}$$

The Laplace transform

$$L_X(s) = \int_0^{\infty} e^{-sx}\; f(x)\; dx \tag{1.17}$$

Mellin transform of a modified Bessel function of the third kind

$$\int_0^{\infty} \omega^{s-1}\; K_v\,(\omega)\, d\omega = 2^{s-2}\,\Gamma\left(\frac{s+v}{2}\right)\;\Gamma\left(\frac{s-v}{2}\right) \tag{1.18}$$

Proof

$$M\,\{f(x); s\} = \int_0^{\infty} x^{s-1}\; f(x)\; dx$$

$$\int_0^{\infty} \omega^{s-1}\; K_v\,(\omega)\, d\omega = \int_0^{\infty} \omega^{s-1}\; \left\{\frac{1}{2}\int_0^{\infty} x^{v-1} e^{-\frac{\omega}{2}(x+\frac{1}{x})} dx\right\} d\omega$$

$$\begin{aligned} &= \frac{1}{2}\int_0^{\infty} x^{v-1}\; \left\{\int_0^{\infty} \omega^{s-1}\; e^{-\frac{\omega}{2}(x+\frac{1}{x})} d\omega\right\} dx \\ &= \frac{1}{2}\int_0^{\infty} x^{v-1}\; \frac{\Gamma(s)}{\left[\frac{1}{2}(x+\frac{1}{x})\right]^s} dx \\ &= \frac{1}{2}\,\Gamma(s)\int_0^{\infty} \frac{x^{v-1}}{\left[\frac{1}{2}(x+\frac{1}{x})\right]^s} dx \\ &= 2^{s-1}\,\Gamma(s)\int_0^{\infty} \frac{x^{s+v-1}}{[x^2+1]^s} dx \\ &= 2^{s-2}\,\Gamma(s)\int_0^{\infty} \frac{x^{s+v-2}}{[x^2+1]^s}\; 2x\; dx \\ &= 2^{s-2}\,\Gamma(s)\int_0^{\infty} \frac{(x^2)^{\frac{s+v}{2}-1}}{[x^2+1]^s}\; 2x\; dx \end{aligned}$$

Now let

$$t = x^2 \quad \text{in which case} \quad dt = 2x\; dx$$

Therefore,

$$\int_{0}^{\infty} \omega^{s-1}\, K_v(\omega)\, d\omega = 2^{s-2}\, \Gamma(s) \int_{0}^{\infty} \frac{t^{\frac{s+v}{2}-1}}{[t+1]^{s}}\, dt$$

$$= 2^{s-2}\, \Gamma(s)\, B(\frac{s+v}{2}, \frac{s-v}{2})$$
$$= 2^{s-2}\, \Gamma\left(\frac{s+v}{2}\right) \Gamma\left(\frac{s-v}{2}\right)$$

Mellin Transform of Tricomi Confluent hyper-geometric function

$$\int_{0}^{\infty} \omega^{s-1}\, \Psi(a;\, c;\, x) dx = \frac{\Gamma s}{\Gamma(a)}\, \frac{\Gamma(a-s)\, \Gamma(s-c+1)}{\Gamma(a-c+1)} \tag{1.19}$$

Proof

$$\int_{0}^{\infty} \omega^{s-1}\, \Psi(a;\, c;\, x) dx = \int_{x=0}^{\infty} x^{s-1} \int_{t=0}^{\infty} \frac{1}{\Gamma(a)} t^{a-1}(1+t)^{c-a-1}\, e^{-xt} dt\, dx$$

$$= \int_{t=0}^{\infty} \frac{1}{\Gamma(a)} t^{a-1}(1+t)^{c-a-1} [\int_{x=0}^{\infty} x^{s-1}\, e^{-xt} dx]\, dt$$
$$= \int_{t=0}^{\infty} \frac{1}{\Gamma(a)} t^{a-1}(1+t)^{c-a-1}\, \frac{\Gamma s}{t^{s}}\, dt$$
$$= \int_{t=0}^{\infty} \frac{\Gamma s}{\Gamma(a)} t^{a-s-1}(1+t)^{c-a-1}\, dt$$
$$= \frac{\Gamma s}{\Gamma(a)} \int_{t=0}^{\infty} \frac{t^{a-s-1}}{(1+t)^{a-c+1}}\, dt$$
$$= \frac{\Gamma s}{\Gamma(a)} \int_{t=0}^{\infty} \frac{t^{a-s-1}}{(1+t)^{(a-s)+s-c+1}}\, dt$$
$$= \frac{\Gamma s}{\Gamma(a)}\, B(a-s, s-c+1)$$
$$= \frac{\Gamma s}{\Gamma(a)}\, \frac{\Gamma(a-s)\, \Gamma(s-c+1)}{\Gamma(a-c+1)}$$

and

$$M[e^{-x}\Psi(a;c;x), s] = \frac{\Gamma(s)\, \Gamma(s-c+1)}{\Gamma(a+s-c+1)} \tag{1.20}$$

Proof

$$M[e^{-x}\Psi(a;c;x),s] = \int_0^\infty x^{s-1}\; e^{-x}\Psi(a;c;x)\;dx$$

$$= \int_{x=0}^\infty x^{s-1} \int_{t=0}^\infty \frac{1}{\Gamma(a)} t^{a-1}(1+t)^{c-a-1}\; e^{-xt}\; e^{-x} dt\; dx$$
$$= \int_{t=0}^\infty \left\{ \frac{1}{\Gamma(a)} t^{a-1}(1+t)^{c-a-1} \int_{x=0}^\infty x^{s-1}\; e^{-x(t+1)}\; dx \right\} dt$$
$$= \int_{t=0}^\infty \frac{1}{\Gamma(a)} t^{a-1}(1+t)^{c-a-1}\; \frac{\Gamma s}{(1+t)^s}\; dt$$
$$= \int_{t=0}^\infty \frac{\Gamma s}{\Gamma(a)} t^{a-1}(1+t)^{c-a-s-1}\;\; dt$$
$$= \frac{\Gamma s}{\Gamma(a)} \int_{t=0}^\infty \frac{t^{(a-1)}}{(1+t)^{a+s-c+1}}\; dt$$
$$= \frac{\Gamma s}{\Gamma(a)}\; B(a, s-c+1)$$
$$= \frac{\Gamma s}{\Gamma(a)}\; \frac{\Gamma(a)\Gamma(s-c+1)}{\Gamma(a+s-c+1)}$$
$$= \frac{\Gamma(s)\;\Gamma(s-c+1)}{\Gamma(a+s-c+1)}$$

The Mellin transform of Kummer's confluent hyper-geometric function

$$\int_0^\infty x^{s-1}\;{}_1F_1(a;c;-x)\;dx = \frac{\Gamma(c)\;\Gamma s}{\Gamma(c-s)}\;\frac{\Gamma(c-a-s)}{\Gamma(c-a)} \tag{1.21}$$

Proof

$$\int_0^\infty x^{s-1}\;{}_1F_1(a;c;-x)\;dx = \int_0^\infty \left\{ x^{s-1} \int_0^1 \frac{t^{a-1}\;(1-t)^{c-a-1}}{B(a,c-a)}\; e^{-xt}\; dt \right\} dx$$

$$= \frac{1}{B(a,c-a)} \int_0^1 \left\{ \int_0^\infty x^{s-1} e^{-xt} dx \right\} t^{a-1}\;(1-t)^{c-a-1} dt$$
$$= \frac{1}{B(a,c-a)} \int_0^1 \frac{\Gamma s}{t^s}\; t^{a-1}\;(1-t)^{c-a-1} dt$$
$$= \frac{\Gamma s}{B(a,c-a)} \int_0^1 t^{a-s-1}\;(1-t)^{c-a-1} dt$$
$$= \frac{\Gamma s}{B(a,c-a)}\; B(a-s, c-a)$$

$$= \frac{\Gamma(c)\,\Gamma s}{\Gamma(c-s)}\,\frac{\Gamma(a-s)}{\Gamma(a)}$$

$$\int_0^\infty x^{s-1}\, e^{-x}\, {}_1F_1(a;c;x)\, dx = \frac{\Gamma(c)\,\Gamma s}{\Gamma(c-s)}\,\frac{\Gamma(c-a-s)}{\Gamma(c-a)} \tag{1.22}$$

Proof

$$\int_0^\infty x^{s-1}\, e^{-x}\, {}_1F_1(a;c;x)\, dx = \int_0^\infty \left\{ x^{s-1}\, e^{-x} \int_0^1 \frac{t^{a-1}\,(1-t)^{c-a-1}}{B(a,c-a)}\, e^{xt}\, dt \right\} dx$$

$$\begin{aligned}
&= \frac{1}{B(a,c-a)} \int_0^1 \left\{ \int_0^\infty x^{s-1} e^{-(1-t)x} dx \right\} t^{a-1}\,(1-t)^{c-a-1} dt \\
&= \frac{1}{B(a,c-a)} \int_0^1 \frac{\Gamma s}{(1-t)^s}\, t^{a-1}\,(1-t)^{c-a-1} dt \\
&= \frac{\Gamma s}{B(a,c-a)} \int_0^1 t^{a-1}\,(1-t)^{c-s-a-1} dt \\
&= \frac{\Gamma s}{B(a,c-a)}\, B(a,c-a-s) \\
&= \Gamma s\, \frac{B(a,c-a-s)}{B(a,c-a)} \\
&= \frac{\Gamma(c)\,\Gamma s}{\Gamma(c-s)}\,\frac{\Gamma(c-a-s)}{\Gamma(c-a)}
\end{aligned}$$

The Mellin transform technique has also been useful as a direct method of obtaining the $r^{th}$ moment of the mixtures.

### 1.6.4 Inequalities

The exponential mixtures are non-increasing functions and the Cauchy-Schwartz and Block-Sarich inequalities have been used to to prove this result.

#### Cauchy-Schwartz Inequality

**Lemma 1.1.** *If $X$ and $Y$ are random variables such that $E(X^2)$ and $E(Y^2)$ exist then*

$$(E(XY))^2 \leq E\left(X^2\right) E\left(Y^2\right)$$

*(Johnson, Kemp and Kotz, 2005, pp 49)*

The Cauchy-Schwartz inequality in lemma 1.1 can be used to prove that the hazard function of an exponential mixture is a non-increasing function as follows:

$$\int_0^\infty \lambda e^{-\lambda x} g(\lambda) d\lambda = \int_0^\infty \lambda e^{-\frac{\lambda x}{2}} e^{-\frac{\lambda x}{2}} g(\lambda) d\lambda$$

$$= \boldsymbol{E}\left[\lambda\, e^{-\frac{\lambda x}{2}}\, e^{-\frac{\lambda x}{2}}\right]$$
$$= \boldsymbol{E}(XY)$$

where,

$$X = \lambda\, e^{-\frac{\lambda x}{2}}$$

and,

$$Y = e^{-\frac{\lambda x}{2}}$$

Therefore,

$$\left[\int_0^\infty \lambda e^{-\lambda x} g(\lambda) d\lambda\right]^2 = [E(XY)]^2 \leq E(X^2)\, E(Y^2)$$

That is

$$\left[\int_0^\infty \lambda e^{-\lambda x} g(\lambda) d\lambda\right]^2 \leq \int_0^\infty \left[\lambda e^{-\frac{\lambda x}{2}}\right]^2 d\lambda \int_0^\infty \left[e^{-\frac{\lambda x}{2}}\right]^2 d\lambda$$

$$\left[\int_0^\infty \lambda e^{-\lambda x} g(\lambda) d\lambda\right]^2 \leq \left[\int_0^\infty \lambda^2\, e^{-\lambda x}\, g(\lambda) d\lambda\right] \left[\int_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda\right]$$

But the hazard function of type I exponential mixture is given by

$$h(x) = \frac{f(x)}{S(x)}$$
$$= \frac{\int_0^\infty \lambda\, e^{-\lambda x}\, g(\lambda) d\lambda}{\int_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda}$$

$$\therefore \qquad h'(x) = \frac{d\, h(x)}{dx}$$
$$= \frac{\left[\int_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda\right] \left[-\int_0^\infty \lambda^2\, e^{-\lambda x}\, g(\lambda) d\lambda\right]}{\left[\int_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda\right]^2} -$$
$$\frac{\left[\int_0^\infty \lambda\, e^{-\lambda x}\, g(\lambda) d\lambda\right] \left[-\int_0^\infty \lambda\, e^{-\lambda x}\, g(\lambda) d\lambda\right]}{\left[\int_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda\right]^2}$$
$$= \frac{\left[\int_0^\infty \lambda\, e^{-\lambda x}\, g(\lambda)\, d\lambda\right]^2 - \left[\int_0^\infty e^{-\lambda x}\, g(\lambda)\, d\lambda\right] \left[\int_0^\infty \lambda^2\, e^{-\lambda x}\, g(\lambda) d\lambda\right]}{\left[\int_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda\right]^2}$$

$$\leq \frac{-2\left[\int\limits_0^\infty \lambda^2\, e^{-\lambda x}\, g(\lambda)\, d\lambda\right]\left[\int\limits_0^\infty e^{-\lambda x}\, g(\lambda)\, d\lambda\right]}{\left[\int\limits_0^\infty e^{-\lambda x}\, g(\lambda) d\lambda\right]^2}$$

$$h'(x) \leq 0.$$

Thus $h(x)$ is a non-increasing function.

**Block-Sarich Inequality**

An alternative approach of showing that the hazard function of an exponential mixture is a decreasing function is to use Poisson transform of a function $\phi(x)$ defined as

$$a_n(s) = \int\limits_0^\infty \frac{e^{-sx}x^n}{n!}\phi(x)dx$$

The theorem by Block-Sarich, quoted by Drzdenko and Yadrenko (2012), is as follows:

(1.23)

**Theorem 1.1.**

If for all $s \geq 0$ and $n \geq 0$

$$a_n^2(s) \leq a_{n+1}(s)a_{n-1}(s) \tag{1.24}$$

then the distribution $F(x)$ has a decreasing failure rate function.

For type I exponential mixture

$$\begin{aligned} a_n(s) &= \int\limits_0^\infty \frac{e^{-sx}x^n}{n!}s(x)dx \\ &= \int\limits_0^\infty \left\{\frac{e^{-sx}x^n}{n!}\int\limits_0^\infty e^{-\lambda x}g(\lambda)d\lambda)\right\}dx \\ &= \int\limits_0^\infty \left\{\int\limits_0^\infty e^{-(s+\lambda)x}x^n]\right\}dx\frac{g(\lambda)}{n!}d\lambda \\ &= \int\limits_0^\infty \frac{\Gamma(n+1)}{(s+\lambda)^{n+1}}\frac{g(\lambda)}{n!}d\lambda \\ &= \int\limits_0^\infty \frac{1}{(s+\lambda)^{n+1}}g(\lambda)d\lambda \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{(s+\lambda)^{\frac{n}{2}}} \frac{1}{(s+\lambda)^{\frac{n}{2}+1}} g(\lambda) d\lambda \\
&= E(XY)
\end{aligned}
$$

where,

$$
\begin{aligned}
X &= \frac{1}{(s+\lambda)^{\frac{n}{2}}} \\
Y &= \frac{1}{(s+\lambda)^{\frac{n}{2}+1}} \\
a_n^2(s) &= [E(XY)]^2 \leq E(X^2)\ E(Y^2)
\end{aligned}
$$

by Cauchy-Schwartz's inequalities

$$
\begin{aligned}
a_n^2(s) &\leq \int_0^\infty \left[\frac{1}{(s+\lambda)^{\frac{n}{2}}}\right]^2 g(\lambda) d\lambda \int_0^\infty \left[\frac{1}{(s+\lambda)^{\frac{n}{2}+1}}\right]^2 g(\lambda) d\lambda \\
\therefore \qquad a_n^2(s) &\leq \int_0^\infty \left[\frac{1}{(s+\lambda)^{n}}\right] g(\lambda) d\lambda \int_0^\infty \left[\frac{1}{(s+\lambda)^{n+2}}\right] g(\lambda) d\lambda \\
a_n^2(s) &\leq a_{n-1}(s)\ a_{n+1}(s)
\end{aligned}
$$

Hence $F(x)$ has a decreasing failure rate function

.

### 1.6.5 Recurrence relation

This is when the $n^{th}$ term of a probability mass function is expressed in in terms of one or more lower terms. Panjer's recursive relations is one such forms and it is given by:

$$
p_n = \left(a + \frac{b}{n}\, p_{n-1}\right) \quad n = 1, 2, 3, \ldots \tag{1.25}
$$

where $a$ and $b$ are real numbers.

This recursive model is known as Panjer's recursive model of class zero denoted by $(a, b, 0)$ class of distributions.

We can extend it to

$$
p_n = \left(a + \frac{b}{n}\right)\, p_{n-1}) \quad n = 2, 3, \ldots
$$

which is Panjer's $(a, b, 1)$ class.

In general, we have

$$\boldsymbol{p_n = \left(a + \frac{b}{n}\, p_{n-1}\right) \quad n = k+1, k+2, \ldots}$$

which is Panjer's $\boldsymbol{(a, b, k)}$ class for $\boldsymbol{k = 0, 1, 2, \ldots}$
Some probability mass functions in chapter 4 take Panjer's recursive form.
In actuarial literature, this recursive relation is due to the original work of Panjer (1981). In statistical literature, however, this relation had been published by Katz (1965) based on his PhD dissertation of 1945.
In this study, compound Poisson distributions have been obtained recursively in terms of the pmf of the independent and identically distributed random variables and the hazard functions of the exponential mixtures.

### 1.6.6 Conditional expectation

$$\boldsymbol{\begin{aligned} E(X^r) &= E\, E(X^r|\Lambda = \lambda) \\ E(X) &= E\, E(X|\Lambda = \lambda) \\ Var(X) &= Var\, E(X|\Lambda) + E\, Var(X|\Lambda) \end{aligned}}$$

## 1.7 Significance of the study

The study of probability distributions is one major area of statistics that involves; construction of distributions, determination of their properties, estimation of parameters and application of the models developed.

Various types of data have emerged that do not fit the known distributions and it is precisely for this reason that alternative methods for constructing distributions have been proposed and utilized. One such method is the mixtures and this work is a contribution to this area. In practice, mixtures have been found useful in actuarial data and financial mathematics and more specifically mixed Poisson distributions and exponential mixtures have been used in the study of insurance claims. For example, the exponential-inverse gamma mixture (Pareto distribution) has been used to model the size component of insurance claims by letting $\boldsymbol{X}$ to be the size of claim by each insured and $\boldsymbol{Y}$ the mean claim size of each insured such that the conditional distribution is $\boldsymbol{f(x|y)}$. Since the mean claim size is different for the different policyholders, the prior distribution for $\boldsymbol{Y}$ can be the inverse gamma.

The most common applications of the exponential distribution is in the field of life-testing, where the lifetime can be usefully represented by an exponential random variable with a relatively simple associated theory. Occasionally, this representation has been found to be inadequate, and hence a modification of the exponential distribution has been essential. In many applications, the hazard rate $\boldsymbol{h(t)}$ is a rate of change function also known as failure rate function in reliability engineering and the force of mortality in life contingency theory. The rate of change is a constant when the time until the next change is exponentially distributed and it is a function of $\boldsymbol{t}$ in the case of exponential mixtures. If $\mathbf{N_t}$ is the number of changes or failures in the time interval $[\boldsymbol{0,t}]$ and $\boldsymbol{T=t}$ is the time until the occurrence of the first change, then $\boldsymbol{p_n(t) = Prob\ (N_t = n)}$ is an infinitely divisible mixed Poisson distribution, which is also a compound Poisson distribution (CPD).

In this study, the Laplace transform of the mixing distributions has been used to obtain the probability density functions, the survival functions and the hazard functions of exponential mixtures. Therefore, Laplace transforms of mixing distributions characterize exponential type I mixtures and hazard functions of an exponential mixtures give rise to infinitely divisible mixed Poisson distributions that are also compound Poisson distributions.

Moments of exponential mixtures have been constructed and specifically the mean of an exponential mixture, which is useful in the construction of the mixing distribution of an equilibrium distribution of an exponential mixture.

The type II exponential-generalized inverse Gaussian distribution by Frangos and Karlis (2004) was fitted to data from a large Greek insurance company that was concerned with the size of car accident claims.

## 1.8 Outline of the thesis

Chapter 1 gives the background of the study, notations, terminologies and formulations, literature review, research problem, objectives, research methodologies, significance of the study and outline of the thesis.

In chapter 2, type I exponential mixtures have been constructed and their properties obtained. It has also been determined that some hazard functions are sums of individual hazard functions. In chapter 3, type II exponential mixtures have been constructed and their properties obtained.

Chapter 4 defines mixed Poisson distribution in terms of hazard function of the exponential mixture. In chapter 5 the link between sums of hazard functions of exponential mixtures and convolutions of infinitely divisible mixed Poisson distributions which are also convolutions of compound Poisson distributions has been explored.

In chapter 6, Laplace transforms of sums of independent continuous random variables and hazard functions using mean excess loss have been obtained. The link between these Laplace transforms and the hazard functions with Poisson mixtures has been explored. Chapter 7 has concluding remarks.

# Chapter 2

# TYPE I EXPONENTIAL MIXTURES AND THEIR PROPERTIES

## 2.1 Introduction

From literature, common mixing distributions used in constructing type I exponential mixtures are: gamma, inverse gamma, inverse Gaussian and Generalized Inverse Gaussian (GIG). The mixture is expressed explicitly in the case of gamma mixing distribution and in terms of modified Bessel function of the third kind in the remaining 3 cases.

There are mixing distributions that have been used in constructing mixed Poisson distributions that are not in the exponential mixtures literature. It also seems that little attention has been given to type I exponential mixtures expressed in terms of confluent hyper-geometric functions.

Laplace transform, Mellin transform, Fourier transform and characteristic function have been related to type I mixed exponential function. For the gamma mixing distribution the probability density function, survival function and hazard function of the mixture have been expressed in terms of Mellin transform. Although the integral transforms have been used for construction of mixtures, they have not been used for obtaining moments of type I exponential mixtures. Hesselager et al. (1997) constructed the hazard function of exponential-Inverse Gaussian and stated that it is one of the special cases of the exponential-Generalized Inverse Gaussian distribution. However, they did not consider other special cases.

To fill up some of these gaps, the objectives of this chapter are; to construct type I exponential mixtures in explicit form, in terms of the Bessel function of the third kind, and in terms of confluent hyper-geometric functions using twenty six (26) mixing distributions; to obtain moments directly using Mellin transform and indirectly using conditional expectation approach; to derive special cases of type I exponential mixtures for Generalized Inverse Gaussian and Generalized Pareto mixing distributions

Hazard functions of type I exponential mixtures are a useful link between exponential and Poisson mixtures and this link has been explored extensively in chapter 4.

## 2.2 The problem in mathematical form

**Remark 2.1.** *The Laplace transform of a distribution characterizes type I exponential mixture.*

Let,

$$\boldsymbol{f(x\,|\lambda) = \lambda\, e^{-\lambda\, x}; \quad x > 0;, \quad \lambda > 0}$$

be the conditional exponential distribution with mean being the reciprocal of the parameter $\boldsymbol{\Lambda}$.

Then,

$$\boldsymbol{f(x) = \int\limits_0^\infty \lambda\, e^{-\lambda\, x}\, g(\lambda)\, d\lambda}$$

is the type I exponential mixture with $\boldsymbol{g(\lambda)}$ being the mixing distribution.

The survival function of type I exponential mixture is given by

$$\begin{aligned} \boldsymbol{S(x)} &= \boldsymbol{\int\limits_0^\infty S(x\,|\lambda)\, g(\lambda)\, d\lambda} \\ &= \boldsymbol{\int\limits_0^\infty e^{-\lambda x}\;\; g(\lambda) d\lambda} \\ &= \boldsymbol{E\left[e^{-\lambda x}\right]} \\ &= \boldsymbol{L_\Lambda(x)} \end{aligned} \tag{2.1}$$

which is the Laplace transform of the mixing distribution.

The hazard function of type I exponential mixture is

$$\begin{aligned} \boldsymbol{h(x)} &= \boldsymbol{\frac{f(x)}{S(x)}} \\ &= \boldsymbol{-\frac{1}{S(x)}\,\frac{d\,S(x)}{dx}} \\ &= \boldsymbol{-\frac{L'_\Lambda(x)}{L_\Lambda(x)}} \end{aligned} \tag{2.2}$$

The $\boldsymbol{rth}$ moment of type I exponential mixture can be obtained directly.

Directly,

$$\boldsymbol{E(X^r) = \int_0^{\infty} x^r \, f(x) \, dx} \tag{2.3}$$

This will require the Mellin transform of the Bessel function of the third kind and Mellin transform of the confluent hyper-geometric function.

Alternatively, the conditional expectations approach can be used as given in the following lemma

Indirectly,

**Lemma 2.1.** *The $\boldsymbol{r^{th}}$ moment of the type I exponential mixture is r factorial times the $\boldsymbol{r^{th}}$ moment of the reciprocal of the mixing distribution, i.e.*

$$\boldsymbol{E(X^r) = r! \, E\left[\frac{1}{\Lambda^r}\right]} \tag{2.4}$$

Proof

$$\boldsymbol{E(X^r) = E \; E(X^r|\Lambda = \lambda)}$$

where

$$\begin{aligned}
\boldsymbol{E(X^r|\Lambda = \lambda)} &= \boldsymbol{\int_0^{\infty} x^r \, f(x|\lambda) \, dx} \\
&= \boldsymbol{\int_0^{\infty} x^r \, \lambda \, e^{-\lambda x} \, dx} \\
&= \boldsymbol{\lambda \int_0^{\infty} x^r \, e^{-\lambda x} \, dx} \\
&= \boldsymbol{\lambda \int_0^{\infty} x^{(r+1)-1} \, e^{-\lambda x} \, dx} \\
&= \boldsymbol{\lambda \frac{\Gamma(r+1)}{\lambda^{r+1}}} \\
&= \boldsymbol{\frac{r!}{\lambda^r}} \\
\therefore \quad \boldsymbol{E(X^r)} &= \boldsymbol{r! \, E\left[\frac{1}{\Lambda^r}\right]}
\end{aligned}$$

where $\boldsymbol{\Lambda}$ is the random variable taking the value $\boldsymbol{\lambda}$.

The mixtures can be categorized into those that are explicit in form, in terms of modified Bessel function of the third kind and in terms of confluent hyper-geometric functions.

## 2.3 Mixtures in explicit form

In this section, mixtures in explicit form have been constructed using various mixing distributions and moments have been obtained using both the direct technique and the conditional expectation approach.

### 2.3.1 Exponential mixing distribution

In the case of the exponential mixing distribution, the mixture which is the exponential-exponential distribution which is also known as Pareto II (Lomax) distribution with parameter $\boldsymbol{\beta}$.

$$g(\lambda) = \beta e^{-\beta\,\lambda} \tag{2.5}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^\infty \lambda e^{-\lambda x}\ \ \beta e^{-\beta\,\lambda} d\lambda \\ &= \beta \int_0^\infty \lambda^{2-1}\, e^{-\lambda\,(x+\beta)}\, d\lambda \\ &= \frac{\beta\,\Gamma(2)}{(x+\beta)^2} \\ &= \frac{\beta}{(x+\beta)^2}, \qquad x>0 \end{aligned} \tag{2.6}$$

The survival function is,

$$\begin{aligned} S(x) &= \int_0^\infty S(x|\lambda)\ \ g(\lambda)\ \ d\lambda \\ &= \int_0^\infty e^{-\lambda\,x}\ \beta e^{-\lambda\beta}\, d\lambda \\ &= \beta \int_0^\infty e^{-\lambda\,(x+\beta)} d\lambda \\ &= \beta \int_0^\infty \lambda^{1-1}\, e^{-\lambda\,(x+\beta)} d\lambda \\ &= \frac{\beta}{x+\beta} \end{aligned} \tag{2.7}$$

The hazard function is,

$$h(x) = \frac{1}{x+\beta} \tag{2.8}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \beta \int_0^\infty \frac{x^r}{(x+\beta)^2}\, dx$$

Let,

$$x = \beta\, t \quad \therefore \quad dx = \beta\, dt$$

$$\begin{aligned} E(X^r) &= \beta^r \int_0^\infty \frac{t^r}{(t+1)^2}\, dt \\ &= \beta^r \int_0^\infty \frac{t^{r+1-1}}{(t+1)^{(r+1)+(1-r)}}\, dt \\ &= \beta^r\, B(r+1, 1-r) \\ &= \beta^r\, \frac{\Gamma(r+1)\, \Gamma(1-r)}{\Gamma(2)} \\ &= r!\beta^r\, \Gamma(1-r) \end{aligned} \tag{2.9}$$

Using conditional expectation approach, we have

$$\begin{aligned} E\left[\frac{1}{\Lambda^r}\right] &= \int_0^\infty \lambda^{-r}\, g(\lambda)\, d\lambda \\ &= \beta \int_0^\infty \lambda^{1-r-1}\, e^{-\beta\,\lambda}\, d\lambda \\ &= \beta\, \frac{\Gamma(1-r)}{\beta^{1-r}} \\ &= \beta^r\, \Gamma(1-r) \end{aligned}$$

$$\therefore \quad E(X^r) = r!\beta^r\, \Gamma(1-r) \tag{2.10}$$

and,

$$E(X) = \beta\, \Gamma(0) = \infty \tag{2.11}$$

### 2.3.2 Gamma I mixing distribution

The gamma I distribution is based on the gamma function in (1.5). Using gamma I as the mixing distribution, the exponential-gamma I mixture, also known as Pareto II (Lomax) distribution with parameters $\alpha$ and $\beta$, is constructed below and the moments have been obtained.

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-\beta\,\lambda}\, \lambda^{\alpha-1}, \quad \lambda > 0; \quad \beta > 0; \quad \alpha > 0 \tag{2.12}$$

The pdf of the mixture is,

$$
\begin{aligned}
f(x) &= \int_0^\infty \lambda e^{-\lambda x} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\,\lambda}\, \lambda^{\alpha-1}\, d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha\, e^{-(x+\beta)\lambda}\, d\lambda \\
&= \frac{\alpha\, \beta^\alpha}{(x+\beta)^{\alpha+1}}, \qquad x > 0 \qquad (2.13)
\end{aligned}
$$

The survival function is,

$$
\begin{aligned}
S\,(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1}\, e^{-\lambda\,(x+\beta)}\, d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\, \frac{\Gamma(\alpha)}{(x+\beta)^\alpha} \\
&= (\frac{\beta}{x+\beta})^\alpha \qquad (2.14)
\end{aligned}
$$

The hazard function is,

$$
\begin{aligned}
h\,(x) &= \frac{\frac{\alpha\,\beta^\alpha}{(x+\beta)^{\alpha+1}}}{(\frac{\beta}{x+\beta})^\alpha} \\
&= \frac{\alpha}{x+\beta} \qquad (2.15)
\end{aligned}
$$

Mc Nolty et al. (1980) stated that this hazard rate is decreasing, since

$$
\begin{aligned}
h'(x) &= \frac{dh(x)}{dx} = \frac{d\alpha(x+\beta)^{-1}}{dx} \\
&= -\alpha(x+\beta)^{-2} \\
&= -\frac{\alpha}{(x+\beta)^2} \\
h'(x) &\le 0
\end{aligned}
$$

Therefore, the distribution has a heavy tail, and thus significantly puts more probability on larger values.

The $r^{th}$ moment about zero is,

$$
E(X^r) = \alpha\, \beta^\alpha \int_0^\infty \frac{x^r}{(x+\beta)^{\alpha+1}}\, dx
$$

Let,

$$
x = \beta\, t \quad \therefore \quad dx = \beta\, dt
$$

$$\begin{aligned}
E(X^r) &= \alpha\,\beta^r \int_0^\infty \frac{t^r}{(t+1)^{\alpha+1}}\,dt \\
&= \alpha\,\beta^r \int_0^\infty \frac{t^{r+1-1}}{(t+1)^{(r+1)+(\alpha-r)}}\,dt \\
&= \alpha\,\beta^r\, B(r+1,\alpha-r) \\
&= \alpha\,\beta^r\,\frac{\Gamma(r+1)\,\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \\
&= r!\,\beta^r\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \qquad (2.16)
\end{aligned}$$

Using conditional expectation approach, we have

$$\begin{aligned}
E\left[\frac{1}{\Lambda^r}\right] &= \int_0^\infty \lambda^{-r}\,\frac{\beta^\alpha\, e^{-\beta\,\lambda}\,\lambda^{\alpha-1}}{\Gamma(\alpha)}\,d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-r-1}\;e^{-\beta\,\lambda}\,d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\,\frac{\Gamma(\alpha-r)}{\beta^{\alpha-r}} \\
&= \beta^r\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)}
\end{aligned}$$

$$\therefore\quad E(X^r) = r!\,\beta^r\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \qquad (2.17)$$

and,

$$E(X) = \frac{\beta}{\alpha-1}\quad \alpha\neq 1 \qquad (2.18)$$

### 2.3.3 Gamma II mixing distribution

The exponential-gamma II mixture, which is also known as Pareto II (Lomax) distribution, with parameters $\alpha$ and $\frac{1}{\beta}$ is constructed as follows:

$$g(\lambda) = \frac{1}{\beta^\alpha\,\Gamma(\alpha)}\,e^{-\frac{\lambda}{\beta}}\,\lambda^{\alpha-1} \qquad (2.19)$$

The pdf of the mixture is,

$$\begin{aligned}
f(x) &= \int_0^\infty \lambda e^{-\lambda x}\,\frac{1}{\beta^\alpha\,\Gamma(\alpha)}\,e^{-\frac{\lambda}{\beta}}\,\lambda^{\alpha-1}\,d\lambda \\
&= \frac{1}{\beta^\alpha\,\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha+1-1}\,e^{-\lambda(x+\frac{1}{\beta})}\,d\lambda \qquad \lambda>0
\end{aligned}$$

$$= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\;\frac{\Gamma(\alpha+1)}{(x+\frac{1}{\beta})^{\alpha+1}}$$

$$= \frac{\alpha\,(\frac{1}{\beta})^{\alpha}}{(x+\frac{1}{\beta})^{\alpha+1}} \qquad x>0 \tag{2.20}$$

The survival function is,

$$S\,(x) = \int_{0}^{\infty} e^{-\lambda\,x}\;\frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\;e^{-\frac{\lambda}{\beta}}\;\lambda^{\alpha-1}\,d\lambda$$

$$= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\int_{0}^{\infty} e^{-\lambda(x+\frac{1}{\beta})}\;\lambda^{\alpha-1}\,d\lambda$$

$$= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\;\frac{\Gamma(\alpha)}{(x+\frac{1}{\beta})^{\alpha}}$$

$$= \left(\frac{\frac{1}{\beta}}{x+\frac{1}{\beta}}\right)^{\alpha} \tag{2.21}$$

and therefore

$$h\,(x) = \frac{\alpha}{x+\frac{1}{\beta}} \tag{2.22}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \alpha\,\left(\frac{1}{\beta}\right)^{\alpha}\int_{0}^{\infty}\frac{x^r}{(x+\left(\frac{1}{\beta}\right))^{\alpha+1}}\,dx$$

Let,

$$x = \left(\frac{1}{\beta}\right)\,t \quad \therefore \quad dx = \left(\frac{1}{\beta}\right)\,dt$$

$$E(X^r) = \alpha\,\left(\frac{1}{\beta}\right)^{r}\int_{0}^{\infty}\frac{t^r}{(t+1)^{\alpha+1}}\,dt$$

$$= \alpha\,\left(\frac{1}{\beta}\right)^{r}\int_{0}^{\infty}\frac{t^{r+1-1}}{(t+1)^{(r+1)+(\alpha-r)}}\,dt$$

$$= \alpha\,\left(\frac{1}{\beta}\right)^{r}\,B(r+1,\alpha-r)$$

$$= \alpha\,\left(\frac{1}{\beta}\right)^{r}\,\frac{\Gamma(r+1)\,\Gamma(\alpha-r)}{\Gamma(\alpha+1)}$$

$$= r!\,\left(\frac{1}{\beta}\right)^{r}\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \tag{2.23}$$

Using conditional expectation approach, we have

$$\boldsymbol{E\left[\frac{1}{\Lambda^r}\right] = \frac{1}{\beta^\alpha\,\Gamma(\alpha)}\int_0^\infty \lambda^{-r}\,e^{-\frac{\lambda}{\beta}}\,\lambda^{\alpha-1}\,d\lambda}$$

$$\boldsymbol{= \frac{1}{\beta^\alpha\,\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-r-1}\,e^{-\frac{\lambda}{\beta}}\,d\lambda}$$

$$\boldsymbol{= \frac{1}{\beta^\alpha\,\Gamma(\alpha)}\,\frac{\Gamma(\alpha-r)}{\left(\frac{1}{\beta}\right)^{\alpha-r}}}$$

$$\boldsymbol{= \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)\,\beta^r}}$$

$$\therefore\quad \boldsymbol{E(X^r) = r!\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)\,\beta^r}} \tag{2.24}$$

and,

$$\boldsymbol{E(X) = \frac{1}{(\alpha-1)\,\beta}\quad \alpha\neq 1} \tag{2.25}$$

### 2.3.4 Shifted gamma (Pearson type III) mixing distribution

The shifted gamma or Pearson type III mixing distribution has pdf;

$$\boldsymbol{g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\,e^{-\beta\,(\lambda-\mu)}\,(\lambda-\mu)^{\alpha-1}} \tag{2.26}$$

and it yields an exponential-shifted gamma distribution which is constructed as follows:,

$$\boldsymbol{f(x) = \int_\mu^\infty \lambda\,e^{-\lambda x}\,\frac{\beta^\alpha}{\Gamma(\alpha)}\,e^{-\beta\,(\lambda-\mu)}\,(\lambda-\mu)^{\alpha-1}\,d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_\mu^\infty \lambda\,e^{-\lambda x}\,e^{-\beta\,(\lambda-\mu)}\,(\lambda-\mu)^{\alpha-1}\,d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_\mu^\infty (\lambda-\mu+\mu)\,(\lambda-\mu)^{\alpha-1}\,e^{-(\lambda-\mu+\mu)\,x}\,e^{-\beta\,(\lambda-\mu)}\,d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)}\,e^{-\mu\,x}\int_\mu^\infty \left[(\lambda-\mu)^{(\alpha+1)-1} + \mu\,(\lambda-\mu)^{\alpha-1}\right]\,e^{-(\lambda-\mu)(x+\beta)}\,d\lambda} \tag{2.27}$$

Let,

$$\boldsymbol{\lambda-\mu = z}\qquad \therefore\quad \boldsymbol{d\lambda = dz}$$

Therefore,

$$\begin{aligned}
f(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\mu\, x} \int\limits_{0}^{\infty} \left[z^{(\alpha+1)-1} + \mu\, z^{\alpha-1}\right] e^{-(x+\beta)\, z}\, dz \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\mu\, x} \left[\frac{\Gamma(\alpha+1)}{(x+\beta)^{\alpha+1}} + \frac{\mu\, \Gamma(\alpha)}{(x+\beta)^{\alpha}}\right] \\
&= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha}}\, e^{-\mu\, x} \left[\frac{\alpha}{(x+\beta)} + \mu\right], \qquad x > 0
\end{aligned} \tag{2.28}$$

The survival function is,

$$\begin{aligned}
S(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int\limits_{\mu}^{\infty} (\lambda-\mu)^{\alpha-1}\, e^{-\lambda x}\, e^{-\beta\, (\lambda-\mu)}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\mu\, x} \int\limits_{\mu}^{\infty} (\lambda-\mu)^{\alpha-1}\, e^{-(\lambda-\mu)(x+\beta)}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\mu\, x} \int\limits_{0}^{\infty} z^{\alpha-1}\, e^{-(x+\beta)\, z}\, dz \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\mu\, x}\, \frac{\Gamma(\alpha)}{(x+\beta)^{\alpha}} \\
&= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha}}\, e^{-\mu\, x}
\end{aligned}$$

and therefore

$$h\,(x) = \mu + \frac{\alpha}{x+\beta} \tag{2.29}$$

Using conditional expectation approach, we have

$$\begin{aligned}
E\left[\frac{1}{\Lambda^{r}}\right] &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int\limits_{\mu}^{\infty} \lambda^{-r}\, e^{-\beta\, (\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int\limits_{\mu}^{\infty} (\lambda-\mu+\mu)^{-r}\, (\lambda-\mu)^{\alpha-1}\, e^{-\beta\, (\lambda-\mu)}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int\limits_{\mu}^{\infty} \sum_{\mu}^{\infty} \binom{-r}{k} (\lambda-\mu)^{-r-k}\, \mu^{k}\, (\lambda-\mu)^{\alpha-1}\, e^{-\beta\, (\lambda-\mu)}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{\mu}^{\infty} \binom{-r}{k} \mu^{k} \int\limits_{\mu}^{\infty} (\lambda-\mu)^{\alpha-r-k-1}\, e^{-\beta\, (\lambda-\mu)}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{\mu}^{\infty} \binom{-r}{k} \mu^{k} \int\limits_{\mu}^{\infty} z^{\alpha-r-k-1}\, e^{-\beta\, z}\, dz
\end{aligned}$$

where,

$$z = \lambda - \mu \quad and \quad dz = d\lambda$$

$$E(X^r) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, r! \sum_{\mu}^{\infty} \binom{-r}{k} \mu^k \frac{\Gamma(\alpha - r - k)}{\beta^{\alpha - r - k}}$$

$$= r!\, \beta^\alpha \sum_{\mu}^{\infty} \binom{-r}{k} (\mu\, \beta)^{r+k} \frac{\Gamma(\alpha - r - k)}{\Gamma(\alpha)} \tag{2.30}$$

### 2.3.5 Half logistic mixing distribution

In the case of half logistic mixing distribution,

$$g(\lambda) = \frac{2\,\mu\, e^{-\mu\lambda}}{(1 + e^{-\mu\lambda})^2} \tag{2.31}$$

the mixture is exponential-half logistic distribution, which is constructed as follows:

$$f(x) = \int_0^\infty \lambda e^{-\lambda x} \frac{2\,\mu\, e^{-\mu\lambda}}{(1 + e^{-\mu\lambda})^2}\, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x} \left\{ 2\mu\, e^{-\mu\lambda} \sum_{k=0}^{\infty} \binom{-2}{k} (e^{-\mu\lambda})^k \right\} d\lambda$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{ \int_0^\infty \lambda e^{-\lambda(x + \mu + \mu k)}\, d\lambda \right\}$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{ \frac{\Gamma(2)}{(x + \mu + \mu k)^2} \right\}$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{ \frac{1}{(x + \mu + \mu k)^2} \right\} \qquad x > 0 \tag{2.32}$$

The survival function is,

$$S(x) = \int_0^\infty e^{-\lambda x} \frac{2\,\mu\, e^{-\mu\lambda}}{(1 + e^{-\mu\lambda})^2}\, d\lambda$$

$$= 2\mu \int_0^\infty e^{-\lambda x - \lambda\mu} \left\{ \sum_{k=0}^{\infty} \binom{-2}{k} (e^{-\mu\lambda})^k \right\} d\lambda$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{ \int_0^\infty e^{-\lambda(x + \mu + \mu k)}\, d\lambda \right\}$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{-e^{-\lambda(x + \mu + \mu k)}}{x + \mu + \mu k} \right]_0^\infty$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{1}{x + \mu + \mu k} \right] \tag{2.33}$$

The hazard function is,,

$$h\,(x) = \frac{\sum\limits_{k=0}^{\infty} \binom{-2}{k} \left\{\frac{1}{(x+\mu+\mu k)^2}\right\}}{\sum\limits_{k=0}^{\infty} \binom{-2}{k} \left\{\frac{1}{x+\mu+\mu k}\right\}} \tag{2.34}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{\int_0^{\infty} \frac{x^r}{(x+\mu+\mu k)^2} dx\right\}$$

Let,

$$x = (\mu+\mu k)t \quad \therefore \quad dx = (\mu+\mu k)dt$$

$$\begin{aligned}
E(X^r) &= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} (\mu+\mu k)^{r-1} \left\{\int_0^{\infty} \frac{t^r}{(1+t)^2} dt\right\} \\
&= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} (\mu+\mu k)^{r-1}\, B(r+1, 1-r) \\
&= 2\mu^r \sum_{k=0}^{\infty} \binom{-2}{k} (k+1)^{r-1}\, B(r+1, 1-r) \\
&= 2\; r!\; \mu^r\, \Gamma(1-r) \sum_{k=0}^{\infty} \binom{-2}{k} (k+1)^{r-1}
\end{aligned} \tag{2.35}$$

Using conditional expectation approach, we have

$$\begin{aligned}
E(\frac{1}{\Lambda^r}) &= \int_0^{\infty} \lambda^{-r} \frac{2\,\mu\, e^{-\mu\,\lambda}}{(1+e^{-\mu\,\lambda})^2} d\lambda \\
&= 2\mu \int_0^{\infty} \lambda^{-r}\; e^{-\mu\lambda} \left\{\sum_{k=0}^{\infty} \binom{-2}{k}\; e^{-\mu\lambda k}\right\} d\lambda \\
&= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{\int_0^{\infty} \lambda^{1-r-1}\; e^{-\mu\lambda(k+1)} d\lambda\right\} \\
&= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \left\{\frac{\Gamma(1-r)}{[\mu(k+1)]^{1-r}}\right\} \\
&= 2\mu^r\; \Gamma(1-r) \sum_{k=0}^{\infty} \binom{-2}{k} (k+1)^{r-1}
\end{aligned}$$

$$E(X^r) = 2\; r!\; \mu^r\, \Gamma(1-r) \sum_{k=0}^{\infty} \binom{-2}{k} (k+1)^{r-1} \tag{2.36}$$

and,

$$E(X) = \infty \tag{2.37}$$

### 2.3.6 Lindley mixing distribution

The Lindley distribution is given as:

$$g(\lambda) = \frac{\theta^2}{\theta+1}\,(\lambda+1)\,e^{-\theta\,\lambda} \tag{2.38}$$

and the pdf of the mixture is,

$$\begin{aligned}
f(x) &= \int_0^\infty \lambda\,e^{-\lambda\,x}\,\frac{\theta^2}{\theta+1}\,(\lambda+1)\,e^{-\theta\,\lambda}\,d\lambda \\
&= \frac{\theta^2}{\theta+1}\int_0^\infty (\lambda^2+\lambda)\,e^{-\lambda\,(x+\theta)}\,d\lambda \\
&= \frac{\theta^2}{\theta+1}\left[\frac{\Gamma(3)}{(x+\theta)^3}+\frac{\Gamma(2)}{(x+\theta)^2}\right] \\
&= \frac{\theta^2}{\theta+1}\left[\frac{2}{(x+\theta)^3}+\frac{1}{(x+\theta)^2}\right] \\
&= \frac{\theta^2}{\theta+1}\,\frac{1}{(x+\theta)^2}\left[\frac{2}{(x+\theta)}+1\right] \\
&= \frac{\theta^2}{\theta+1}\,\frac{1}{(x+\theta)^2}\,\frac{2+x+\theta}{(x+\theta)} \\
&= \frac{\theta^2}{\theta+1}\,\frac{2+x+\theta}{(x+\theta)^3} \qquad x>0;
\end{aligned} \tag{2.39}$$

The survival function is,

$$\begin{aligned}
S(x) &= \int_0^\infty e^{-\lambda\,x}\,\frac{\theta^2}{\theta+1}\,(\lambda+1)\,e^{-\theta\,\lambda}\,d\lambda \\
&= \frac{\theta^2}{\theta+1}\int_0^\infty (\lambda+1)\,e^{-\lambda\,(x+\theta)}\,d\lambda \\
&= \frac{\theta^2}{\theta+1}\left[\frac{1}{(x+\theta)^2}+\frac{1}{(x+\theta)}\right] \\
&= \frac{\theta^2}{\theta+1}\,\frac{1+x+\theta}{(x+\theta)^2}
\end{aligned} \tag{2.40}$$

Therefore,

$$\begin{aligned}
h(x) &= \frac{1}{x+\theta}\,\frac{2+x+\theta}{1+x+\theta} \\
&= \frac{A}{x+\theta}+\frac{B}{x+\theta+1} \\
A+B=1 &\qquad B=2 \quad and \quad A=-1
\end{aligned}$$

hence

$$h(x) = \frac{2}{x+\theta} - \frac{1}{x+\theta+1} \tag{2.41}$$

which is the difference of two hazard functions of Pareto II (Lomax) distribution

The $r^{th}$ moment about zero is,

$$\begin{aligned} E(X^r) &= \frac{\theta^2}{\theta+1} \int_0^\infty x^r \, \frac{2+x+\theta}{(x+\theta)^3} \, dx \\ &= \frac{\theta^2}{\theta+1} \left\{ (2+\theta) \int_0^\infty \frac{x^r}{(x+\theta)^3} \, dx + \int_0^\infty \frac{x^{r+1}}{(x+\theta)^3} \, dx \right\} \end{aligned}$$

Let,

$$\begin{aligned} x &= \theta\, t \quad \therefore \quad dx = \theta\, dt \\ E(X^r) &= \frac{\theta^2}{\theta+1} \left\{ (2+\theta)\theta^{r-2} \int_0^\infty \frac{t^r}{(1+t)^3} \, dt + \theta^{r-1} \int_0^\infty \frac{t^{r+1}}{(1+t)^3} \, dt \right\} \\ &= \frac{\theta^r}{\theta+1} \left\{ (2+\theta)\, B(r+1, 2-r) + \theta\, B(r+2, 1-r) \right\} \\ &= \frac{\theta^r}{\theta+1} \left\{ (2+\theta)\, \frac{\Gamma(r+1)\,\Gamma(2-r)}{\Gamma(3)} + \theta\, \frac{\Gamma(r+2)\,\Gamma(1-r)}{\Gamma(3)} \right\} \\ &= \frac{r!\,\theta^r}{2(\theta+1)} \left\{ (2+\theta)\,\Gamma(2-r) + \theta\,(r+1)\,\Gamma(1-r) \right\} \\ &= \frac{r!\,\theta^r}{2(\theta+1)}\, \Gamma(1-r)\,[2+2\theta-2r] \\ &= \frac{r!\,\theta^r}{\theta+1}\,[1+\theta-r]\,\Gamma(1-r) \end{aligned} \tag{2.42}$$

Using conditional expectation approach, we have

$$\begin{aligned} E\left[\frac{1}{\Lambda^r}\right] &= \frac{\theta^2}{\theta+1} \int_0^\infty \lambda^{-r}\,(\lambda+1)\, e^{-\theta\,\lambda}\, d\lambda \\ &= \frac{\theta^2}{\theta+1} \left\{ \int_0^\infty \lambda^{1-r}\, e^{-\theta\,\lambda}\, d\lambda + \int_0^\infty \lambda^{-r}\, e^{-\theta\,\lambda}\, d\lambda \right\} \\ &= \frac{\theta^2}{\theta+1} \left\{ \frac{\Gamma(2-r)}{\theta^{2-r}} + \frac{\Gamma(1-r)}{\theta^{1-r}} \right\} \\ \therefore \quad E(X^r) &= \frac{r!\,\theta^r}{\theta+1}\,[1+\theta-r]\,\Gamma(1-r) \end{aligned} \tag{2.43}$$

and,

$$E(X) = \infty \tag{2.44}$$

### 2.3.7 Generalized Lindley mixing distribution

The generalized Lindley distribution is given as:

$$g(\lambda) = \frac{\theta^2\,(\theta\lambda)^{\alpha-1}\,(\alpha+\lambda)\,e^{-\theta\,\lambda}}{(\theta+1)\,\Gamma(\alpha+1)} \quad \lambda > 0 \quad \theta > 0 \tag{2.45}$$

In the subsection, the exponential-generalized Lindley distribution has been constructed and the moments obtained. It has been shown that the exponential-Lindley distribution is a special case of this exponential mixture when $\alpha = 1$.

The pdf of the mixture is,

$$\begin{aligned}
f(x) &= \int_0^\infty \lambda\,e^{-\lambda\,x}\,\frac{\theta^2\,(\theta\lambda)^{\alpha-1}\,(\alpha+\lambda)\,e^{-\theta\,\lambda}}{(\theta+1)\,\Gamma(\alpha+1)}\,d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)} \int_0^\infty \left[\alpha\,\lambda^{(\alpha+1)-1}+\lambda^{(\alpha+2)-1}\right]\,e^{-(x+\theta)\,\lambda}\,d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left[\frac{\alpha\,\Gamma(\alpha+1)}{(x+\theta)^{\alpha+1}}+\frac{\Gamma(\alpha+2)}{(x+\theta)^{\alpha+2}}\right] \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\,(x+\theta)^{\alpha+1}}\left[\alpha+\frac{\alpha+1}{x+\theta}\right] \\
&= \frac{1}{\theta+1}\left(\frac{\theta}{x+\theta}\right)^{\alpha+1}\left[\frac{\alpha x+\alpha\theta+\alpha+1}{x+\theta}\right] \\
&= \frac{1}{\theta+1}\left(\frac{\theta}{x+\theta}\right)^{\alpha+1}\left[\frac{\alpha(x+\theta+1)+1}{x+\theta}\right] \\
&= \left(\frac{\theta}{x+\theta}\right)^{\alpha+1}\left[\frac{\alpha(x+\theta+1)+1}{(\theta+1)(x+\theta)}\right]
\end{aligned} \tag{2.46}$$

The survival function is,

$$\begin{aligned}
S(x) &= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)} \int_0^\infty \left[\alpha\,\lambda^{\alpha-1}+\lambda^{(\alpha+1)-1}\right]\,e^{-(x+\theta)\,\lambda}\,d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left[\frac{\alpha\,\Gamma(\alpha)}{(x+\theta)^{\alpha}}+\frac{\Gamma(\alpha+1)}{(x+\theta)^{\alpha+1}}\right] \\
&= \left(\frac{\theta}{x+\theta}\right)^{\alpha}\,\frac{\theta}{\theta+1}\,\frac{x+\theta+1}{x+\theta}
\end{aligned} \tag{2.47}$$

and,

$$\begin{aligned}
h(x) &= \frac{\alpha(x+\theta+1)+1}{(x+\theta)(x+\theta+1)} \\
&= \frac{A}{x+\theta}+\frac{B}{x+\theta+1}
\end{aligned}$$

by partial fraction technique

$$\alpha(x+\theta+1)+1 = A(x+\theta+1)+B(x+\theta)$$
$$A+B=\alpha \quad , \quad A=\alpha+1 \quad and \quad B=-1$$

hence

$$h(x) = \frac{\alpha+1}{x+\theta} - \frac{1}{x+\theta+1} \tag{2.48}$$

which is the difference of two hazard functions of Pareto II (Lomax) distributions

The $r^{th}$ moment about zero is,

$$\begin{aligned}
E(X^r) &= \frac{\theta^{\alpha+1}}{\theta+1}\int_0^\infty \frac{x^r}{(x+\theta)^{\alpha+1}}\,\frac{\alpha(x+\theta+1)+1}{x+\theta} \\
&= \frac{\theta^{\alpha+1}}{\theta+1}\int_0^\infty \frac{x^r\left[\alpha(x+\theta+1)+1\right]}{(x+\theta)^{\alpha+2}} \\
&= \frac{\theta^{\alpha+1}}{\theta+1}\int_0^\infty \frac{x^r\left[\alpha(x+\theta)+(\alpha+1)\right]}{(x+\theta)^{\alpha+2}} \\
&= \frac{\alpha\,\theta^{\alpha+1}}{\theta+1}\int_0^\infty \frac{x^r}{(x+\theta)^{\alpha+1}} + \frac{(\alpha+1)\theta^{\alpha+1}}{\theta+1}\int_0^\infty \frac{x^r}{(x+\theta)^{\alpha+2}} \\
&= \frac{\alpha\,\theta^{r+1}}{\theta+1}\int_0^\infty \frac{t^r}{(1+t)^{\alpha+1}}dt + \frac{(\alpha+1)\theta^{r}}{\theta+1}\int_0^\infty \frac{t^r}{(1+t)^{\alpha+2}}dt \\
&= \frac{\alpha\,\theta^{r+1}}{\theta+1}\,B(r+1,\alpha-r) + \frac{(\alpha+1)\theta^{r}}{\theta+1}\,B(r+1,\alpha-r+1)
\end{aligned}$$

$$\therefore \quad E(X^r) = r!\,\frac{\theta^r}{\theta+1}\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)}\left\{\theta+\frac{\alpha-r}{\alpha}\right\} \tag{2.49}$$

Using conditional expectation approach, we have

$$\begin{aligned}
E\left[\frac{1}{\Lambda^r}\right] &= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\left\{\int_0^\infty \lambda^{\alpha-r+1-1}\,e^{-\theta\,\lambda}\,d\lambda + \alpha\int_0^\infty \lambda^{\alpha-r-1}\,e^{\theta\,\lambda}\,d\lambda\right\} \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\left\{\frac{\Gamma(\alpha-r+1)}{\theta^{\alpha-r+1}} + \frac{\alpha\,\Gamma(\alpha-r)}{\theta^{\alpha-r}}\right\} \\
&= \frac{\theta^{r+1}}{\theta+1}\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} + \frac{\theta^r}{\alpha(\theta+1)}\,\frac{\Gamma(\alpha-r+1)}{\Gamma(\alpha)} \\
&= \frac{\theta^r}{\theta+1}\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)}\left\{\theta+\frac{\alpha-r}{\alpha}\right\}
\end{aligned}$$

$$\therefore \quad E(X^r) = r!\,\frac{\theta^r}{\theta+1}\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)}\left\{\theta+\frac{\alpha-r}{\alpha}\right\}$$

$$\boldsymbol{E(X)} = \frac{\boldsymbol{\theta}}{\boldsymbol{\theta+1}} \frac{\boldsymbol{\Gamma(\alpha-1)}}{\boldsymbol{\Gamma(\alpha)}} \left\{\boldsymbol{\theta} + \frac{\boldsymbol{\alpha-1}}{\boldsymbol{\alpha}}\right\} \tag{2.50}$$

$$= \frac{\boldsymbol{\theta}}{\boldsymbol{\theta+1}} \left[\frac{\boldsymbol{1}}{\boldsymbol{\theta(\alpha-1)}} + \frac{\boldsymbol{1}}{\boldsymbol{\alpha}}\right] \quad \boldsymbol{\alpha \neq 1} \tag{2.51}$$

## 2.4 Mixtures in terms of modified Bessel function of the third kind

### 2.4.1 Inverse gamma mixing distribution

The inverse gamma distribution has pdf:

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\frac{\beta}{\lambda}}\, \lambda^{-\alpha-1}, \quad \lambda > 0, \quad \beta > 0, \quad \alpha > 0 \tag{2.52}$$

Using this mixing distribution, the exponential-inverse gamma distribution has been constructed and the moments obtained.

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^{\infty} \lambda\, e^{-\lambda\, x}\ \frac{1}{\lambda^{\alpha+1}}\, \frac{\beta^{\alpha}}{\Gamma(\alpha)}\ e^{-\frac{\beta}{\lambda}}\, d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{-\alpha}\ e^{-x(\lambda+\frac{\beta}{x}\,\frac{1}{\lambda})}\, d\lambda \end{aligned}$$

Let $\lambda = \sqrt{(\frac{\beta}{x})}\, z$ and therefore $d\lambda = \sqrt{(\frac{\beta}{x})}\, dz$

$$\begin{aligned} f\,(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{(\frac{\beta}{x})}\right)^{-(\alpha-1)} \int_0^{\infty} z^{-(\alpha-1)-1}\, e^{-\sqrt{x\beta}(z+\frac{1}{z})}\, dz \\ &= \frac{2}{\Gamma(\alpha)} \sqrt{(\frac{\beta}{x})} \left(\sqrt{(\beta\, x)}\right)^{\alpha} K_{\alpha-1}(2\sqrt{\beta x}) \end{aligned} \tag{2.53}$$

(2.) The survival function is

$$\begin{aligned} S(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{(\frac{\beta}{x})}\right)^{-\alpha} \int_0^{\infty} z^{-\alpha-1}\, e^{-\frac{2\sqrt{(\beta\, x)}}{2}[z+\frac{1}{z}]}\, dz \\ &= \frac{2\,(\sqrt{\beta\, x})^{\alpha}}{\Gamma(\alpha)}\, K_{\alpha}\,(2\sqrt{\beta\, x}) \end{aligned} \tag{2.54}$$

The hazard function is,,

$$h(x) = \sqrt{(\frac{\beta}{x})}\, \frac{K_{\alpha-1}\,(2\sqrt{\beta\, x})}{K_{\alpha}\,(2\sqrt{\beta\, x})} \tag{2.55}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{2}{\Gamma(\alpha)} \int_0^{\infty} x^r \sqrt{(\frac{\beta}{x})} \left(\sqrt{(\beta\, x)}\right)^{\alpha} K_{\alpha-1}(2\sqrt{\beta x})\, dx$$

$$= \frac{2}{\Gamma(\alpha)} \beta^{\frac{1}{2}+\frac{\alpha}{2}} \int_0^\infty x^{r+\frac{1}{2}+\frac{\alpha}{2}} K_{\alpha-1}(2\sqrt{\beta x})\, dx$$

Let,

$$\omega^2 = 4\beta x \quad \therefore \quad dx = \frac{\omega}{2\beta}\, d\omega$$

$$E(X^r) = \frac{1}{\Gamma(\alpha)\, \beta^r\, 2^{2r+\alpha-1}} \int_0^\infty \omega^{2r+\alpha+1-1} K_{\alpha-1}(\omega)\, d\omega$$

By virtue of (1.18) and assuming $s = 2r + \alpha + 1$ and $v = \alpha - 1$

$$E(X^r) = \frac{1}{\Gamma(\alpha)\, \beta^r\, 2^{2r+\alpha-1}}\, 2^{2r+\alpha-1}\, \Gamma(\alpha + r)\Gamma(r+1)$$

$$= r!\, \frac{r!}{\beta^r} \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \tag{2.56}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{-r-\alpha-1}\, e^{-\beta\, \frac{1}{\lambda}}\, d\lambda$$

Let,

$$t = \frac{1}{\lambda} \quad d\lambda = -\frac{dt}{t^2}$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha+r-1}\, e^{-\beta t}\, dt$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha + r)}{\beta^{\alpha+r}}$$

$$= \frac{\beta^\alpha}{\beta^{\alpha+r}} \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}$$

$$= \frac{1}{\beta^r} \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}$$

$$\therefore \quad E(X^r) = \frac{r!}{\beta^r} \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \tag{2.57}$$

and,

$$E(X) = \frac{1}{\beta} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)}$$

$$= \frac{\alpha}{\beta} \tag{2.58}$$

### 2.4.2 Pearson type V mixing distribution

The exponential mixture has been constructed using Pearson type V mixing distribution and the moments obtained. It can be seen that the exponential-inverse gamma distribution is a special case of the exponential-Pearson type V distribution when $c = 0$.

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\frac{\beta}{\lambda - c}}\, (\lambda - c)^{-(\alpha+1)} \qquad \lambda > c \quad \alpha,\, \beta > 0 \tag{2.59}$$

The pdf of the mixture is,

$$\begin{aligned}
f(x) &= \int_{c}^{\infty} \lambda\, e^{-\lambda\, x}\, \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\frac{\beta}{\lambda - c}}\, (\lambda - c)^{-(\alpha+1)}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{c}^{\infty} \lambda\, (\lambda - c)^{-(\alpha+1)}\, e^{-\lambda\, x - \frac{\beta}{\lambda - c}}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{c}^{\infty} (\lambda - c + c)\, (\lambda - c)^{-(\alpha+1)}\, e^{-x(\lambda - c + c) - \frac{\beta}{\lambda - c}}\, d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-cx} \int_{c}^{\infty} \left[(\lambda - c)^{-(\alpha-1)-1} + c\, (\lambda - c)^{-\alpha-1}\right] e^{-x\,(\lambda - c) - \frac{\beta}{\lambda - c}}\, d\lambda
\end{aligned}$$

$$\lambda - c = t \quad \text{and therefore} \quad d\lambda = dt \tag{2.60}$$

$$\begin{aligned}
f\,(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-cx} \int_{0}^{\infty} \left[t^{-(\alpha-1)-1} + ct^{-\alpha-1}\right] e^{-x\, t - \frac{\beta}{t}}\, dt \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-cx} \int_{0}^{\infty} \left[t^{-(\alpha-1)-1} + ct^{-\alpha-1}\right] e^{-x(t + \frac{\beta}{x}\,\frac{1}{t})}\, dt
\end{aligned}$$

Put,

$$t = \sqrt{\frac{\beta}{x}}\, z \quad \text{and therefore} \quad dt = \sqrt{\frac{\beta}{x}}\, dz \tag{2.61}$$

$$\begin{aligned}
f(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-cx} \int_{0}^{\infty} \left[\left(\sqrt{\frac{\beta}{x}}\right)^{-(\alpha-1)} z^{-(\alpha-1)-1} + c\left(\sqrt{\frac{\beta}{x}}\right)^{-\alpha} z^{-\alpha-1}\right] e^{-\sqrt{\beta x}\left(z + \frac{1}{z}\right)} dz \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-cx} \left[2\left(\sqrt{\frac{\beta}{x}}\right)^{-(\alpha-1)} K_{-(\alpha-1)}\,(2\sqrt{\beta\, x}) + 2c\left(\sqrt{\frac{\beta}{x}}\right)^{-\alpha} K_{-\alpha}\,(2\sqrt{\beta\, x})\right] \\
&= \frac{2\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{x}}\right)^{-\alpha} e^{-cx} \left[\left(\sqrt{\frac{\beta}{x}}\right) K_{-(\alpha-1)}\,(2\sqrt{\beta\, x}) + c\, K_{-\alpha}\,(2\sqrt{\beta\, x})\right]
\end{aligned} \tag{2.62}$$

(2.) The survival function is

$$\boldsymbol{S(x) = \int_c^\infty e^{-\lambda\, x}\, \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-\frac{\beta}{\lambda-c}}\, (\lambda-c)^{-(\alpha+1)}\, d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_c^\infty (\lambda-c)^{-(\alpha+1)}\, e^{-\lambda\, x - \frac{\beta}{\lambda-c}}\, d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_c^\infty (\lambda-c)^{-(\alpha+1)}\, e^{-x(\lambda-c+c) - \frac{\beta}{\lambda-c}}\, d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-cx} \int_c^\infty (\lambda-c)^{-(\alpha+1)} e^{-x\,(\lambda-c) - \frac{\beta}{\lambda-c}}\, d\lambda}$$

Using substitution (2.60)

$$\boldsymbol{S(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-cx} \int_0^\infty t^{-(\alpha+1)}\, e^{-x\,t - \frac{\beta}{t}}\, dt}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-cx} \int_0^\infty t^{-(\alpha+1)}\, e^{-x(t + \frac{\beta}{x}\frac{1}{t})}\, dt}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{x}}\right)^{-\alpha} e^{-cx} \int_0^\infty z^{-\alpha-1}\, e^{-\sqrt{\beta\, x}(z+\frac{1}{z})}\, dz}$$

$$\boldsymbol{= \frac{2\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{x}}\right)^{-\alpha} e^{-cx}\, K_\alpha(2\sqrt{\beta\, x})}$$

$$\boldsymbol{= \frac{2}{\Gamma(\alpha)} \left(\sqrt{\beta\, x}\right)^{\alpha} e^{-cx}\, K_\alpha(2\sqrt{\beta\, x})} \tag{2.63}$$

The hazard function is,,

$$\boldsymbol{h(x) = c + \sqrt{\frac{\beta}{x}}\, \frac{K_{\alpha-1}(2\sqrt{\beta\, x})}{K_\alpha(2\sqrt{\beta\, x})}} \tag{2.64}$$

Using conditional expectation approach, we have

$$\boldsymbol{E\left[\frac{1}{\Lambda^r}\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_c^\infty \lambda^{-r}\, e^{-\frac{\beta}{\lambda-c}}\, (\lambda-c)^{-(\alpha+1)} d\lambda}$$

$$\boldsymbol{= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_c^\infty \frac{e^{-\frac{\beta}{\lambda-c}}}{\lambda^r\, (\lambda-c)^{-(\alpha+1)}} d\lambda}$$

Let,

$$\boldsymbol{z = \lambda - c \quad \therefore \quad \lambda = z + c}$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (z+c)^{-r}\, e^{-\frac{\beta}{z}}\, z^{-(\alpha+1)} dz$$

If

$$z = \frac{1}{t} \quad \therefore \quad dz = -\frac{dt}{t^2}$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (\frac{1}{t}+c)^{-r}\, e^{-\beta t}\, t^{(\alpha+1)} \frac{dt}{t^2}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (1+ct)^{-r}\, e^{-\beta t}\, t^{(\alpha+r-1)} dt$$

Put,

$$y = ct \quad \therefore \quad dt = \frac{dy}{c}$$

$$E(X^r) = r!\, \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{y}{c}\right)^{\alpha+r-1} (1+y)^{-r}\, e^{-\frac{\beta}{c}y}\, \frac{dy}{c}$$

$$= r!\, \frac{\beta^\alpha}{\Gamma(\alpha)}\, \frac{1}{c^{\alpha+r}} \int_0^\infty (y)^{\alpha+r-1}\, (1+y)^{-r}\, e^{-\frac{\beta}{c}y}\, dy$$

$$= \frac{r!}{c^r} \left(\frac{\beta}{c}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty (y)^{\alpha+r-1} \sum_{k=0}^\infty \binom{-r}{k} y^k\, e^{-\frac{\beta}{c}y}\, dy$$

$$= \frac{r!}{c^r} \left(\frac{\beta}{c}\right)^\alpha \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \left\{\binom{-r}{k} \int_0^\infty (y)^{k+\alpha+r-1}\, e^{-\frac{\beta}{c}y}\, dy\right\}$$

$$= \frac{r!}{c^r} \left(\frac{\beta}{c}\right)^\alpha \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \left\{\binom{-r}{k} \frac{\Gamma(k+\alpha+r)}{\left(\frac{\beta}{c}\right)}\right\}$$

$$= \frac{r!}{\beta^r}\, \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \binom{-r}{k} \left(\frac{c}{\beta}\right)^k \Gamma(k+\alpha+r)$$

$$= \frac{r!}{\beta^r}\, \frac{1}{\Gamma(\alpha)} \left\{\Gamma(\alpha+r) + \sum_{k=1}^\infty \binom{-r}{k} \left(\frac{c}{\beta}\right)^k \Gamma(k+\alpha+r)\right\} \tag{2.65}$$

### 2.4.3 Inverse Gaussian mixing distribution

In this subsection, we construct the exponential-inverse Gaussian distribution and obtain the moments.

$$g(\lambda) = \left(\frac{\Phi}{2\,\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left\{-\Phi\frac{(\lambda-\mu)^2}{2\lambda\mu^2}\right\} \quad \text{for} \quad \lambda > 0 \quad -\infty < \mu < \infty \tag{2.66}$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \lambda^{-\frac{3}{2}} \exp\left\{-\frac{\Phi}{2\lambda\mu^2}\left[\lambda^2 - 2\mu\lambda + \mu^2\right]\right\}$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \lambda^{-\frac{3}{2}} \exp\left\{-\frac{\Phi\lambda}{2\mu^2} + \frac{\Phi}{\mu} - \frac{\Phi}{2\lambda}\right\}$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\frac{\Phi}{\mu}} \lambda^{-\frac{3}{2}} \exp\left\{-\frac{\Phi}{2}\left(\frac{\lambda}{\mu^2} + \frac{1}{\lambda}\right)\right\}$$

Let,

$$\mu^2 = \frac{\Phi}{\Psi}$$

Then,

$$g(\lambda) = \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}}\,\lambda^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\left(\Psi\,\lambda + \Phi\,\frac{1}{\lambda}\right)\right\} \tag{2.67}$$

The pdf of the mixture is,

$$f(x) = \int_0^\infty \lambda\, e^{-\lambda\, x} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}}\,\lambda^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\left(\Psi\,\lambda + \Phi\,\frac{1}{\lambda}\right)\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{-\frac{1}{2}} \exp\left\{-\lambda\,x - \frac{1}{2}\,\Psi\lambda - \frac{\Phi}{2\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\left(x + \frac{1}{2}\,\Psi\right)\lambda - \frac{\Phi}{2}\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\,\left(x + \frac{1}{2}\,\Psi\right)\left[\lambda + \frac{\Phi}{2x+\Psi}\,\frac{1}{\lambda}\right]\right\} d\lambda$$

$$\lambda = \sqrt{\frac{\Phi}{2x+\Psi}}\,z \quad \text{and therefore} \quad d\lambda = \sqrt{\frac{\Phi}{2x+\Psi}}\,dz \tag{2.68}$$

$$f(x) = \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{\frac{1}{2}} \int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\,\sqrt{\Phi\,\left(2x+\Psi\right)}\left[z + \frac{1}{z}\right]\right\} dz$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\sqrt{\Phi\,\left(2x+\Psi\right)} \tag{2.69}$$

2.) The survival function is,

$$S(x) = \int_0^\infty e^{-\lambda\, x} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}}\,\lambda^{-\frac{3}{2}} \exp\left\{-\frac{\Phi}{2}\left(\frac{\lambda}{\mu^2} + \frac{1}{\lambda}\right)\right\}$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{-\frac{1}{2}-1} \exp\left\{-\lambda\,x - \frac{1}{2}\,\Psi\lambda - \frac{\Phi}{2\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^{\infty} \lambda^{-\frac{1}{2}-1} \exp\left\{-\left(x + \frac{1}{2}\,\Psi\right)\lambda - \frac{\Phi}{2}\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^{\infty} \lambda^{-\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\,\left(2x + \frac{1}{2}\,\Psi\right)\left[\lambda + \frac{\Phi}{2x + \frac{1}{2}\,\Psi}\,\frac{1}{\lambda}\right]\right\} d\lambda$$

Using the substitution (2.68)

$$S(x) = \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x + \Psi}}\right)^{-\frac{1}{2}} \int_0^{\infty} z^{-\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\,\sqrt{\Phi\,\left(2x + \Psi\right)}\left[z + \frac{1}{z}\right]\right\} dz$$

$$= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x + \Psi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}\sqrt{\Phi\,\left(2x + \Psi\right)}$$

$$= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{2x + \Psi}{\Phi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\sqrt{\Phi\,\left(2x + \Psi\right)} \tag{2.70}$$

The hazard function is,,

$$h(x) = \sqrt{\frac{\Phi}{2x + \Psi}}$$

Hesselager et al. (1998) used the following parameterization

$$\sqrt{\frac{\Phi}{\Psi}} = \sqrt{\frac{c}{b}} \quad \Phi = 2c \quad \text{and} \quad \Psi = 2b$$

Therefore,

$$h(x) = \sqrt{\frac{2c}{2x + 2c\,\frac{b}{c}}}$$

$$= \sqrt{\frac{c}{x + b}} \tag{2.71}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^{\infty} x^r \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x + \Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\sqrt{\Phi\,\left(2x + \Psi\right)}\,dx$$

Let

$$\omega = \sqrt{\Phi\,\left(2x + \Psi\right)} \quad \text{so that} \quad x = 0 \Rightarrow \omega = \sqrt{\Psi\,\Phi} \quad \text{and} \quad x = \infty \Rightarrow \omega = \infty$$

The limits of integration therefore change from $(0, \infty)$ to $(\sqrt{\Psi\,\Phi}, \infty)$. This makes Lemma 1.1 not applicable since it only applies for the limits $(0, \infty)$.

In this case the $\mathbf{r^{th}}$ moment using conditional expectation approach is the only method that is applicable.

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}} \int_0^\infty \lambda^{-r-\frac{3}{2}} \exp\left\{-\frac{\Phi}{2}\,\frac{\Psi}{\Phi}\left(\lambda + \frac{\Phi}{\Psi}\,\frac{1}{\lambda}\right)\right\}$$

Let,

$$\lambda = \sqrt{\left(\frac{\Phi}{\Psi}\right)}z \quad \therefore \quad d\lambda = \sqrt{\left(\frac{\Phi}{\Psi}\right)}dz$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}}}{\sqrt{\left(\frac{\Phi}{\Psi}\right)}^{r+\frac{1}{2}}} \int_0^\infty z^{-r-\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\Phi\,\Psi}\left(z + \frac{1}{z}\right)\right\}$$

$$= 2\,\frac{\left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}}}{\sqrt{\left(\frac{\Phi}{\Psi}\right)}^{r+\frac{1}{2}}}\, K_{r+\frac{1}{2}}(\sqrt{\Phi\,\Psi})$$

$$\therefore \quad E(X^r) = \frac{r!\,\left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}}}{\sqrt{\left(\frac{\Phi}{\Psi}\right)}^{r+\frac{1}{2}}}\, K_{r+\frac{1}{2}}(\sqrt{\Phi\,\Psi}) \tag{2.72}$$

and

$$E(X) = \frac{\left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}}}{\sqrt{\left(\frac{\Phi}{\Psi}\right)}^{\frac{3}{2}}}\, K_{\frac{3}{2}}(\sqrt{\Phi\,\Psi})$$

Using (1.7) and (1.8)

$$E(X) = \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}} \sqrt{\left(\frac{\Phi}{\Psi}\right)}^{-\frac{3}{2}} \left(1 + \frac{1}{\sqrt{\Phi\,\Psi}}\right) K_{\frac{1}{2}}(\sqrt{\Phi\,\Psi})$$

$$= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\,\Psi}} \sqrt{\left(\frac{\Phi}{\Psi}\right)}^{-\frac{3}{2}} \left(1 + \frac{1}{\sqrt{\Phi\,\Psi}}\right) \left(\frac{\pi}{2\,\sqrt{\Phi\,\Psi}}\right)^{\frac{1}{2}} e^{-\sqrt{\Phi\,\Psi}}$$

$$= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} \sqrt{\left(\frac{\Phi}{\Psi}\right)}^{-\frac{3}{2}} \left(1 + \frac{1}{\sqrt{\Phi\,\Psi}}\right) \left(\frac{\pi}{2\,\sqrt{\Phi\,\Psi}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} \sqrt{\left(\frac{\Phi}{\Psi}\right)}^{-\frac{3}{2}} \left(1 + \frac{1}{\sqrt{\Phi\,\Psi}}\right) \left(\frac{\pi}{2\,\Psi\,\sqrt{\left(\frac{\Phi}{\Psi}\right)}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{2\,\Phi}{\pi}\cdot\frac{\pi}{2\Psi}\cdot\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{1}{2}} \sqrt{\left(\frac{\Phi}{\Psi}\right)}^{-\frac{3}{2}} \left(1+\frac{1}{\sqrt{\Phi\,\Psi}}\right)$$

$$= \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{\frac{1}{2}} \sqrt{\left(\frac{\Phi}{\Psi}\right)}^{-\frac{3}{2}} \left(1+\frac{1}{\sqrt{\Phi\,\Psi}}\right)$$

$$= \sqrt{\left(\frac{\Phi}{\Psi}\right)} \left(1+\frac{1}{\sqrt{\Phi\,\Psi}}\right)$$

$$= \sqrt{\left(\frac{\Psi}{\Phi}\right)} + \frac{1}{\Phi} \tag{2.73}$$

We can also obtain $f(x)$ in (2.69) explicitly using (1.7) as follows:

$$f(x) = \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\sqrt{\Phi\ (2x+\Psi)}$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{\frac{1}{2}} \left(\frac{\pi}{2\,\sqrt{\Phi\ (2x+\Psi)}}\right)^{\frac{1}{2}} e^{-\sqrt{\Phi\,(2x+\Psi)}} \tag{2.74}$$

### 2.4.4 Reciprocal inverse Gaussian mixing distribution

Using reciprocal inverse Gaussian mixing distribution, the exponential-inverse Gaussian distribution is constructed and moments are obtained.

$$g(\lambda) = \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}}\, \lambda^{-\frac{1}{2}}\ \exp\left\{-\frac{1}{2}\left(\Phi\,\lambda+\Psi\,\frac{1}{\lambda}\right)\right\} \quad for \quad \lambda>0 \tag{2.75}$$

The pdf of the mixture is,

$$f(x) = \int_0^\infty \lambda\, e^{-\lambda\,x} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}}\, \lambda^{-\frac{1}{2}}\ \exp\left\{-\frac{\Phi}{2}\lambda-\frac{1}{2}\,\Psi\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{1}{2}}\ \exp\left\{-\lambda\,(x+\frac{\Phi}{2})-\frac{1}{2}\,\Psi\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{3}{2}-1}\ \exp\left\{-\frac{1}{2}\,(2x+\Phi)\,\lambda-\frac{1}{2}\,\Psi\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{3}{2}-1}\ \exp\left\{-\frac{1}{2}\,(2x+\Phi)\,\left[\lambda+\frac{\Psi}{(2x+\Phi)}\,\frac{1}{\lambda}\right]\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{3}{2}} \int_0^\infty z^{\frac{3}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\Psi(2x+\Phi)}\left[z+\frac{1}{z}\right]\right\} dz$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{3}{2}} 2\,K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)$$

$$= \left[\left(\frac{\Phi}{2\,\pi}\right)\left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)\right]^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \sqrt{\frac{\Psi}{(2x+\Phi)}}\, 2\,K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)$$

$$= \left[\left(\frac{2\,\Phi}{\pi}\right)\left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)\right]^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \sqrt{\frac{\Psi}{(2x+\Phi)}}\, K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right) \quad (2.76)$$

The survival function is,

$$S(x) = \int_0^\infty e^{-\lambda\,x} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}}\, \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\Phi}{2}\lambda - \frac{1}{2}\,\Psi\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{-\frac{1}{2}} \exp\left\{-\lambda\,(x+\frac{\Phi}{2}) - \frac{1}{2}\,\Psi\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\,(2x+\Phi)\,\lambda - \frac{1}{2}\,\Psi\,\frac{1}{\lambda}\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \int_0^\infty \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\,(2x+\Phi)\,\left[\lambda + \frac{\Psi}{(2x+\Phi)}\,\frac{1}{\lambda}\right]\right\} d\lambda$$

$$= \left(\frac{\Phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} \int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\Psi(2x+\Phi)}\left[z+\frac{1}{z}\right]\right\} dz$$

$$= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} 2\,K_{\frac{1}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)$$

$$= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right) \quad (2.77)$$

Therefore,

$$h(x) = \sqrt{\frac{\Psi}{(2x+\Phi)}}\,\frac{K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)}{K_{\frac{1}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)}$$

But

$$K_{\frac{3}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}\, e^{-\omega}\, [1 + \frac{1}{\omega}]$$

Therefore,

$$\begin{aligned} h(x) &= \sqrt{\frac{\Psi}{(2x+\Phi)}} \left(1 + \sqrt{\frac{1}{\Psi(2x+\Phi)}}\right) \\ &= \sqrt{\frac{\Psi}{(2x+\Phi)}} + \frac{1}{2x+\Phi} \end{aligned} \tag{2.78}$$

which is the sum of a hazard function of an exponential-inverse Gaussian and that of exponential-gamma (Pareto II) distributions.
Explicit forms for $f(x)$ and $S(x)$ are:

$$f(x) = \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{3}{2}} \left[1 + \frac{1}{\sqrt{\Psi(2x+\Phi)}}\right] \left[\frac{\pi}{2\sqrt{\Psi(2x+\Phi)}}\right]^{\frac{1}{2}} e^{-\sqrt{\Psi(2x+\Phi)}}$$

$$= \left(\frac{\Phi}{\sqrt{\Psi(2x+\Phi)}}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right) \left[1 + \frac{1}{\sqrt{\Psi(2x+\Phi)}}\right] e^{-\sqrt{\Psi(2x+\Phi)}}$$

$$= \left(\frac{\Phi}{\sqrt{\Psi(2x+\Phi)}}\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}} + \frac{1}{2x+\Psi}\right) e^{-\sqrt{\Psi(2x+\Phi)}}$$

$$= \sqrt{\frac{\Phi}{2x+\Phi)}}\, e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}} + \frac{1}{2x+\Psi}\right) e^{-\sqrt{\Psi(2x+\Phi)}} \tag{2.79}$$

$$\begin{aligned} S(x) &= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} \left[\frac{\pi}{2\sqrt{\Psi(2x+\Phi)}}\right]^{\frac{1}{2}} e^{-\sqrt{\Psi(2x+\Phi)}} \\ &= \left(\frac{\Phi}{\sqrt{\Psi(2x+\Phi)}}\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}}\, e^{-\sqrt{\Psi(2x+\Phi)}} \\ &= \sqrt{\frac{\Psi}{(2x+\Phi)}}\, e^{\sqrt{\Psi\Phi}}\, e^{-\sqrt{\Psi(2x+\Phi)}} \end{aligned} \tag{2.80}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r \left[\left(\frac{2\,\Phi}{\pi}\right) \left(\sqrt{\frac{\Psi}{(2x+\Phi)}}\right)\right]^{\frac{1}{2}} e^{\sqrt{\Psi\,\Phi}} \sqrt{\frac{\Psi}{(2x+\Phi)}}\, K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)\, dx$$

Let

$$\omega = \sqrt{\Psi(2x+\Phi)} \quad \text{so that} \quad x = 0 \Rightarrow \omega = \sqrt{\Psi\,\Phi} \quad \text{and} \quad x = \infty \Rightarrow \omega = \infty$$

The limits of integration therefore change from $(0, \infty)$ **to** $(\sqrt{\Psi\,\Phi}, \infty)$. This makes Lemma 1.1 not applicable since it only applies for the limits $(0, \infty)$.

In this case the $\mathbf{r^{th}}$ moment using conditional expectation approach is the only method that is applicable.

(5.) Using conditional expectation approach

$$\begin{aligned} E\left[\frac{1}{\Lambda^r}\right] &= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \int_0^\infty \lambda^{-r-\frac{1}{2}} \exp\left\{-\frac{\Phi}{2}\lambda - \frac{\Phi}{2\,\frac{\Phi}{\Psi}}\,\frac{1}{\lambda}\right\} g(\lambda)d\lambda \\ &= \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \int_0^\infty \lambda^{-r+\frac{1}{2}-1} \exp\left\{-\frac{\Phi}{2}\lambda - \frac{\Phi}{2\,\frac{\Phi}{\Psi}}\,\frac{1}{\lambda}\right\} g(\lambda)d\lambda \end{aligned}$$

Let,

$$\lambda = \frac{1}{\left(\sqrt{\frac{\Phi}{\Psi}}\right)} z \quad \therefore \quad d\lambda = \frac{1}{\sqrt{\Phi\Psi}} dz$$

$$\begin{aligned} E(X^r) &= r!\,\left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{r-\frac{1}{2}} \int_0^\infty z^{-(r-\frac{1}{2})-1} \exp\left\{-\frac{1}{2}\,\sqrt{\Phi\Psi}\left(z+\frac{1}{z}\right)\right\} dz \\ &= r!\,2\,\left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{r-\frac{1}{2}} K_{-(r-\frac{1}{2})}(\sqrt{\Phi\Psi}) \\ &= r!\,\left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{r-\frac{1}{2}} K_{-(r-\frac{1}{2})}(\sqrt{\Phi\Psi}) \end{aligned} \tag{2.81}$$

and,

$$\begin{aligned} E(X) &= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Phi\Psi}) \\ &= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{\frac{1}{2}} \left(\frac{\pi}{2\sqrt{\Phi\Psi}}\right)^{\frac{1}{2}} e^{-\sqrt{\Phi\Psi}} \\ &= \left(\frac{2\,\Phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{\frac{1}{2}} \left(\frac{\pi}{2\sqrt{\Phi\Psi}}\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{\Phi}{\Psi}} \end{aligned} \tag{2.82}$$

### 2.4.5 Generalized Inverse Gaussian (GIG) mixing distribution

The Generalized Inverse Gaussian (GIG) distribution is based on the modified Bessel function of the third kind with index $\boldsymbol{v}$ as given in equation (1.6), and it may be constructed as follows:

Let

$$\begin{aligned}
\omega &= \sqrt{\Phi\,\Psi} \\
K_v(\sqrt{\Phi\,\Psi}) &= \frac{1}{2}\int_0^\infty x^{v-1} e^{-\frac{\sqrt{\Phi\,\Psi}}{2}(x+\frac{1}{x})}\,dx \\
&= \frac{1}{2}\int_0^\infty x^{v-1} e^{-\frac{\Phi}{2}\sqrt{\frac{\Psi}{\Phi}}(x+\frac{1}{x})}\,dx \\
&= \frac{1}{2}\int_0^\infty x^{v-1} e^{-\frac{\Phi}{2}\left(\sqrt{\frac{\Psi}{\Phi}}x+\sqrt{\frac{\Psi}{\Phi}}\frac{1}{x}\right)}\,dx
\end{aligned}$$

Let,

$$\begin{aligned}
x &= \sqrt{\frac{\Psi}{\Phi}}\,z \quad \therefore \quad dx = \sqrt{\frac{\Psi}{\Phi}}\,dz \\
K_v(\sqrt{\Phi\,\Psi}) &= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v z^{v-1}\, e^{-\frac{\Phi}{2}\left(\frac{\Psi}{\Phi}z+\frac{1}{z}\right)}\,dz \\
&= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v z^{v-1}\, e^{-\frac{1}{2}\left(\Psi z+\Phi\frac{1}{z}\right)}\,dz \\
\therefore \quad I &= \int_0^\infty \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \frac{z^{v-1}\, e^{-\frac{1}{2}\left(\Psi z+\Phi\frac{1}{z}\right)}}{2\,K_v(\sqrt{\Phi\,\Psi})}\,dz
\end{aligned}$$

Therefore, the GIG mixing distribution is given by

$$g(\lambda) = \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \frac{\lambda^{v-1}\, e^{-\frac{1}{2}\left(\Psi\lambda+\Phi\frac{1}{\lambda}\right)}}{2\,K_v(\sqrt{\Phi\,\Psi})}, \quad \lambda > 0 \tag{2.83}$$

with the parameters taking values in one of the ranges.

1. $\Phi > 0 \quad \Psi \geq 0$ if $\mathbf{v} < 0$
2. $\Phi > 0 \quad \Psi > 0$ if $\mathbf{v} = 0$
3. $\Phi \geq 0 \quad \Psi = 0$ if $\mathbf{v} > 0$

(2.84)

Using the GIG mixing distribution, the exponential-GIG distribution is constructed and moments have been obtained.

The pdf of the mixture is,

$$f(x) = \int_0^\infty \lambda e^{-\lambda x} \quad \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}\lambda^{v-1} \quad \exp\left\{-\frac{1}{2}(\frac{\Phi}{\lambda}+\Psi\lambda)\right\} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \int_0^\infty \lambda^{v+1-1}\exp\left\{-\lambda x-\frac{1}{2}(\frac{\Phi}{\lambda}+\Psi\lambda)\right\} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \int_0^\infty \lambda^{v+1-1}\exp\left\{-\frac{1}{2}(2\lambda x+\Psi\lambda+\frac{\Phi}{\lambda})\right\} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \int_0^\infty \lambda^{v+1-1}\exp\left\{-\frac{2x+\Psi}{2}(\lambda+\frac{\Phi}{2x+\Psi}\ \frac{1}{\lambda})\right\} \quad d\lambda$$

$$\lambda = \sqrt{\frac{\Phi}{2x+\Psi}}\, z \quad \text{and therefore} \quad d\lambda = \sqrt{\frac{\Phi}{2x+\Psi}}\, dz \tag{2.85}$$

$$f(x) = \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{v+1} \int_0^\infty z^{v+1-1}\exp\left\{\frac{\sqrt{2x\Phi+\Phi\Psi}}{2}(z+\frac{1}{z})\right\} dz$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{K_v(\sqrt{\Phi\Psi})} \quad \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{v+1} \quad K_{v+1}\,(\sqrt{\Phi(2x+\Psi)})$$

$$= \sqrt{\frac{\Phi}{\Psi}}\ \left[\sqrt{\frac{\Psi}{2x+\Psi}}\right]^{v+1} \frac{K_{v+1}\,(\sqrt{\Phi(2x+\Psi)})}{K_v(\sqrt{\Phi\Psi})} \tag{2.86}$$

The survival function is,

$$S(x) = \int_0^\infty e^{-\lambda x} \quad \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}\lambda^{v-1} \quad \exp\left\{-\frac{1}{2}(\frac{\Phi}{\lambda}+\Psi\lambda)\right\} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \int_0^\infty \lambda^{v-1}\exp\left\{-\lambda x-\frac{1}{2}(\frac{\Phi}{\lambda}+\Psi\lambda)\right\} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \int_0^\infty \lambda^{v-1}\exp\left\{-\frac{1}{2}(2\lambda x+\Psi\lambda+\frac{\Phi}{\lambda})\right\} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \int_0^\infty \lambda^{v-1}\exp\left\{-\frac{2x+\Psi}{2}(\lambda+\frac{\Phi}{2x+\Psi}\ \frac{1}{\lambda})\right\} \quad d\lambda$$

Using the substitution (2.85)

$$S(x) = \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \quad \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{v} \int_0^\infty z^{v-1}\exp\left\{\frac{\sqrt{2x\Phi+\Phi\Psi}}{2}(z+\frac{1}{z})\right\} \quad dz$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{K_v(\sqrt{\Phi\Psi})} \quad \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{v} K_v\,(\sqrt{\Phi(2x+\Psi)})$$

$$= (\frac{\Psi}{\Phi})^{\frac{v}{2}} \left(\frac{\Phi}{2x+\Psi}\right)^{\frac{v}{2}} \frac{K_v\,(\sqrt{\Phi(2x+\Psi)})}{K_v(\sqrt{\Phi\Psi})}$$

$$= \left(\frac{\Psi}{2x+\Psi}\right)^{\frac{v}{2}} \frac{K_v\,(\sqrt{\Phi(2x+\Psi)})}{K_v(\sqrt{\Phi\Psi})} \tag{2.87}$$

The hazard function is,

$$h(x) = \frac{\frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{K_v(\sqrt{\Phi\Psi})} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{v+1} K_{v+1}\,(\sqrt{\Phi(2x+\Psi)})}{\frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{K_v(\sqrt{\Phi\Psi})} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{v} K_v\,(\sqrt{\Phi(2x+\Psi)})}$$

$$= \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right) \frac{K_{v+1}\,(\sqrt{\Phi(2x+\Psi)})}{K_v\,(\sqrt{\Phi(2x+\Psi)})} \tag{2.88}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r \sqrt{\frac{\Phi}{\Psi}} \left[\sqrt{\frac{\Psi}{2x+\Psi}}\right]^{v+1} \frac{K_{v+1}\,(\sqrt{\Phi(2x+\Psi)})}{K_v(\sqrt{\Phi\Psi})}\, dx$$

Let

$$\omega = \sqrt{\Psi(2x+\Phi)} \quad \text{so that} \quad x = 0 \Rightarrow \omega = \sqrt{\Psi\,\Phi} \quad \text{and} \quad x = \infty \Rightarrow \omega = \infty$$

The limits of integration therefore change from $(0, \infty)$ **to** $(\sqrt{\Psi\,\Phi}, \infty)$. This makes Lemma 1.1 not applicable since it only applies for the limits $(0, \infty)$.

In this case the $\mathbf{r^{th}}$ moment using conditional expectation approach is the only method that is applicable.

(5.) Using conditional expectation approach

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty \lambda^{v-r-1} \exp\left\{-\frac{1}{2}(\frac{\Phi}{\lambda} + \Psi\lambda)\right\} d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty \lambda^{v-r-1} \exp\left\{-\frac{1}{2}(\Psi\lambda + \frac{\Phi}{\lambda})\right\} d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty \lambda^{v-r-1} \exp\left\{-\frac{\Psi}{2}(\lambda + \frac{\Phi}{\Psi}\frac{1}{\lambda})\right\} d\lambda$$

Let $\lambda = \sqrt{\frac{\Phi}{\Psi}}\,\mathrm{z} \quad \therefore \quad \mathrm{d}\lambda = \sqrt{\frac{\Phi}{\Psi}}\,\mathrm{dz}$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{v-r} \int_0^\infty z^{v-r-1} \exp\left\{\frac{\sqrt{\Phi\Psi}}{2}(z + \frac{1}{z})\right\} dz$$

$$= \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{-r} \frac{K_{v-r}(\sqrt{\Psi\Phi})}{K_v(\sqrt{\Psi\Phi})}$$

$$\therefore \quad E(X^r) = r! \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r} \frac{K_{v-r}(\sqrt{\Psi\Phi})}{K_v(\sqrt{\Psi\Phi})} \tag{2.89}$$

$$E(X) = \left(\sqrt{\frac{\Psi}{\Phi}}\right) \frac{K_{v-1}(\sqrt{\Psi\Phi})}{K_v(\sqrt{\Psi\Phi})} \tag{2.90}$$

### 2.4.6 Special cases of GIG mixing distribution

The exponential-inverse Gaussian, the exponential-reciprocal inverse Gaussian, gamma I and Pareto II distributions are special cases of the exponential-generalized inverse Gaussian when $v = -\frac{1}{2}$, $v = \frac{1}{2}$, $\Phi \geq 0 \quad \Psi = 0$ if $v > 0$ and $\Psi = 0$, $v < 0$ respectively.

**a) When $v = -\frac{1}{2}$**

$$f(x) = \sqrt{\frac{\Phi}{2x+\Psi}} \left(\sqrt{\frac{\Psi}{2x+\Psi}}\right)^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}{K_{-\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

$$= \sqrt{\frac{\Phi}{2x+\Psi}} \left(\sqrt{\frac{\Psi}{2x+\Psi}}\right)^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}{\left(\frac{\pi}{2\sqrt{\Phi\Psi}}\right)^{\frac{1}{2}} e^{-\sqrt{\Phi\Psi}}}$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2\sqrt{\Phi\Psi}}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Psi}{\Psi}}\right)^{\frac{1}{2}} \sqrt{\frac{\Phi}{2x+\Psi}}\, K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2\sqrt{\Phi\Psi}}{\pi} \sqrt{\frac{2x+\Psi}{\Psi}}\right)^{\frac{1}{2}} \sqrt{\frac{\Phi}{2x+\Psi}}\, K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2}{\pi} \sqrt{\frac{\Phi\Psi(2x+\Psi)}{\Psi}}\right)^{\frac{1}{2}} \sqrt{\frac{\Phi}{2x+\Psi}}\, K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2}{\pi} \sqrt{\Phi(2x+\Psi)} \frac{\Phi}{2x+\Psi}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2\Phi}{\pi} \sqrt{\frac{\Phi}{2x+\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right) \tag{2.91}$$

$$\boldsymbol{S(x)} = \left(\sqrt{\frac{\Psi}{2x+\Psi}}\right)^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}{K_{-\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

$$= \left(\sqrt{\frac{\Psi}{2x+\Psi}}\right)^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}{\left(\frac{\pi}{2\sqrt{\Phi\Psi}}\right)^{\frac{1}{2}} e^{-\sqrt{\Phi\Psi}}}$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2\sqrt{\Phi\Psi}}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Psi}{\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2\sqrt{\Phi\Psi}}{\pi}\sqrt{\frac{2x+\Psi}{\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2\sqrt{\Phi\Psi}}{\pi}\sqrt{\frac{2x+\Psi}{\Psi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{2}{\pi}\sqrt{\Phi(2x+\Psi)}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right) \tag{2.92}$$

The hazard function is,

$$\boldsymbol{h(x)} = \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right) \frac{K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}{K_{\frac{1}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}$$

$$= \left(\sqrt{\frac{\Phi}{2x+\Psi}}\right) \tag{2.93}$$

$$\boldsymbol{E(X^r)} = r!\left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r} \frac{K_{-\frac{1}{2}-r}(\sqrt{\Psi}\Phi)}{K_{-\frac{1}{2}}(\sqrt{\Psi}\Phi)} \tag{2.94}$$

$$\boldsymbol{E(X)} = \left(\sqrt{\frac{\Psi}{\Phi}}\right) \frac{K_{-\frac{3}{2}}(\sqrt{\Psi}\Phi)}{K_{-\frac{1}{2}}(\sqrt{\Psi}\Phi)}$$

$$= \left(\sqrt{\frac{\Psi}{\Phi}}\right)\left[1+\frac{1}{\sqrt{\Phi\Psi}}\right]$$

$$= \left(\sqrt{\frac{\Psi}{\Phi}}\right) + \frac{1}{\Phi} \tag{2.95}$$

which is the exponential-inverse Gaussian distribution.

**b) When $v = \frac{1}{2}$**

$$\boldsymbol{f(x)} = \sqrt{\frac{\Phi}{2x+\Psi}}\left(\sqrt{\frac{\Psi}{2x+\Psi}}\right)^{\frac{1}{2}} \frac{K_{\frac{3}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}{K_{\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

But the modified Bessel function for the Reciprocal inverse Gaussian is $\boldsymbol{K_{\frac{3}{2}}\left(\sqrt{\Phi(2x+\Psi)}\right)}$

Therefore $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are interchanged and thus

$$\begin{aligned}
\boldsymbol{f(x)} &= \sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{2x+\Phi}}}\left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{2x+\Phi}}}\right)^{\frac{1}{2}} \frac{\boldsymbol{K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)}}{\boldsymbol{K_{-\frac{1}{2}}(\sqrt{\Psi\Phi})}} \\
&= \sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{2x+\Phi}}}\left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{2x+\Phi}}}\right)^{\frac{1}{2}} \frac{\boldsymbol{K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)}}{\left(\frac{\boldsymbol{\pi}}{\boldsymbol{2\sqrt{\Psi\Phi}}}\right)^{\frac{1}{2}} \boldsymbol{e^{-\sqrt{\Psi\Phi}}}} \\
&= \boldsymbol{e^{\sqrt{\Psi\Phi}}}\left(\frac{\boldsymbol{2\sqrt{\Psi\Phi}}}{\boldsymbol{\pi}}\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{2x+\Phi}}}\right)^{\frac{1}{2}} \sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{2x+\Phi}}}\, \boldsymbol{K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)} \\
&= \boldsymbol{e^{\sqrt{\Psi\Phi}}}\left(\frac{\boldsymbol{2\Phi}}{\boldsymbol{\pi}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{2x+\Phi}}}\right)^{\frac{1}{2}} \sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{2x+\Phi}}}\, \boldsymbol{K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)} \\
&= \boldsymbol{e^{\sqrt{\Psi\Phi}}}\left(\frac{\boldsymbol{2\Phi}}{\boldsymbol{\pi}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{2x+\Phi}}}\right)^{\frac{3}{2}} \boldsymbol{K_{\frac{3}{2}}\left(\sqrt{\Psi(2x+\Phi)}\right)} \qquad (2.96)
\end{aligned}$$

Once again by interchanging $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ in $\boldsymbol{E(X^r)}$, we have

$$\boldsymbol{E(X^r)} = \boldsymbol{r!}\left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{\Psi}}}\right)^{\boldsymbol{r}} \frac{\boldsymbol{K_{\frac{1}{2}-r}(\sqrt{\Psi\Phi})}}{\boldsymbol{K_{\frac{1}{2}}(\sqrt{\Psi\Phi})}} \qquad (2.97)$$

$$\boldsymbol{E(X)} = \left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{\Psi}}}\right) \qquad (2.98)$$

which is the exponential-reciprocal inverse Gaussian distribution.

## 2.5 Mixtures in terms confluent hyper-geometric functions

### 2.5.1 Beta I mixing distribution

The beta I distribution is based on the beta I function in (1.3), and the exponential-beta I mixture is constructed below together with the moments.

$$g(\lambda) = \frac{\lambda^{p-1}(1-\lambda)^{q-1}}{B(p,q)} \; 0 < \lambda < 1, \; p,q > 0 \tag{2.99}$$

The pdf of the mixture is,

$$\begin{aligned}
f(x) &= \int_0^1 \lambda e^{-\lambda x} \frac{\lambda^{p-1}(1-\lambda)^{q-1}}{B(p,q)} d\lambda \\
&= \frac{1}{B(p,q)} \int_0^1 \lambda^{p+1-1}(1-\lambda)^{q-1} e^{-\lambda x} d\lambda \\
&= \frac{1}{B(p,q)} \int_0^1 \lambda^{p+1-1}(1-\lambda)^{(p+q+1)-(p+1)-1} e^{-\lambda x} d\lambda \\
&= \frac{B(p+1,q)}{B(p,q)} \, {}_1F_1(p+1, p+q+1, -x) \\
&= \frac{p}{p+q} \, {}_1F_1(p+1; p+q+1; -x)
\end{aligned} \tag{2.100}$$

The survival function is,

$$\begin{aligned}
S(x) &= \int_0^1 e^{-\lambda x} \frac{\lambda^{p-1}(1-\lambda)^{q-1}}{B(p,q)} d\lambda \\
&= \frac{1}{B(p,q)} \int_0^1 \lambda^{p-1}(1-\lambda)^{(p+q)-(p)-1} e^{-\lambda x} d\lambda \\
&= {}_1F_1(p; p+q; -x)
\end{aligned} \tag{2.101}$$

and the hazard function is

$$h(x) = \frac{p}{p+q} \frac{{}_1F_1(p+1; p+q+1; -x)}{{}_1F_1(p; p+q; -x)} \tag{2.102}$$

The $r^{th}$ moment about zero is,

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r \frac{p}{p+q} \, {}_1F_1(p+1; p+q+1; -x) dx \\
&= \frac{p}{p+q} \int_0^\infty x^r \, {}_1F_1(p+1; p+q+1; -x) dx
\end{aligned}$$

Applying (1.21)

$$\int_0^\infty x^{s-1}\, {}_1F_1(a;c;-x)dx = \frac{\Gamma(c)\,\Gamma s}{\Gamma(c-s)}\,\frac{\Gamma(a-s)}{\Gamma(a)}$$

$$\begin{aligned} E(X^r) &= \frac{p}{p+q}\int_0^\infty x^{(r+1)-1}\, {}_1F_1(p+1;p+q+1;-x)dx \\ &= \frac{p}{p+q}\,\frac{\Gamma(p+q+1)\,\Gamma(r+1)}{\Gamma(p+q-r)}\,\frac{\Gamma(p-r)}{\Gamma(p+1)} \\ &= pr!\,\frac{\Gamma(p+q)\,\Gamma(p-r)}{\Gamma(p+q-r)\,p\,\Gamma(p)} \\ &= r!\,\frac{\Gamma(p+q)\,\Gamma(p-r)}{\Gamma(p+q-r)\,\Gamma(p)} \end{aligned} \tag{2.103}$$

Using conditional expectation approach, we have

$$\begin{aligned} E\left[\frac{1}{\Lambda^r}\right] &= \int_0^1 \frac{\lambda^{p-r-1}(1-\lambda)^{q-1}}{B(p,q)}d\lambda \\ &= \frac{B(p-r,q)}{B(p,q)} \\ &= \frac{\Gamma(p-r)}{\Gamma(p)}\,\frac{\Gamma(p+q)}{\Gamma(p+q-r)} \end{aligned}$$

$$\therefore\quad E(X^r) = r!\,\frac{\Gamma(p-r)}{\Gamma(p)}\,\frac{\Gamma(p+q)}{\Gamma(p+q-r)} \tag{2.104}$$

and,

$$E(X) = \frac{p+q-1}{p-1} \tag{2.105}$$

### 2.5.2 Uniform mixing distribution

Using uniform mixing distribution, the exponential-uniform distribution is constructed and moments are obtained.

$$g(\lambda) = \frac{1}{b-a} \quad a \le \lambda \le b \tag{2.106}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_a^b \lambda e^{-\lambda x}\,\frac{1}{b-a}\,d\lambda \\ &= \frac{1}{b-a}\left\{\int_0^b \lambda\, e^{-\lambda x}\,d\lambda - \int_0^a \lambda\, e^{-\lambda x}\,d\lambda\right\} \end{aligned}$$

$$= \frac{1}{b-a}\left\{\int_0^b \lambda^{2-1}\, e^{-\lambda x}\, d\lambda - \int_0^a \lambda^{2-1}\, e^{-\lambda x}\ d\lambda\right\}$$

$$= \frac{1}{b-a}\ \left\{\frac{\gamma(2,\ bx)}{x^2} - \frac{\gamma(2,\ ax)}{x^2}\right\}$$

$$= \frac{1}{(b-a)\ x^2}\ \{\gamma(2,\ bx) - \gamma(2,\ ax)\}$$

But

$$\gamma(a,x) = \frac{x^a}{a}\ e^{-x}\ {}_1F_1(a;\ a+1;\ -x)$$

we have

$$f(x) = \frac{1}{(b-a)\ x^2}\ \left\{\frac{(bx)^2}{2}\ e^{-bx}\ {}_1F_1(2;\ 3;\ -bx) - \frac{(ax)^2}{2}\ e^{-ax}\ {}_1F_1(2;\ 3;\ -ax)\right\}$$

$$= \frac{1}{2\ (b-a)}\ \left\{b^2\ e^{-bx}\ {}_1F_1(1;\ 3;\ bx) - a^2\ e^{-ax}\ {}_1F_1(1;\ 3;\ ax)\right\} \tag{2.107}$$

The survival function is,

$$S(x) = \frac{1}{b-a}\ \int_a^b\ e^{-\lambda x}\ d\lambda$$

$$= \frac{1}{b-a}\left[\frac{e^{-x\lambda}}{-x}\right]_a^b$$

$$= -\frac{1}{x(b-a)}\ \left[e^{-ax} - e^{-bx}\right]$$

$$= \frac{\left[e^{-ax} - e^{-bx}\right]}{x(b-a)} \tag{2.108}$$

The hazard function is,,

$$h(x) = \frac{x}{2\ (e^{-ax} - e^{-bx})}\ \left\{b^2\ e^{-bx}\ {}_1F_1(1;\ 3;\ bx) - a^2\ e^{-ax}\ {}_1F_1(1;\ 3;\ ax)\right\} \tag{2.109}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty\ x^r\ \frac{1}{2\ (b-a)}\ \left\{b^2\ e^{-bx}\ {}_1F_1(1;\ 3;\ bx) - a^2\ e^{-ax}\ {}_1F_1(1;\ 3;\ ax)\right\} dx$$

$$= \frac{1}{2\ (b-a)}\ \int_0^\infty\ x^r\ \left\{b^2\ e^{-bx}\ {}_1F_1(1;\ 3;\ bx) - a^2\ e^{-ax}\ {}_1F_1(1;\ 3;\ ax)\right\} dx$$

Let us consider

$$I = b^2 \int_0^\infty x^r \, e^{-bx} \, {}_1F_1(1;\, 3;\, bx) dx$$

Put

$$y = bx \quad \text{so that} \quad x = \frac{y}{b} \quad and \quad dx = \frac{dy}{b}$$

$$\therefore \quad I = b^2 \int_0^\infty \left(\frac{y}{b}\right)^r \, e^{-y} \, {}_1F_1(1;\, 3;\, y) \frac{dy}{b}$$

$$= \frac{b^2}{b^{r+1}} \int_0^\infty y^r \, e^{-y} \, {}_1F_1(1;\, 3;\, y) dy$$

$$= b^{1-r} \int_0^\infty y^r \, e^{-y} \, {}_1F_1(1;\, 3;\, y) dy$$

Applying (1.22)

$$I = b^{1-r} \, \frac{\Gamma(r+1)\, \Gamma(3)\, \Gamma(3-1-r-1)}{\Gamma(3-1)\, \Gamma(3-r-1)}$$

$$= b^{1-r} \, \frac{\Gamma(r+1)\, 2\Gamma(2)\, \Gamma(1-r)}{\Gamma(2)\, \Gamma(2-r)}$$

$$= b^{1-r} \, \frac{r!\, 2}{\Gamma(1-r)(1-r)\, \Gamma(1-r)}$$

$$= \frac{2\, r!\, b^{1-r}}{(1-r)}$$

$$\therefore \quad E(X^r) = \frac{1}{2\,(b-a)} \left\{ \frac{2\, r!\, b^{1-r}}{(1-r)} - \frac{2\, r!\, a^{1-r}}{(1-r)} \right\}$$

$$= \frac{r!}{(b-a)} \left\{ \frac{b^{1-r}}{(1-r)} - \frac{a^{1-r}}{(1-r)} \right\} \tag{2.110}$$

Using conditional expectation approach, we have

$$E\left(\frac{1}{\Lambda^r}\right) = \frac{1}{b-a} \int_a^b \lambda^{-r} \, d\lambda$$

$$= \frac{1}{b-a} \left[ \frac{\lambda^{1-r}}{1-r} \right]_a^b$$

$$= \frac{1}{b-a} \left[ \frac{b^{1-r}}{1-r} - \frac{a^{1-r}}{1-r} \right]$$

$$\therefore \quad E(X^r) = r! \, \frac{1}{b-a} \left[ \frac{b^{1-r}}{1-r} - \frac{a^{1-r}}{1-r} \right] \tag{2.111}$$

and,

$$E(X) = \infty \tag{2.112}$$

### 2.5.3 Beta II mixing distribution

The beta II distribution is based on the beta II function in (1.4) the exponential-beta II mixture has been constructed below and the moments obtained.

$$g(\lambda) = \frac{\lambda^{p-1}}{B(p,q)\ (1+\lambda)^{p+q}}, \quad \lambda > 0; \quad p,\ q > 0 \tag{2.113}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^\infty \lambda e^{-\lambda x}\ \frac{\lambda^{p-1}}{B(p,q)\ (1+\lambda)^{p+q}}\ d\lambda \\ &= \frac{1}{B(p,q)} \int_0^\infty \frac{\lambda^p}{(1+\lambda)^{p+q}}\ e^{-\lambda x}\ d\lambda \end{aligned}$$

Using the Tricomi function in (1.11):

$$\Psi(a;c;x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{(c-a)-1}\ e^{-xt}\ dt \quad for\ \ a,b,>0$$

$$\begin{aligned} f(x) &= \frac{\Gamma(p+1)}{B(p,q)}\ \Psi(p+1;2-q;x) \\ &= \frac{p\,\Gamma(p+q)}{\Gamma q}\ \Psi(p+1;2-q;x) \end{aligned} \tag{2.114}$$

and the survival function is

$$\begin{aligned} S(x) &= \int_0^\infty e^{-\lambda x}\ \frac{\lambda^{p-1}}{B(p,q)\ (1+\lambda)^{p+q}}\ d\lambda \\ &= \frac{1}{B(p,q)} \int_0^\infty \frac{\lambda^{p-1}}{(1+\lambda)^{p+q}}\ e^{-\lambda x}\ d\lambda \\ &= \frac{\Gamma(p)}{B(p,q)}\ \Psi(p;1-q;x) \\ &= \frac{\Gamma(p+q)}{\Gamma q}\ \Psi(p,1-q;x) \end{aligned} \tag{2.115}$$

Therefore the hazard function is

$$h(x) = \frac{p\ \Psi(p+1;2-q;x)}{\Psi(p,1-q;x)} \tag{2.116}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{p\,\Gamma(p+q)}{\Gamma q} \int_0^\infty x^r\ \Psi(p+1;2-q;x)dx$$

$$= \frac{p\,\Gamma(p+q)}{\Gamma q}\, M(\Psi(p+1; 2-q; x),\ r+1)$$

But from (1.19)

$$M\,[\Psi(a;\ c;\ x),\ s] = \frac{\Gamma(s)\,\Gamma(a-s)\,\Gamma(s-c+1)}{\Gamma(a)\,\Gamma(a-c+1)}$$

$$\therefore \quad E(X^r) = \frac{p\,\Gamma(p+q)}{\Gamma q}\ \frac{\Gamma(r+1)\,\Gamma(p+1-r-1)\,\Gamma(r+1-2+q+1)}{\Gamma(p+1)\,\Gamma(p+1-2+q+1)}$$

$$= \frac{p\,\Gamma(p+q)}{\Gamma q}\ \frac{\Gamma(r+1)\,\Gamma(p-r)\,\Gamma(q+r)}{\Gamma(p+1)\,\Gamma(p+q)}$$

$$= r!\ \frac{\Gamma(p-r)}{\Gamma(p)}\ \frac{\Gamma(q+r)}{\Gamma(q)} \tag{2.117}$$

Alternatively

$$E\left[\frac{1}{\Lambda^r}\right] = \int_0^\infty \frac{\lambda^{p-r-1}}{B(p,q)\ (1+\lambda)^{p+q}} d\lambda$$

$$= \frac{B(p-r, q+r)}{B(p,q)}$$

$$= \frac{\Gamma(p-r)}{\Gamma(p)}\ \frac{\Gamma(q+r)}{\Gamma(q)}$$

$$\therefore \quad E(X^r) = r!\ \frac{\Gamma(p-r)}{\Gamma(p)}\ \frac{\Gamma(q+r)}{\Gamma(q)} \tag{2.118}$$

and,

$$E(X) = \frac{q}{p-1} \tag{2.119}$$

### 2.5.4 Scaled beta mixing distribution

Scaled beta distribution can be obtained from the classical beta distribution with parameters $\alpha$ and $\beta$, using the change of variable technique as follows:

Let

$$t = \frac{\lambda}{\mu}$$

where $t$ comes from the classical beta distribution.

Therefore, the distribution of $\lambda$ is given by

$$g(\lambda) = \frac{\left(\frac{\lambda}{\mu}\right)^{\alpha-1}\left(1-\frac{\lambda}{\mu}\right)^{\beta-1}}{\mu\, B(\alpha,\beta)}$$

$$= \frac{\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{\mu^{\alpha+\beta-1}\, B(\alpha,\beta)} \quad 0 < \lambda < \mu;\ \alpha,\beta > 0 \tag{2.120}$$

The pdf of the mixture is,

$$f(x) = \int_0^{\mu} \lambda\, e^{-\lambda x} \frac{\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{\mu^{\alpha+\beta-1}\, B(p,q)}\, d\lambda$$

$$= \frac{1}{\mu^{\alpha+\beta-1}\, B(\alpha,\beta)} \int_0^{\mu} \lambda^{\alpha+1-1}(\mu-\lambda)^{\beta-1}\, e^{-\lambda x}\, d\lambda$$

Let,

$$\lambda = \mu z \quad \Longrightarrow \quad z = \frac{\lambda}{\mu} \quad \text{and} \quad d\lambda = \mu dz \tag{2.121}$$

$$f(x) = \frac{\mu}{B(\alpha,\beta)} \int_0^{1} z^{\alpha+1-1}(1-z)^{\beta-1}\, e^{-\mu z x}\, dz$$

$$= \frac{\mu\, B(\alpha+1,\beta)}{B(\alpha,\beta)}\, {}_1F_1(\alpha+1;\alpha+\beta+1;-\mu x)$$

$$= \frac{\mu\, \alpha}{\alpha+\beta}\, {}_1F_1(\alpha+1;\alpha+\beta+1;-\mu x) \tag{2.122}$$

The survival function is,

$$S(x) = \int_0^{\mu} e^{-\lambda x} \frac{\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{\mu^{\alpha+\beta-1}\, B(\alpha,\beta)}\, d\lambda$$

$$= \frac{1}{\mu^{\alpha+\beta-1}\, B(\alpha,\beta)} \int_0^{\mu} \lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}\, e^{-\lambda x}\, d\lambda$$

Using substitution (2.121)

$$S(x) = \frac{1}{B(\alpha,\beta)} \int_0^{1} z^{\alpha-1}(1-z)^{\beta-1}\, e^{-\mu z x}\, dz$$

$$= {}_1F_1(\alpha;\alpha+\beta;-\mu x) \tag{2.123}$$

The hazard function is,,

$$h(x) = \frac{\mu\, B(\alpha+1,\beta)}{B(\alpha,\beta)}\ \frac{F(\alpha+1,\alpha+\beta+1,-\mu x)}{F(\alpha,\alpha+\beta,-\mu x)}$$

$$= \frac{\mu\, \alpha}{\alpha+\beta}\ \frac{{}_1F_1(\alpha+1,\alpha+\beta+1,-\mu x)}{{}_1F_1(\alpha;\alpha+\beta;-\mu x)} \tag{2.124}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^{\infty} x^r\, \frac{\mu\, \alpha}{\alpha+\beta}\, {}_1F_1(\alpha+1;\alpha+\beta+1;-\mu x) dx$$

$$= \frac{\mu\,\alpha}{\alpha+\beta}\int_0^\infty x^{(r+1)-1}\;{}_1F_1(\alpha+1;\alpha+\beta+1;-\mu x)dx$$

Let

$$y = \mu x \quad \text{so that} \quad x = \frac{y}{\mu} \quad and \quad dx = \frac{dy}{\mu}$$

$$\therefore \quad E(X^r) = \frac{\mu\,\alpha}{\alpha+\beta}\int_0^\infty \left(\frac{y}{\mu}\right)^{(r+1)-1}\;{}_1F_1(\alpha+1;\alpha+\beta+1;-y)\frac{dy}{\mu}$$

$$= \frac{\mu\,\alpha}{\alpha+\beta}\,\frac{1}{\mu^{r+1}}\int_0^\infty y^{(r+1)-1}\;{}_1F_1(\alpha+1;\alpha+\beta+1;-y)dy$$

Applying (1.21)

$$E(X^r) = \frac{\alpha}{\mu^r(\alpha+\beta)}\int_0^\infty y^{(r+1)-1}\;{}_1F_1(\alpha+1;\alpha+\beta+1;-y)dy$$

$$= \frac{\alpha}{\mu^r(\alpha+\beta)}\,\frac{\Gamma(r+1)\,\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-r-1)}\,\frac{\Gamma(\alpha+1-r-1)}{\Gamma(\alpha+1)}$$

$$= \frac{\alpha}{\mu^r(\alpha+\beta)}\,\frac{\Gamma(r+1)\,\Gamma(\alpha+\beta+1)\,\Gamma(\alpha-r)}{\Gamma(\alpha+\beta-r)\Gamma(\alpha+1)}$$

$$= \frac{\alpha}{\mu^r(\alpha+\beta)}\,\frac{r!\,\Gamma(\alpha+\beta+1)\,\Gamma(\alpha-r)}{\Gamma(\alpha+\beta-r)\Gamma(\alpha)}$$

$$= \frac{r!\,\Gamma(\alpha+\beta)\,\Gamma(\alpha-r)}{\Gamma(\alpha)\,\mu^r\,\Gamma(\alpha+\beta-r)}$$

$$= \frac{r!\,B(\beta,\alpha-r)}{\mu^r\,B(\alpha,\beta)} \tag{2.125}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \int_0^\mu \frac{\lambda^{\alpha-r-1}(\mu-\lambda)^{\beta-1}}{\mu^{\alpha+\beta-1}\,B(\alpha,\beta)}d\lambda$$

$$= \frac{1}{\mu^{\alpha+\beta-1}\,B(\alpha,\beta)}\int_0^\mu \lambda^{\alpha-r-1}(\mu-\lambda)^{\beta-1}\,d\lambda$$

Let,

$$\lambda = \mu z \quad \therefore \quad d\lambda = \mu\,dz$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\mu^{\alpha-r-1}}{\mu^{\alpha+\beta-1}\,B(\alpha,\beta)}\int_0^1 z^{\alpha-r-1}(\mu-\mu z)^{\beta-1}\,\mu dz$$

$$E(X^r) = \frac{\mu^{\alpha-r-1}\,\mu^{\beta-1}\,\mu}{\mu^{\alpha+\beta-1}\,B(\alpha,\beta)}\,B(\alpha-r,\beta)$$

$$= \frac{r!}{\mu^r} \frac{B(\alpha - r, \beta)}{B(\alpha, \beta)}$$

and

$$\begin{aligned} E(X) &= \frac{B(\alpha-1,\beta)}{\mu\, B(\alpha,\beta)} \\ &= \frac{\Gamma(\alpha-1)\,\Gamma(\beta)\,\Gamma(\alpha+\beta)}{\mu\,\Gamma(\alpha+\beta-1)\,\Gamma(\alpha)\,\Gamma(\beta)} \\ &= \frac{1}{\mu}\left\{\frac{\alpha+\beta-1}{(\alpha-1)}\right\}, \quad \alpha \neq 1 \end{aligned} \tag{2.126}$$

### 2.5.5 Full beta mixing distribution

Kempton (1975) mixed two Gamma distributions to obtain what he called full beta model given by:

$$\begin{aligned} g(\lambda) &= \int_0^\infty \frac{a^p}{\Gamma(p)}\, e^{-a\lambda}\, \lambda^{p-1}\, \frac{1}{b^q\,\Gamma(q)}\, e^{-\frac{a}{b}}\, a^{q-1}\, da \\ &= \frac{b^p}{B(p,q)}\, \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}} \quad \lambda > 0; \quad p,q > 0 \end{aligned} \tag{2.127}$$

The pdf of the mixture is,

$$f(x) = \frac{b^p}{B(p,q)} \int_0^\infty e^{-\lambda x} \lambda^{p+1-1}\, (1+b\lambda)^{-p-q}\, d\lambda$$

Let,

$$z = b\lambda \quad \Longrightarrow \quad \lambda = \frac{z}{b} \quad \text{and} \quad d\lambda = \frac{dz}{b} \tag{2.128}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{b^p}{B(p,q)}\, \frac{1}{b^{p+1}} \int_0^\infty z^{p+1-1}\, (1+z)^{-p-q}\, e^{-\frac{x}{b}\, z}\, dz \\ &= \frac{\Gamma(p+1)}{b\, B(p,q)}\, \Psi(p+1; 2-q; \frac{x}{b}) \end{aligned} \tag{2.129}$$

The survival function is,

$$S(x) = \frac{b^p}{B(p,q)} \int_0^\infty e^{-\lambda x} \lambda^{p-1}\, (1+b\lambda)^{-p-q}\, d\lambda$$

Using substitution (2.128)

$$S(x) = \frac{b^p}{B(p,q)}\, \frac{1}{b^p} \int_0^\infty z^{p-1}\, (1+z)^{-p-q}\, e^{-\frac{x}{b}\, z}\, dz$$

$$= \frac{\Gamma(p)}{B(p,q)}\ \Psi(p;1-q;\frac{x}{b})$$

$$= \frac{\Gamma(p+q)}{\Gamma(q)}\ \Psi(p;1-q;\frac{x}{b}) \tag{2.130}$$

The hazard function is,,

$$h(x) = \frac{p}{b}\ \frac{\Psi(p+1;2-q;\frac{x}{b})}{\Psi(p;1-q;\frac{x}{b})} \tag{2.131}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{\Gamma(p+1)}{b\,B(p,q)} \int_0^\infty x^r\ \Psi(p+1;2-q;\frac{x}{b})\ dx$$

Let,

$$t = \frac{x}{b} \quad \therefore \quad x = t\,b \quad \text{and} \quad dx = b\,dt$$

$$E(X^r) = \frac{\Gamma(p+1)}{b\,B(p,q)}\ b^{r+1} \int_0^\infty t^r\ \Psi(p+1;2-q;t)\ dt$$

But from (1.19)

$$M\ [\Psi(a;\ c;\ x),\ s] = \frac{\Gamma(s)\,\Gamma(a-s)\,\Gamma(s-c+1)}{\Gamma(a)\,\Gamma(a-c+1)}$$

$$\therefore \quad E(X^r) = \frac{b^r\ \Gamma(p+1)}{B(p,q)}\ \frac{\Gamma(r+1)\,\Gamma(p+1-r-1)\,\Gamma(r+1-2+q+1)}{\Gamma(p+1)\,\Gamma(p+1-2+q+1)}$$

$$= r!\ b^r\ \frac{\Gamma(p-r)}{\Gamma(p)}\ \frac{\Gamma(q+r)}{\Gamma(q)} \tag{2.132}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \int_0^\infty \frac{b^p}{B(p,q)}\ \frac{\lambda^{p-r-1}}{(1+b\lambda)^{p+q}} d\lambda$$

$$= \frac{b^p}{B(p,q)} \int_0^\infty \frac{\lambda^{p-r-1}}{(1+b\lambda)^{p+q}} d\lambda$$

Let,

$$y = b\,\lambda \quad \therefore \quad dy = b\,d\lambda$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{b^r}{B(p,q)} \int_0^\infty \frac{y^{p-r-1}}{(1+y)^{p+q}} dy$$

$$= \frac{b^r}{B(p,q)}\ B(p-r,q+r)$$

$$= b^r \, \frac{B(p-r, q+r)}{B(p,q)}$$

$$\therefore \quad E(X^r) = r!\, b^r \, \frac{B(p-r, q+r)}{B(p,q)} \tag{2.133}$$

and,

$$E(X) = b\, \frac{B(p-1, q+1)}{B(p,q)}$$

$$= b\, \frac{q}{p-1} \tag{2.134}$$

### 2.5.6 Pearson type I mixing distribution

Pearson type I distribution can be obtained from the classical beta distribution with parameters $\alpha$ and $\beta$, using the change of variable technique as follows: Consider

$$a < \lambda < b$$

$$\therefore \quad 0 < \lambda - a < b - a$$

$$\therefore \quad 0 < \frac{\lambda - a}{b - a} < 1$$

Let

$$t = \frac{\lambda - a}{b - a} \quad \therefore \quad \frac{dt}{d\lambda} = \frac{1}{b-a}$$

The pdf of $\lambda$ is given by

$$g(\lambda) = \frac{\left(\frac{\lambda-a}{b-a}\right)^{p-1} \left[1 - \frac{\lambda-a}{b-a}\right]^{q-1}}{(b-a)\, B(p,q)}$$

$$= \frac{1}{B(p,q)} \, \frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}} \, \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \, \frac{1}{b-a}, \quad a < \lambda < b \tag{2.135}$$

The pdf of the mixture is,

$$f(x) = \frac{1}{B(p,q)} \int_a^b \lambda\, e^{-\lambda x} \, \frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}} \, \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \, \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{B(p,q)} \int_a^b \frac{\lambda}{b-a} \left(\frac{\lambda-a}{b-a}\right)^{p-1} \left(\frac{b-\lambda}{b-a}\right)^{q-1} e^{-\lambda x}\, d\lambda$$

$$= \frac{1}{B(p,q)} \int_a^b \frac{\lambda - a + a}{b-a} \left(\frac{\lambda-a}{b-a}\right)^{p-1} \left(\frac{b-\lambda}{b-a}\right)^{q-1} e^{-\lambda x}\, d\lambda$$

$$= \frac{1}{B(p,q)} \int_a^b \left\{\frac{\lambda-a}{b-a} + \frac{a}{b-a}\right\} \left(\frac{\lambda-a}{b-a}\right)^{p-1} \left\{1 - \frac{\lambda-a}{b-a}\right\}^{q-1} e^{-\lambda x}\, d\lambda$$

Let,

$$\frac{\lambda-a}{b-a}=t \quad \text{and hence} \quad \lambda=a+(b-a)\,t \quad and \quad d\lambda=(b-a)\,dt \tag{2.136}$$

$$f(x)=\frac{1}{B(p,q)}\int_0^1\left(t+\frac{a}{b-a}\right)t^{p-1}\,(1-t)^{q-1}\,e^{-x\,(a+(b-a)\,t}\,(b-a)\,dt$$
$$=\frac{b-a}{B(p,q)}\int_0^1\left(t+\frac{a}{b-a}\right)t^{p-1}\,(1-t)^{q-1}\,e^{-a\,x}\,e^{-(b-a)xt}\,dt$$
$$=\frac{(b-a)\,e^{-a\,x}}{B(p,q)}\int_0^1 t^{(p+1)-1}\,(1-t)^{q-1}\,e^{-(b-a)xt}\,dt+$$
$$\frac{(b-a)\,e^{-a\,x}}{B(p,q)}\int_0^1\frac{a}{b-a}\,t^{p-1}\,(1-t)^{q-1}\,e^{-(b-a)xt}\,dt$$
$$=\frac{(b-a)\,e^{-a\,x}}{B(p,q)}\,\frac{B(p+1,q)}{B(p+1,q)}\int_0^1 t^{(p+1)-1}\,(1-t)^{(p+q-1)-(p+1)-1}\,e^{-(b-a)xt}\,dt+$$
$$\frac{(b-a)\,e^{-a\,x}}{B(p,q)}\int_0^1\frac{a}{b-a}\,t^{p-1}\,(1-t)^{p+q-p-1}\,e^{-(b-a)xt}\,dt$$

$$=\frac{p}{p+q}(b-a)\,e_1^{-ax}F_1\,(p+1;p+q+1;-(b-a)x)+ae_1^{-ax}F_1\,(p;p+q;-(b-a)x) \tag{2.137}$$

The survival function is,

$$S(x)=\frac{1}{B(p,q)}\int_a^b e^{-\lambda x}\,\frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}}\,\frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}}\,\frac{1}{b-a}\,d\lambda$$
$$=\frac{1}{(b-a)B(p,q)}\int_a^b\left(\frac{\lambda-a}{b-a}\right)^{p-1}\left(\frac{b-\lambda}{b-a}\right)^{q-1}e^{-\lambda x}\,d\lambda$$
$$=\frac{e^{-a\,x}}{(b-a)\,B(p,q)}\int_0^1 t^{p-1}\,(1-t)^{q-1}\,e^{-(b-a)xt}\,(b-a)\,dt$$
$$=e^{-a\,x}\int_0^1\frac{t^{p-1}\,(1-t)^{q-1}\,e^{-(b-a)xt}}{B(p,q)}\,dt$$
$$=\frac{e^{-a\,x}}{(b-a)\,B(p,q)}\int_0^1 t^{p-1}\,(1-t)^{q-1}\,e^{-(b-a)xt}\,(b-a)\,dt$$
$$=e^{-a\,x}\int_0^1\frac{t^{p-1}\,(1-t)^{p+q-p-1}\,e^{-(b-a)xt}}{B(p,q)}\,dt$$

$$= e^{-ax}\ {}_1F_1\left(p; p+q; -(b-a)x\right) \tag{2.138}$$

The hazard function is,,

$$h(x) = a + \frac{p(b-a)}{p+q}\ \frac{{}_1F_1\left(p+1; p+q+1; -(b-a)x\right)}{{}_1F_1\left(p; p+q; -(b-a)x\right)} \tag{2.139}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{1}{B(p,q)} \int_a^b \lambda^{-r}\ \frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}}\ \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}}\ \frac{1}{b-a}\ d\lambda$$

$$= \frac{1}{B(p,q)} \int_a^b \frac{\lambda^{-r}}{b-a}\ \left(\frac{\lambda-a}{b-a}\right)^{p-1}\ \left(\frac{b-\lambda}{b-a}\right)^{q-1}\ d\lambda$$

$$= \frac{1}{B(p,q)} \int_a^b \frac{(\lambda-a+a)^{-r}}{b-a}\ \left(\frac{\lambda-a}{b-a}\right)^{p-1}\ \left(\frac{b-\lambda}{b-a}\right)^{q-1}\ d\lambda$$

$$= \frac{1}{B(p,q)}\frac{(b-a)^{-r}}{b-a} \int_a^b \left\{\frac{\lambda-a}{b-a} + \frac{a}{b-a}\right\}^{-r}\ \left(\frac{\lambda-a}{b-a}\right)^{p-1}\ \left\{1 - \frac{\lambda-a}{b-a}\right\}^{q-1}\ d\lambda$$

Using the substitution (2.136)

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{1}{B(p,q)} \int_0^1 \frac{1}{b-a}\ \left(t + \frac{a}{b-a}\right)^{-r}\ t^{p-1}\ (1-t)^{q-1}\ (b-a)\ dt$$

$$= \frac{1}{(b-a)^r\ B(p,q)} \int_0^1 \left(t + \frac{a}{b-a}\right)^{-r}\ t^{p-1}\ (1-t)^{q-1}\ dt$$

$$= \frac{1}{(b-a)^r\ B(p,q)} \sum_{k=0}^{\infty} \binom{-r}{k}\ \left(\frac{a}{b-a}\right)^{-r-k} \int_0^1 t^{p+k-1}\ (1-t)^{q-1}\ dt$$

$$= \frac{1}{(b-a)^r\ B(p,q)} \sum_{k=0}^{\infty} \binom{-r}{k}\ \left(\frac{a}{b-a}\right)^{-r-k}\ B(p+k,q)$$

$$\therefore\quad E(X^r) = r!\ \frac{1}{(b-a)^r\ B(p,q)} \sum_{k=0}^{\infty} \binom{-r}{k}\ \left(\frac{a}{b-a}\right)^{-r-k}\ B(p+k,q) \tag{2.140}$$

and,

$$E(X) = \frac{b-a}{B(p,q)} \sum_{k=0}^{1} \binom{-2}{k}\ \left(\frac{a}{b-a}\right)^{-1-k}\ B(p+k,q) \tag{2.141}$$

### 2.5.7 Shifted Gamma (Pearson type III) distribution

In section 2.3.4 shifted gamma mixture of type I exponential distribution has been obtained in explicit form. This mixture can also be expressed in terms of confluent hyper-geometric function as shown below:

The mixing distribution is

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1}; \quad \lambda > \mu > 0; \quad \alpha, \beta > 0 \tag{2.142}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{\mu}^{\infty} e^{\lambda x}\, e^{-\beta(\lambda-\mu)}\, \lambda\, (\lambda-\mu)^{\alpha-1}\, d\lambda \\ &= \frac{\beta^{\alpha}\, e^{-\mu x}}{\Gamma(\alpha)} \int_{\mu}^{\infty} \lambda\, (\lambda-\mu)^{\alpha-1}\, e^{-(\beta+x)(\lambda-\mu)} d\lambda \end{aligned}$$

Let,

$$z = \lambda - \mu \implies \lambda = z + \mu \quad d\lambda = dz \tag{2.143}$$

$$f(x) = \frac{\beta^{\alpha}\, e^{-\mu x}}{\Gamma(\alpha)} \int_{0}^{\infty} (z+\mu)\, z^{\alpha-1}\, e^{-(\beta+x)z}\, dz$$

Put,

$$z = \mu\, y \implies dz = \mu\, dy$$

$$\begin{aligned} \therefore \quad f(x) &= \frac{\mu\, (\mu\beta)^{\alpha}\, e^{-\mu x}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha-1}\, (y+1)^{1}\, e^{-\mu\, (\beta+x)\, y}\, dy \\ &= \frac{\mu\, (\mu\beta)^{\alpha}\, e^{-\mu x}\, \Gamma(\alpha)}{\Gamma(\alpha)}\, \Psi[\alpha,\, \alpha+2,\, \mu(\beta+x)] \\ &= \mu\, (\mu\beta)^{\alpha}\, e^{-\mu x}\, \Psi[\alpha,\, \alpha+2,\, \mu(\beta+x)] \end{aligned} \tag{2.144}$$

The survival function is,

$$\begin{aligned} S(x) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} e^{\lambda x}\, e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1}\, d\lambda \\ &= \frac{\beta^{\alpha}\, e^{-\mu x}}{\Gamma(\alpha)} \int_{0}^{\infty} (\lambda-\mu)^{\alpha-1}\, e^{-(\beta+x)(\lambda-\mu)} d\lambda \end{aligned}$$

Using substitution (2.143)

$$S(x) = \frac{\beta^{\alpha}\, e^{-\mu x}}{\Gamma(\alpha)} \int_{0}^{\infty} (z+\mu)^{1-1}\, z^{\alpha-1}\, e^{-(\beta+x)z}\, dz$$

Put,

$$z = \mu\, y \quad \Longrightarrow \quad dz = \mu\, dy$$

$$S(x) = \frac{(\mu\beta)^{\alpha}\, e^{-\mu x}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha-1}\, (y+1)^{1-1}\, e^{-\mu\,(\beta+x)\,y}\, dy$$

$$S(x) = \frac{(\mu\beta)^{\alpha}\, e^{-\mu x}\, \Gamma(\alpha)}{\Gamma(\alpha)}\, \Psi[\alpha,\ \alpha+1,\ \mu(\beta+x)]$$

$$= (\mu\beta)^{\alpha}\, e^{-\mu x}\, \Psi[\alpha,\ \alpha+1,\ \mu(\beta+x)] \tag{2.145}$$

The hazard function is,,

$$h(x) = \frac{\mu\, \Psi[\alpha,\ \alpha+2,\ \mu(\beta+x)]}{\Psi[\alpha,\ \alpha+1,\ \mu(\beta+x)]} \tag{2.146}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{\mu\,(\mu\beta)^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} x^{r}\, e^{-\mu x} \left\{ \int_{0}^{\infty} y^{\alpha-1}\, e^{-\mu\,(\beta+x)\,y}\, dy \right\} dx$$

$$= \frac{\mu\,(\mu\beta)^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha-1}\, e^{-\mu\beta y}\, dy \left\{ \int_{0}^{\infty} x^{r+1-1}\, e^{-x(1+y)\mu}\, dy \right\} dx$$

$$= \frac{\mu\,(\mu\beta)^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha-1}\, e^{-\mu\beta y}\, dy\, \frac{\Gamma(r+1)}{[\mu(1+y)]^{r+1}}$$

$$= \frac{\mu\,(\mu\beta)^{\alpha}}{\Gamma(\alpha)}\, \frac{r!}{\mu^{r+1}} \int_{0}^{\infty} y^{\alpha-1}(1+y)^{-r-1}\, e^{\mu\beta y}\, dy$$

$$= \frac{\mu\,(\mu\beta)^{\alpha}}{\Gamma(\alpha)}\, \frac{r!}{\mu^{r+1}}\, \Gamma(\alpha)\Psi[\alpha, \alpha-r, \mu\beta]$$

$$= r!\, \frac{(\mu\beta)^{\alpha}}{\mu^{r}}\, \Psi[\alpha, \alpha-r, \mu\beta] \tag{2.147}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \int_{\mu}^{\infty} \lambda^{-r}\, \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1} d\lambda$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{\mu}^{\infty} \lambda^{-r}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1}\, d\lambda$$

Using substitution (2.143)

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} (z+\mu)^{-r}\, e^{-\beta z}\, z^{\alpha-1}\, dz$$

and,

$$z = \mu t \quad \therefore \quad dz = \mu\, dt$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\mu^{\alpha-r}\, \beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(1+t)^{-r}\, e^{-\beta\mu t}\, dt$$

$$= \frac{\mu^{\alpha-r}\, \beta^\alpha}{\Gamma(\alpha)} \Gamma(\alpha)\, \Psi(\alpha, \alpha-r+1, \beta\mu)$$

$$= \frac{(\mu\beta)^\alpha}{\mu^r} \Psi(\alpha, \alpha-r+1, \beta\mu)$$

$$\therefore \quad E(X^r) = r!\, \frac{(\mu\beta)^\alpha}{\mu^r} \Psi(\alpha, \alpha-r+1, \beta\mu) \tag{2.148}$$

and,

$$E(X) = \frac{(\mu\beta)^\alpha}{\mu} \Psi(\alpha, \alpha; \beta\mu) \tag{2.149}$$

### 2.5.8 Pearson type VI mixing distribution

The exponential-Pearson type VI distribution is constructed below and the moments obtained:

$$g(\lambda) = \frac{\left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \frac{1}{d-c}}{B(a, b-a)\, \left(1+\frac{\lambda-d}{d-c}\right)^b}, \quad \lambda > d \tag{2.150}$$

The pdf of the mixture is,

$$f(x) = \frac{1}{B(a, b-a)} \int_d^\infty \frac{\lambda-d+d}{d-c} \left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \left(1+\frac{\lambda-d}{d-c}\right)^{-b} e^{-x\lambda}\, d\lambda$$

$$= \frac{1}{B(a, b-a)} \int_d^\infty \left\{\left(\frac{\lambda-d}{d-c}\right)^{(b-a+1)-1} + \frac{d}{d-c}\left(\frac{\lambda-d}{d-c}\right)^{b-a-1}\right\} \left(1+\frac{\lambda+d}{d-c}\right)^{-b} e^{-x\lambda} d\lambda$$

Let,

$$\frac{\lambda-d}{d-c} = z \quad so \quad that \quad \lambda = d + (d-c)\, z \quad and \quad d\lambda = (d-c)\, dz \tag{2.151}$$

Therefore,

$$f(x) = \frac{1}{B(a, b-a)} \int_0^\infty \left\{z^{(b-a+1)-1} + \frac{d}{d-c}\, z^{b-a-1}\right\} (1+z)^{-b} e^{-x[d+(d-c)z]} (d-c) dz$$

$$= \frac{(d-c)\, e^{-dx}}{B(a, b-a)} \int_0^\infty z^{(b-a+1)-1} (1+z)^{-b} e^{-(d-c)xz} dz +$$

$$\frac{(d-c)\,e^{-dx}}{B(a,b-a)}\int_0^\infty z^{b-a-1}\,\frac{d}{d-c}\,(1+z)^{-b}\,e^{-(d-c)xz}dz$$

$$=\frac{(d-c)\,e^{-dx}}{B(a,b-a)}\frac{\Gamma(b-a+1)}{\Gamma(b-a+1)}\int_0^\infty z^{(b-a+1)-1}(1+z)^{(2-a)-(b-a+1)-1}e^{-(d-c)xz}dz+$$

$$\frac{(d-c)\,e^{-dx}}{B(a,b-a)}\,\frac{\Gamma(b-a)}{\Gamma(b-a)}\int_0^\infty z^{(b-a)-1}\,\frac{d}{d-c}\,(1+z)^{(1-a)-(b-a)-1}\,e^{-(d-c)xz}dz$$

$$=\frac{\Gamma(b-a+1)\,(d-c)\,e^{-dx}}{B(a,b-a)}\,\Psi(b-a+1;2-a;(d-c)x)+$$

$$\frac{\Gamma(b-a)\,(d-c)\,e^{-dx}}{B(a,b-a)}\,\Psi(b-a;1-a;(d-c)x)$$

$$=(d-c)(b-a)\,\frac{\Gamma b}{\Gamma(a)}\,e^{-dx}\,\Psi(b-a+1;2-a;(d-c)x)+$$

$$d\,\frac{\Gamma b}{\Gamma(a)}\,e^{-dx}\,\Psi(b-a;1-a;(d-c)x) \tag{2.152}$$

The survival function is,

$$S(x)=\frac{1}{B(a,b-a)}\int_d^\infty \frac{1}{d-c}\left(\frac{\lambda-d}{d-c}\right)^{b-a-1}\left(1+\frac{\lambda-d}{d-c}\right)^{-b}e^{-x\lambda}\,d\lambda$$

$$=\frac{1}{(d-c)B(a,b-a)}\int_d^\infty \left(\frac{\lambda-d}{d-c}\right)^{b-a-1}\left(1+\frac{\lambda+d}{d-c}\right)^{-b}e^{-x\lambda}\,d\lambda$$

Using the substitution (2.151)

$$S(x)=\frac{1}{(d-c)B(a,b-a)}\int_0^\infty z^{b-a-1}\,(1+z)^{-b}\,e^{-x[d+(d-c)z]}(d-c)dz$$

$$=\frac{e^{-dx}}{B(a,b-a)}\int_0^\infty z^{b-a-1}(1+z)^{-b}e^{-(d-c)xz}dz$$

$$=\frac{e^{-dx}}{B(a,b-a)}\,\frac{\Gamma(b-a)}{\Gamma(b-a)}\int_0^\infty z^{b-a-1}(1+z)^{(1-a)-(b-a)-1}\,e^{-(d-c)xz}dz$$

$$=\frac{\Gamma b}{\Gamma(a)}\,e^{-dx}\,\Psi(b-a;1-a;(d-c)x)$$

Therefore,

$$h(x)=d+(d-c)(b-a)\,\frac{\Psi(b-a+1;2-a;(d-c)x)}{\Psi(b-a;1-a;(d-c)x)} \tag{2.153}$$

The rth moment is,

$$E(X^r)=r!\int_d^\infty \lambda^{-r}\,\frac{\left(\frac{\lambda-d}{d-c}\right)^{b-a-1}\frac{1}{d-c}}{B(a,b-a)\,\left(1+\frac{\lambda-d}{d-c}\right)^{b}}d\lambda$$

$$= \frac{r!}{B(a,b-a)} \int_{d}^{\infty} \frac{\left[\left(\frac{\lambda-d+d}{d-c}\right)(d-c)\right]^{-r} \left(\frac{\lambda-d}{d-c}\right)^{b-a-1}}{\left(1+\frac{\lambda-d}{d-c}\right)^{b}} \frac{d\lambda}{d-c}$$

Using the substitution (2.151)

$$\begin{aligned}
E(X^r) &= \frac{r!}{B(a,b-a)} \int_{0}^{\infty} \left(z+\frac{d}{d-c}\right)^{-r} (d-c)^{-r} \frac{z^{b-a-1}}{(1+z)^b} \, dz \\
&= \frac{r!}{(d-c)^r \, B(a,b-a)} \int_{0}^{\infty} \left(z+\frac{d}{d-c}\right)^{-r} \frac{z^{b-a-1}}{(1+z)^b} \, dz \\
&= \frac{r!}{(d-c)^r \, B(a,b-a)} \int_{0}^{\infty} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d}{d-c}\right)^{-r-k} \frac{z^{k+b-a-1}}{(1+z)^b} \, dz \\
&= \frac{r!}{(d-c)^r \, B(a,b-a)} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d}{d-c}\right)^{-r-k} \int_{0}^{\infty} \frac{z^{k+b-a-1}}{(1+z)^b} \, dz \\
&= \frac{r!}{(d-c)^r \, B(a,b-a)} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d}{d-c}\right)^{-r-k} B(b+k-a, a-k) \\
&= \frac{r!}{(d-c)^r} \frac{\Gamma b}{\Gamma(a)\,\Gamma(b-a)} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d}{d-c}\right)^{-r-k} \frac{\Gamma(b+k-a)\,\Gamma(a-k)}{\Gamma b} \\
&= \frac{r!}{(d-c)^r} \frac{1}{\Gamma(a)\,\Gamma(b-a)} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d}{d-c}\right)^{-r-k} \Gamma(b+k-a)\,\Gamma(a-k) \\
&= \frac{r!}{d^r} \frac{1}{\Gamma(a)\,\Gamma(b-a)} \sum_{k=0}^{\infty} \binom{-r}{k} \left(\frac{d-c}{d}\right)^{k} \Gamma(b+k-a)\,\Gamma(a-k)
\end{aligned} \tag{2.154}$$

and

$$E(X) = \frac{1}{d} \frac{1}{\Gamma(a)\,\Gamma(b-a)} \sum_{k=0}^{\infty} \binom{-1}{k} \left(\frac{d-c}{d}\right)^{k} \Gamma(b+k-a)\,\Gamma(a-k) \tag{2.155}$$

### 2.5.9 Right truncated gamma mixing distribution

Right truncated gamma distribution is based on incomplete gamma function described in (1.14) defined by

$$\gamma(b,p) = \int_{0}^{p} e^{-t}\, t^{b-1}\, dt$$

Now, consider the integral

$$\int_{0}^{p} e^{-a\lambda}\, \lambda^{b-1}\, d\lambda$$

Let

$$t = a\lambda \quad \therefore \quad \lambda = \frac{t}{a} \quad \text{and} \quad d\lambda = \frac{dt}{a}$$

$$\int_0^p e^{-a\lambda}\,\lambda^{b-1}\,d\lambda = \int_0^{ap} e^{-t}\,(\frac{t}{a})^{b-1}\,\frac{dt}{a}$$

$$= \frac{1}{a^b}\int_0^{ap} e^{-t}\,t^{b-1}\,dt$$

$$= \frac{1}{a^b}\,\gamma(b, ap)$$

$$\therefore \quad \int_0^p \frac{a^b\,e^{-a\lambda}\lambda^{b-1}}{\gamma(b, ap)}d\lambda = 1$$

Therefore

$$g(\lambda) = \frac{a^b}{\gamma(b, ap)}\,e^{-a\,\lambda}\,\lambda^{b-1} \quad 0 < \lambda < p \quad a, b > 0 \tag{2.156}$$

The pdf of the mixture is,

$$f(x) = \int_0^p \lambda\,e^{-\lambda x}\,\frac{a^b}{\gamma(b, ap)}\,e^{-a\,\lambda}\,\lambda^{b-1}\,d\lambda$$

$$= \frac{a^b}{\gamma(b, ap)}\int_0^p \lambda^{(b+1)-1}\,e^{-(x+a)\lambda}\,d\lambda$$

$$= \frac{a^b}{\gamma(b, ap)}\,\frac{\gamma(b+1, (x+a)p)}{(x+a)^{b+1}}$$

$$= \frac{a^b}{(x+a)^{b+1}}\,\frac{\gamma(b+1, (x+a)p)}{\gamma(b, ap)}$$

$$= \frac{a^b}{(x+a)^{b+1}}\,\frac{(b+1)^{-1}[(x+a)p]^{b+1}\,e^{-(x+a)p}\,{}_1F_1(1; b+2; (x+a)p)}{b^{-1}\,ap^b\,e^{-ap}\,{}_1F_1(1; b+1; ap}$$

$$= \frac{a^b}{(x+a)^{b+1}}\,\frac{b}{b+1}\,\frac{(x+a)^{b+1}\,p^{b+1}\,e^{-xp}\,e^{-ap}\,{}_1F_1(1; b+2; (x+a)p)}{a^b\,p^b\,e^{-ap}\,{}_1F_1(1; b+1; ap}$$

$$= \frac{bp}{b+1}\,e^{-px}\,\frac{{}_1F_1(1; b+2; (x+a)p)}{{}_1F_1(1; b+1; ap)} \tag{2.157}$$

The survival function is,

$$S(x) = \int_0^p e^{-\lambda x}\,\frac{a^b}{\gamma(b, ap)}\,e^{-a\,\lambda}\,\lambda^{b-1}\,d\lambda$$

$$= \frac{a^b}{\gamma(b,ap)} \int_0^p \lambda^{(b-1} e^{-(x+a)\lambda} d\lambda$$

$$= \frac{a^b}{\gamma(b,ap)} \frac{\gamma(b,(x+a)p)}{(x+a)^b}$$

$$= \frac{a^b}{(x+a)^b} \frac{\gamma(b,(x+a)p)}{\gamma(b,ap)}$$

$$= \frac{a^b}{(x+a)^{b+1}} \frac{b^{-1}[(x+a)p]^{b+1} e^{-(x+a)p} {}_1F_1(1;b+1;(x+a)p)}{b^{-1} ap^b e^{-ap} {}_1F_1(1;b+1;ap}$$

$$= e^{-px} \frac{{}_1F_1(1;b+1;(x+a)p)}{{}_1F_1(1;b+1;ap)} \tag{2.158}$$

Therefore,

$$h(x) = \frac{bp}{b+1} \frac{{}_1F_1(1;b+2;(x+a)p)}{{}_1F_1(1;b+1;ap)} \tag{2.159}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \int_0^p \lambda^{-r} \frac{a^b}{\gamma(b,ap)} e^{-a\lambda} \lambda^{b-1} d\lambda$$

$$= \frac{a^b}{\gamma(b,ap)} \int_0^p \lambda^{-r+b-1} e^{-a\lambda} d\lambda$$

$$= \frac{a^b}{\gamma(b,ap)} \int_0^p \lambda^{-r+b-1} e^{-a\lambda} d\lambda$$

$$= \frac{a^b}{\gamma(b,ap)} \frac{\gamma(b-r,ap)}{a^{b-r}}$$

$$= a^r \frac{\gamma(b-r,ap)}{\gamma(b,ap)}$$

$$\therefore \quad E(X^r) = r!\, a^r \frac{\gamma(b-r,ap)}{\gamma(b,ap)}$$

and,

$$E(X) = a \frac{\gamma(b-1,ap)}{\gamma(b,ap)} \tag{2.160}$$

### 2.5.10 Left truncated gamma mixing distribution

The left truncated gamma distribution can be obtained from the two parameter gamma distribution as follows:

Since

$$\int_{0}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta y}\, y^{\alpha-1}\, dy = 1$$

$$\int_{0}^{\lambda_0} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta y}\, y^{\alpha-1}\, dy + \int_{\lambda_0}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta y}\, y^{\alpha-1}\, dy = 1$$

Therefore

$$\begin{aligned}\int_{\lambda_0}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta y}\, y^{\alpha-1}\, dy &= 1 - \int_{0}^{\lambda_0} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta y}\, y^{\alpha-1}\, dy \\ &= 1 - \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, \frac{\gamma(\alpha, \beta\lambda_0)}{\beta^{\alpha}} \\ &= 1 - \frac{\gamma(\alpha, \beta\lambda_0)}{\Gamma(\alpha)}\end{aligned}$$

and,

$$\int_{\lambda_0}^{\infty} \frac{\beta^{\alpha}\, y^{\alpha-1}\, e^{-\beta y}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}\, dy = 1$$

The required pdf is

$$g(\lambda) = \frac{\beta^{\alpha}\, \lambda^{\alpha-1}\, e^{-\beta\lambda}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \quad \lambda_0 < \lambda < \infty \quad \alpha, \beta > 0 \tag{2.161}$$

The pdf of the mixture is,

$$\begin{aligned}f(x) &= \int_{\lambda_0}^{\infty} \lambda\, e^{-\lambda x}\, \frac{\beta^{\alpha}\, y^{\alpha-1}\, e^{-\beta y}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}\, d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \int_{\lambda_0}^{\infty} \lambda^{(\alpha+1)-1}\, e^{-(x+\beta)\lambda}\, d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \left\{ 1 - \int_{0}^{\lambda_0} \lambda^{(\alpha+1)-1}\, e^{-(x+\beta)\lambda} \right\} d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \left\{ 1 - \frac{\gamma(\alpha+1, (x+\beta)\lambda_0)}{(x+\beta)^{\alpha+1}} \right\} \\ &= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha+1}}\, \frac{1}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \left\{ (X+\beta)^{\alpha+1} - \gamma(\alpha+1, (x+\beta)\lambda_0) \right\} \\ &= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha+1}}\, \frac{(x+\beta)^{\alpha+1} - \gamma(\alpha+1, (x+\beta)\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}\end{aligned} \tag{2.162}$$

$$= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha+1}} \frac{(x+\beta)^{\alpha+1} - (\alpha+1)^{-1}[(x+\beta)\lambda_0]^{\alpha+1} e_1^{-(x+\beta)\lambda_0} F_1(1;\alpha+2;(x+\beta)\lambda_0)}{\Gamma(\alpha) - \alpha^{-1}(\beta\lambda_0)^{\alpha}\, e_1^{-\beta\,\lambda_0} F_1(1;\alpha+1;\beta\lambda_0)}$$

$$= \frac{\alpha\,\beta^{\alpha}}{\alpha+1} \left\{ \frac{(\alpha+1) - \lambda_0^{\alpha+1}\, e^{-(x+\beta)\lambda_0}\; {}_1F_1(1;\alpha+2;(x+\beta)\lambda_0)}{\alpha\Gamma(\alpha) - (\beta\lambda_0)^{\alpha}\, e^{-\beta\,\lambda_0}\; {}_1F_1(1;\alpha+1;\beta\lambda_0)} \right\} \tag{2.163}$$

The survival function is,

$$S(x) = \int_{\lambda_0}^{\infty} e^{-\lambda x}\, \frac{\beta^{\alpha}\,\lambda^{\alpha-1}\, e^{-\beta\lambda}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)}\, d\lambda$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \int_{\lambda_0}^{\infty} \lambda^{\alpha-1}\, e^{-(x+\beta)\lambda}\, d\lambda$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \left\{ 1 - \frac{\gamma(\alpha,(x+\beta)\lambda_0)}{(x+\beta)^{\alpha}} \right\}$$

$$= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha}} \frac{(x+\beta)^{\alpha} - \alpha^{-1}[(x+\beta)\lambda_0]^{\alpha}\, e^{-(x+\beta)\lambda_0}\; {}_1F_1(1;\alpha+1;(x+\beta)\lambda_0)}{\Gamma(\alpha) - \alpha^{-1}(\beta\lambda_0)^{\alpha}\, e^{-\beta\,\lambda_0}\; {}_1F_1(1;\alpha+1;\beta\lambda_0)}$$

$$= \beta^{\alpha} \left\{ \frac{\alpha - \lambda_0^{\alpha}\, e^{-(x+\beta)\lambda_0}\; {}_1F_1(1;\alpha+1;(x+\beta)\lambda_0)}{\alpha\Gamma(\alpha) - (\beta\lambda_0)^{\alpha}\, e^{-\beta\,\lambda_0}\; {}_1F_1(1;\alpha+1;\beta\lambda_0)} \right\} \tag{2.164}$$

The hazard function,

$$h(x) = \frac{\alpha}{\alpha+1} \left\{ \frac{(\alpha+1) - \lambda_0^{\alpha+1}\, e^{-(x+\beta)\lambda_0}\; {}_1F_1(1;\alpha+2;(x+\beta)\lambda_0)}{\alpha - \lambda_0^{\alpha}\, e^{-(x+\beta)\,\lambda_0}\; {}_1F_1(1;\alpha+1;(x+\beta)\lambda_0)} \right\} \tag{2.165}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \int_{\lambda_0}^{\infty} \lambda^{-r}\, \frac{\beta^{\alpha}\,\lambda^{\alpha-1}\, e^{-\beta\lambda}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} d\lambda$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \int_{\lambda_0}^{\infty} \lambda^{\alpha-r-1}\, e^{-\beta\lambda} d\lambda$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \int_{\lambda_0}^{\infty} \lambda^{\alpha-r-1}\, e^{-\beta\lambda} d\lambda$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \left\{ 1 - \frac{\gamma(\alpha-r,\beta\lambda_0)}{\beta^{\alpha-r}} \right\}$$

$$\therefore \quad E(X^r) = r!\,\beta^r \left\{ \frac{\beta^{\alpha-r} - \gamma(\alpha-r,\beta\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \right\} \tag{2.166}$$

and,

$$E(X) = \beta\, \frac{\beta^{\alpha-1} - \gamma(\alpha-1,\beta\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha,\beta\lambda_0)} \tag{2.167}$$

### 2.5.11 Gamma truncated from both sides mixing distribution

The gamma truncated from both sides distribution can be obtained from the one parameter gamma distribution as follows:

$$\int_a^b e^{-\beta y}\, y^{\alpha-1}\, dy = \int_0^b e^{-\beta y}\, y^{\alpha-1}\, dy - \int_0^a e^{-\beta y}\, y^{\alpha-1}\, dy$$
$$= \frac{\gamma(\alpha, b\beta)}{\beta^\alpha} - \frac{\gamma(\alpha, a\beta)}{\beta^\alpha}$$

Therefore,

$$\int_a^b \frac{\beta^\alpha\, y^{\alpha-1}\, e^{-\beta y}}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)}\, dy = 1$$

The required mixing pdf is

$$g(\lambda) = \frac{\beta^\alpha\, \lambda^{\alpha-1}\, e^{-\beta\lambda}}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \quad a < \lambda < b; \quad \alpha, \beta > 0 \tag{2.168}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \frac{\beta^\alpha}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \int_a^b \lambda\, e^{-\lambda x}\, e^{-\beta\lambda}\, \lambda^{\alpha-1}\, d\lambda \\ &= \frac{\beta^\alpha}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \int_a^b \lambda^{(\alpha+1)-1}\, e^{-(x+\beta)\lambda}\, d\lambda \\ &= \frac{\beta^\alpha}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \left\{ \frac{\gamma(\alpha+1, (x+\beta)b}{(x+\beta)^{\alpha+1}} - \frac{\gamma(\alpha+1, (x+\beta)a}{(x+\beta)^{\alpha+1}} \right\} \\ &= \frac{\beta^\alpha}{(x+\beta)^{\alpha+1}} \left\{ \frac{\gamma(\alpha+1, (x+\beta)b) - \gamma(\alpha+1, (x+\beta)a)}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \right\} \end{aligned} \tag{2.169}$$

The survival function is,

$$\begin{aligned} S(x) &= \frac{\beta^\alpha}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \int_a^b e^{-\lambda x}\, e^{-\beta\lambda}\, \lambda^{\alpha-1}\, d\lambda \\ &= \frac{\beta^\alpha}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \left\{ \frac{\gamma(\alpha, (x+\beta)b}{(x+\beta)^{\alpha}} - \frac{\gamma(\alpha, (x+\beta)a}{(x+\beta)^{\alpha}} \right\} \\ &= \frac{\beta^\alpha}{(x+\beta)^{\alpha}} \left\{ \frac{\gamma(\alpha, (x+\beta)b) - \gamma(\alpha, (x+\beta)a)}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \right\} \end{aligned} \tag{2.170}$$

The hazard function is,,

$$h(x) = \frac{1}{x+\beta} \left\{ \frac{\gamma(\alpha+1, (x+\beta)b) - \gamma(\alpha+1, (x+\beta)a)}{\gamma(\alpha, (x+\beta)b) - \gamma(\alpha, (x+\beta)a)} \right\} \tag{2.171}$$

Using conditional expectation approach, we have

$$\boldsymbol{E}\left[\frac{1}{\boldsymbol{\Lambda}^r}\right] = \int_a^b \boldsymbol{\lambda}^{-r}\ \frac{\beta^{\alpha}\ \boldsymbol{\lambda}^{\alpha-1}\ e^{-\beta\boldsymbol{\lambda}}}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)}$$

$$= \frac{\beta^{\alpha}}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \int_a^b \boldsymbol{\lambda}^{\alpha-r-1}\ e^{-\beta\boldsymbol{\lambda}} d\boldsymbol{\lambda}$$

$$= \frac{\beta^{\alpha}}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)}\ \frac{1}{\beta^{\alpha-r}} \int_a^b \beta^{\alpha-r}\ \boldsymbol{\lambda}^{\alpha-r-1}\ e^{-\beta\boldsymbol{\lambda}} d\boldsymbol{\lambda}$$

$$= \beta^r\ \frac{\gamma(\alpha - r, b\beta) - \gamma(\alpha - r, a\beta)}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)}$$

$$\therefore \quad \boldsymbol{E}(\boldsymbol{X}^r) = r!\ \beta^r\ \frac{\gamma(\alpha - r, b\beta) - \gamma(\alpha - r, a\beta)}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \tag{2.172}$$

and,

$$\boldsymbol{E}(\boldsymbol{X}) = \beta\ \frac{\gamma(\alpha - 1, b\beta) - \gamma(\alpha - 1, a\beta)}{\gamma(\alpha, b\beta) - \gamma(\alpha, a\beta)} \tag{2.173}$$

### 2.5.12 Truncated Pearson type III mixing distribution

The truncated Pearson type III distribution is obtained from the Pearson differential equation as follows:
Pearson differential equation are given as

$$\frac{1}{y}\ \frac{dy}{dx} = \frac{-(a+x)}{c_0 + c_1 x + c_2\ x^2}$$

where,

$$y = f(x)$$

is a probability density function

Pearson type III corresponds to the case of $c_2 = 0$ and $c_1 \neq 0$

$$\therefore \quad \frac{1}{y}\ \frac{dy}{dx} = \frac{-(a+x)}{c_0 + c_1 x}$$

$$= \frac{-(x+a)}{c_1 x + c_0}$$

$$= -\frac{1}{c_1}\ \left[\frac{x+a}{x + \frac{c_0}{c_1}}\right]$$

$$= -\frac{1}{c_1}\left[\frac{x+\frac{c_0}{c_1}-\frac{c_0}{c_1}+a}{x+\frac{c_0}{c_1}}\right]$$

$$= -\frac{1}{c_1}\left[1-\frac{\frac{c_0}{c_1}-a}{x+\frac{c_0}{c_1}}\right]$$

$$= -\frac{1}{c_1}+\frac{\frac{c_0}{c_1}-a}{c_1x+c_0}$$

$$\therefore \quad \int \frac{dy}{y} = \int\left[-\frac{1}{c_1}+\frac{\frac{c_0}{c_1}-a}{c_1x+c_0}\right]dx$$

$$= -\frac{x}{c_1}+\frac{1}{c_1}\left(\frac{c_0}{c_1}-a\right)\log\left(c_1x+c_0\right)+\log\, k$$

$$= -\frac{x}{c_1}+m\,\log\left(c_1x+c_0\right)+\log\, k$$

where

$$m = \frac{1}{c_1}\left(\frac{c_0}{c_1}-a\right)$$

$$\therefore \log y = \log e^{-\frac{x}{c(1)}}+\log(c_1x+c_0)^m+\log k$$

$$= \log k e^{-\frac{x}{c_1}}\,(c_1x+c_0)^m$$

$$y = k e^{-\frac{x}{c_1}}\,(c_1x+c_0)^m$$

$$= k e^{-\frac{x}{c_1}}\,(c_0+c_1x)^m$$

Let

$$c_1 = -c_0 \quad \text{and} \quad m = \frac{1}{c_0}(a+1)$$

$$\therefore \quad y = k e^{\frac{x}{c_0}}\,(1-x)^m\, c_0^m$$

$$i.e. \quad f(x) = k\, c_0^m\, e^{-\frac{x}{c_0}}\,(1-x)^m, \quad 0<x<1$$

implying that

$$\int_0^1 f(x)\,dx = k\,c_0^m \int_0^1 e^{\frac{x}{c_0}}\,(1-x)^m\,dx = 1$$

$$k\,c_0^m \int_0^1 x^{1-1}\,(1-x)^m\,e^{-\frac{x}{c_0}}\,dx = 1$$

$$k\,c_0^m \int_0^1 x^{1-1}\,(1-x)^{(2+m)-1-1}\,e^{-\frac{x}{c_0}}\,dx = 1$$

$$k\,c_0^m\,B(1,(2+m)-1)\int_0^1 \frac{x^{1-1}\,(1-x)^{(2+m)-1-1}\,e^{-\frac{x}{c_0}}}{B(1,(2+m)-1)}\,dx = 1$$

$$k\, c_0^m\, B(1,m+1)\; {}_1F_1(1;m+2;\frac{1}{c_0}) = 1$$

$$k\, c_0^m\, \frac{\Gamma 1\, \Gamma(m+1)}{\Gamma(m+2)}\; {}_1F_1(1;m+2;\frac{1}{c_0}) = 1$$

$$\therefore \quad k\, c_0^m = \frac{1}{B(1,m+1)\; {}_1F_1(1;m+2;\frac{1}{c_0})}$$

$$\frac{1}{B(1,m+1){}_1F_1(1;m+2;\frac{1}{c_0})} B(1,m+1) \int_0^1 \frac{x^{1-1}\,(1-x)^m\, e^{-\frac{x}{c_0}}}{B(1,(m+1)}\, dx = 1$$

$$\int_0^1 \frac{x^{1-1}\,(1-x)^m\, e^{-\frac{x}{c_0}}}{B(1,m+1)\; {}_1F_1(1;m+2;\frac{1}{c_0})}\, dx = 1$$

$$\int_0^1 \frac{(1-x)^m\, e^{-\alpha x}}{B(1,m+1)\; {}_1F_1(1;m+2;-\alpha)}\, dx = 1$$

where

$$\frac{1}{c_0} = -\alpha \quad \text{and} \quad m = \alpha(a+1)$$

Therefore the truncated Pearson type III mixing distribution is given by

$$g(\lambda) = \frac{(1-\lambda)^m\, e^{-\alpha\lambda}}{B(1,m+1)\; {}_1F_1(1;m+2;-\alpha)} \quad 0 < \lambda < 1 \quad m > 0 \qquad (2.174)$$

which is basically a confluent hyper-geometric probability distribution function with parameters $a = 1$ and $c = m+2$

The pdf of the mixture is,

$$f(x) = \int_0^1 \lambda\, e^{-\lambda x}\, \frac{(1-\lambda)^m\, e^{-\alpha\lambda}}{B(1,m+1)\; {}_1F_1(1;m+2;-\alpha)}\, d\lambda$$

$$= \frac{\int_0^1 \lambda^{2-1}\,(1-\lambda)^m\, e^{-(x+\alpha)\lambda}\, d\lambda}{B(1,m+1)\; {}_1F_1(1;m+2;-\alpha)}$$

$$= \frac{B(2,m+3-2)\, \int_0^1 \frac{\lambda^{2-1}\,(1-\lambda)^m\, e^{-(x+\alpha)\lambda}}{B(2,m+3-2)}\, d\lambda}{B(1,m+1)\; {}_1F_1(1;m+2;-\alpha)}$$

$$= \frac{B(2,m+1)}{B(1,m+1)}\, \frac{{}_1F_1(2;m+3;-(x+\alpha))}{{}_1F_1(1;m+2;-\alpha}$$

$$= \frac{1}{m+2} \frac{{}_1F_1(2;m+3;-(x+\alpha))}{{}_1F_1(1;m+2;-\alpha)} \tag{2.175}$$

(2.) The survival function is

$$S(x) = \int_0^1 e^{-\lambda x} \frac{(1-\lambda)^m e^{-\alpha\lambda}}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)} d\lambda$$

$$= \frac{\int_0^1 \lambda^{1-1} (1-\lambda)^m e^{-(x+\alpha)\lambda} d\lambda}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)}$$

$$= \frac{\int_0^1 \lambda^{1-1} (1-\lambda)^{m+2-1-1} e^{-(x+\alpha)\lambda} d\lambda}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)}$$

$$= \frac{B(1,m+2-1)}{B(1,m+1)} \frac{{}_1F_1(2;m+2;-(x+\alpha))}{{}_1F_1(1;m+2;-\alpha}$$

$$= \frac{{}_1F_1(2;m+2;-(x+\alpha))}{{}_1F_1(1;m+2;-\alpha)} \tag{2.176}$$

The hazard function is,

$$h(x) = \frac{1}{m+2} \frac{{}_1F_1(2;m+3;-(x+\alpha))}{{}_1F_1(1;m+2;-(x+\alpha))} \tag{2.177}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \int_0^1 \lambda^{-r} \frac{(1-\lambda)^m e^{-\alpha\lambda}}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)} d\lambda$$

$$= \int_0^1 \frac{\lambda^{1-r+1} (1-\lambda)^m e^{-\alpha\lambda}}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)} d\lambda$$

$$= B(1-r,m+1) \frac{{}_1F_1(1-r;2-r+m;-\alpha)}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)}$$

$$E(X^r) = r!\, B(1-r,m+1) \frac{{}_1F_1(1-r;2-r+m;-\alpha)}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)} \tag{2.178}$$

and,

$$E(X) = B(0,m+1) \frac{{}_1F_1(0;2-r+m;-\alpha)}{B(1,m+1)\,{}_1F_1(1;m+2;-\alpha)}$$

$$= \infty \tag{2.179}$$

### 2.5.13 Pareto I mixing distribution

Willmot (1993) called the Pareto I distribution, shifted Pareto distribution and its pdf is given as:

$$g(\lambda) = \frac{\alpha\,\beta^\alpha}{\lambda^{\alpha+1}}; \quad \lambda > \beta > 0 \quad \alpha > 0 \tag{2.180}$$

The exponential-Pareto I mixture is constructed as follows:

The pdf of the mixture is,

$$f(x) = \alpha\,\beta^{\alpha} \int_{\beta}^{\infty} \lambda^{1-\alpha-1}\, e^{-\lambda x}\, d\lambda$$

Let,

$$\lambda = z + \beta \quad \Longrightarrow \quad z = \lambda - \beta \quad \text{and} \quad d\lambda = dz$$

so that

$$f(x) = \alpha\,\beta^{\alpha}\, e^{-\beta x} \int_{0}^{\infty} (z+\beta)^{1-\alpha-1}\, e^{-zx}\, dz$$

Put,

$$z = \beta y \quad \Longrightarrow \quad dz = \beta dy \tag{2.181}$$

and therefore

$$\begin{aligned} f(x) &= \alpha\,\beta^{\alpha}\, e^{-\beta x}\, \beta \int_{0}^{\infty} \beta^{1-\alpha-1}\, (y+1)^{1-\alpha-1}\, e^{-\beta\, x\, y}\, dy \\ &= \alpha\,\beta\, e^{-\beta x} \int_{0}^{\infty} y^{1-1}\, (y+1)^{1-\alpha-1}\, e^{-\beta\, x\, y}\, dy \\ &= \alpha\,\beta\, e^{-\beta x} \quad \Psi(1; 2-\alpha; \beta x) \end{aligned} \tag{2.182}$$

The survival function is,

$$S(x) = \alpha\,\beta^{\alpha} \int_{0}^{\infty} \lambda^{-\alpha-1}\, e^{-\lambda x}\, d\lambda$$

Let,

$$\lambda = z + \beta \quad \Longrightarrow \quad z = \lambda - \beta \quad \text{and} \quad d\lambda = dz$$

and therefore

$$S(x) = \alpha\,\beta^{\alpha}\, e^{-\beta x} \int_{0}^{\infty} (z+\beta)^{-\alpha-1}\, e^{-zx}\, dz$$

Using the substitution (2.181)

$$z = \beta y \quad \Longrightarrow \quad dz = \beta dy$$

so that

$$\begin{aligned}
S(x) &= \alpha\, \beta^{\alpha}\, e^{-\beta x}\, \beta \int_0^{\infty} \beta^{-\alpha-1}\, (y+1)^{-\alpha-1}\, e^{-\beta\, x\, y}\, dy \\
&= \alpha\, e^{-\beta x} \int_0^{\infty} y^{1-1}\, (y+1)^{-\alpha-1}\, e^{-\beta\, x\, y}\, dy \\
&= \alpha\, e^{-\beta\, x} \quad \Psi(1; 1-\alpha; \beta x) \qquad (2.183)
\end{aligned}$$

The hazard function is,,

$$h(x) = \beta\, \frac{\Psi(1; 2-\alpha; \beta x)}{\Psi(1; 1-\alpha; \beta x)} \qquad (2.184)$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \alpha\, \beta \int_0^{\infty} x^r\, e^{-\beta x} \quad \Psi(1; 2-\alpha; \beta x) dx$$

Let,

$$\begin{aligned}
y &= \beta x \quad \therefore \quad dy = \beta dx \\
E(X^r) &= \alpha\, \beta \int_0^{\infty} (\frac{y}{\beta})^r\, e^{-y} \quad \Psi(1; 2-\alpha; y) \frac{dy}{\beta} \\
&= \frac{\alpha}{\beta^r} \int_0^{\infty} y^{(r+1)-1}\, e^{-y} \quad \Psi(1; 2-\alpha; y) dy
\end{aligned}$$

But from (1.20)

$$M[e^{-x}\, \Psi(a; c; x), s] = \frac{\Gamma(s)\Gamma(s-c+1)}{\Gamma(a+s-c+1)}$$

Hence

$$\begin{aligned}
E(X^r) &= \frac{\alpha}{\beta^r}\, \frac{\Gamma(r+1)\Gamma(r+1-2+\alpha+1)}{\Gamma(1+r+1-2+\alpha+1)} \\
&= \frac{\alpha}{\beta^r}\, \frac{\Gamma(r+1)\Gamma(\alpha+r)}{\Gamma(\alpha+r+1)} \\
&= \frac{r!}{\beta^r}\, \frac{\alpha}{\alpha+r} \qquad (2.185)
\end{aligned}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \alpha\, \beta^{\alpha} \int_{\beta}^{\infty} \lambda^{-r-\alpha-1}\, d\lambda$$

$$= \boldsymbol{\alpha\, \beta^{\alpha} \left[\frac{\lambda^{-r-\alpha}}{-r-\alpha}\right]_{\beta}^{\infty}}$$

$$= \boldsymbol{\frac{\alpha\, \beta^{\alpha}}{r+\alpha}\, \frac{1}{\beta^{r+\alpha}}}$$

$$= \boldsymbol{\frac{\alpha}{\alpha+r}\, \frac{1}{\beta^{r}}}$$

$$\therefore \quad \boldsymbol{E(X^r) = r!\, \frac{\alpha}{\alpha+r}\, \frac{1}{\beta^{r}}} \tag{2.186}$$

and,

$$\boldsymbol{E(X) = \frac{\alpha}{\beta\, (\alpha+1)}} \tag{2.187}$$

### 2.5.14 Pareto II (Lomax) mixing distribution

Exponential-Pareto II distribution has been constructed in this subsection and the moments obtained. Pareto II distribution is also known as Lomax distribution and,

$$\boldsymbol{g(\lambda) = \frac{\alpha\, \beta^{\alpha}}{(\lambda+\beta)^{\alpha+1}}; \quad \lambda > 0 \quad \alpha, \beta > 0} \tag{2.188}$$

The pdf of the mixture is,

$$\boldsymbol{f(x) = \alpha\, \beta^{\alpha} \int_0^{\infty} \lambda\, (\lambda+\beta)^{-\alpha-1}\, e^{-\lambda x}\, d\lambda}$$

Let,

$$\boldsymbol{\lambda = \beta u} \quad \text{so that} \quad \boldsymbol{d\lambda = \beta\, du} \tag{2.189}$$

and therefore

$$\boldsymbol{f(x) = \alpha\, \beta^{\alpha} \int_0^{\infty} \beta\, \mu\, \beta^{-\alpha} (1+u)^{-\alpha-1}\, e^{-\beta\, u\, x}\, d\, u}$$

$$\boldsymbol{= \alpha\, \beta \int_0^{\infty} u\, (1+u)^{-\alpha-1}\, e^{-\beta\, u\, x}\, d\, u}$$

$$\boldsymbol{= \alpha\, \beta\, \Psi(2; 2-\alpha; \beta\, x)} \tag{2.190}$$

The survival function is,

$$\boldsymbol{S(x) = \alpha\, \beta^{\alpha} \int_0^{\infty} (\lambda+\beta)^{-\alpha-1}\, e^{-\lambda x}\, d\lambda}$$

Using the substitution (2.189)

$$\boldsymbol{S(x) = \alpha\, \beta^{\alpha} \int_0^{\infty} \beta^{-\alpha} (1+u)^{-\alpha-1}\, e^{-\beta\, u\, x}\, d\, u}$$

$$= \alpha \int_0^\infty u^{1-1}\,(1+u)^{-\alpha-1}\,e^{-\beta\,u\,x}\,d\,u$$

$$= \alpha\,\Psi(1;1-\alpha;\beta\,x) \tag{2.191}$$

The hazard function is„

$$h(x) = \beta\,\frac{\Psi(2;2-\alpha;\beta\,x]}{\Psi(1;1-\alpha;\beta\,x)} \tag{2.192}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \alpha\,\beta \int_0^\infty x^r\,\Psi(2;2-\alpha;\beta\,x)dx$$

Let,

$$y = \beta x \quad \therefore \quad dy = \beta dx$$

$$E(X^r) = \alpha\,\beta \int_0^\infty (\frac{y}{\beta})^r\,\Psi(2;2-\alpha;y)\frac{dy}{\beta}$$

$$= \frac{\alpha}{\beta^r} \int_0^\infty y^{(r+1)-1}\,\Psi(2;2-\alpha;y)dy$$

But from (1.19)

$$M[\Psi(a;c;x),s] = \frac{\Gamma(s)\Gamma(a-s)\Gamma(s-c+1)}{\Gamma(a)\Gamma(a-c+1)}$$

Hence

$$E(X^r) = \frac{\alpha}{\beta^r}\,\frac{\Gamma((r+1))\Gamma(2-r-1)\Gamma(r+1-2+\alpha+1)}{\Gamma(2)\Gamma(2-2+\alpha+1)}$$

$$= \frac{\alpha}{\beta^r}\,\frac{\Gamma((r+1))\Gamma(1-r)\Gamma(\alpha+r)}{\Gamma(\alpha+1)}$$

$$= r!\,\frac{\alpha}{\beta^r}\,B(1-r,\alpha+r) \tag{2.193}$$

Using conditional expectation approach, we have

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\alpha\,\beta^\alpha}{\beta^{r+\alpha}} \int_0^\infty \frac{t^{-r}}{(1+t)^{\alpha+1}}\,dt$$

$$= \frac{\alpha}{\beta^r}\,B(1-r,\alpha+r)$$

$$\therefore \quad E(X^r) = r!\,\frac{\alpha}{\beta^r}\,B(1-r,\alpha+r) \tag{2.194}$$

and,

$$E(X) = \frac{\alpha}{\beta}\,B(0,\alpha+1)$$

$$= \infty \tag{2.195}$$

### 2.5.15 Generalized Pareto mixing distribution

Generalized Pareto distribution is a Gamma I mixture of a Gamma I distribution as shown below:

Let

$$g(\lambda) = \int_0^\infty g(\lambda|\kappa)\, f(\kappa)\, d\kappa$$

where

$$g(\lambda|\kappa) = \frac{\kappa^\beta}{\Gamma(\beta)}\, e^{-\kappa\lambda}\, \lambda^{\beta-1} \tag{2.196}$$

The pdf of the mixture is,

$$\begin{aligned} f(\kappa) &= \frac{\mu^\alpha}{\Gamma(\alpha)}\, e^{-\mu\kappa}\, \kappa^{\alpha-1} \\ g(\lambda) &= \int_0^\infty \frac{\kappa^\beta}{\Gamma(\beta)}\, e^{-\kappa\lambda}\, \lambda^{\beta-1} \frac{\mu^\alpha}{\Gamma(\alpha)}\, e^{-\mu\kappa}\, \kappa^{\alpha-1} d\kappa \\ &= \frac{\mu^\alpha\, \lambda^{\beta-1}}{B(\alpha,\beta)\, (\lambda+\mu)^{\alpha+\beta}}; \quad \lambda > 0 \quad \alpha,\beta > 0 \qquad (2.197) \\ f(x) &= \frac{\mu^\alpha}{B(\alpha,\beta} \int_0^\infty \lambda^{(\beta+1)-1}\, (\lambda+\mu)^{-\alpha-\beta}\, e^{-\lambda x}\, d\lambda \end{aligned}$$

Let,

$$\lambda = \mu\, z \quad \text{which implies that} \quad d\lambda = \mu\, dz \tag{2.198}$$

And so,

$$\begin{aligned} f(x) &= \frac{\mu}{B(\alpha,\beta} \int_0^\infty z^\beta\, (1+z)^{-\alpha-\beta}\, e^{-\mu x\, z}\, dz \\ &= \frac{\mu\, \Gamma(\beta+1)}{B(\alpha,\beta}\, \Psi(\beta+1, 2-\alpha, \mu x) \\ &= \frac{\mu\, \beta\, \Gamma(\alpha+\beta)}{\Gamma(\alpha)}\, \Psi(\beta+1; 2-\alpha; \mu\, x) \qquad (2.199) \end{aligned}$$

The survival function is,

$$S(x) = \frac{\mu^\alpha}{B(\alpha,\beta} \int_0^\infty \lambda^{\beta-1}\, (\lambda+\mu)^{-\alpha-\beta}\, e^{-\lambda x}\, dx$$

Using the substitution (2.198)

$$\boldsymbol{S(x) = \frac{1}{B(\alpha,\beta}\int_0^\infty z^{\beta-1}\,(1+z)^{-\alpha-\beta}\,e^{-\mu z\,x}\,dz}$$
$$\boldsymbol{= \frac{\Gamma(\beta)}{B(\alpha,\beta}\,\Psi(\beta,1-\alpha,\mu\,x)}$$
$$\boldsymbol{= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\,\Psi(\beta;1-\alpha;\mu\,x)} \tag{2.200}$$

The hazard function is,

$$\boldsymbol{h(x) = \frac{\mu\,\beta\,\Psi(\beta+1;2-\alpha;\mu\,x)}{\Psi(\beta;1-\alpha;\mu\,x)}} \tag{2.201}$$

The $\boldsymbol{r^{th}}$ moment about zero is,

$$\boldsymbol{E(X^r) = \int_0^\infty x^r\,\frac{\mu\,\beta\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\,\Psi(\beta+1;2-\alpha;\mu\,x)\,dx}$$

Let,

$$\boldsymbol{y = \mu x \quad \therefore \quad dy = \mu dx}$$
$$\boldsymbol{E(X^r) = \frac{\mu\,\beta\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\int_0^\infty \left(\frac{y}{\mu}\right)^r\,\Psi(\beta+1;2-\alpha;y)\,\frac{dy}{\mu}}$$
$$\boldsymbol{= \frac{\beta\,\Gamma(\alpha+\beta)}{\mu^r\,\Gamma(\alpha)}\int_0^\infty y^{(r+1)-1}\,\Psi(\beta+1;2-\alpha;y)\,dy}$$
$$\boldsymbol{= \frac{\beta\,\Gamma(\alpha+\beta)}{\mu^r\,\Gamma(\alpha)}\,\frac{\Gamma((r+1))\Gamma(\beta+1-r-1)\Gamma(r+1-2+\alpha+1)}{\Gamma(\beta+1)\Gamma(\beta+1-2+\alpha+1)}}$$
$$\boldsymbol{= \frac{\beta\,\Gamma(\alpha+\beta)}{\mu^r\,\Gamma(\alpha)}\,\frac{\Gamma((r+1))\Gamma(\beta-r)\Gamma(\alpha+r)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}}$$
$$\boldsymbol{= \frac{r!\,\Gamma(\alpha+\beta)}{\mu^r\,\Gamma(\alpha)\Gamma(\beta)}\,\frac{\Gamma(\beta-r)\Gamma(\alpha+r)}{\Gamma(\alpha+\beta)}}$$
$$\boldsymbol{= \frac{r!}{\mu^r}\,\frac{B(\alpha+r,\beta-r)}{B(\alpha,\beta)}} \tag{2.202}$$

Using conditional expectation approach, we have

$$\boldsymbol{E\left[\frac{1}{\Lambda^r}\right] = \int_0^\infty \lambda^{-r}\,\frac{\mu^\alpha\,\lambda^{\beta-1}}{B(\alpha,\beta)\,(\lambda+\mu)^{\alpha+\beta}}\,d\lambda}$$
$$\boldsymbol{= \frac{\mu^\alpha}{B(\alpha,\beta)\,\int_0^\infty \lambda^{\beta-r-1}\,(\lambda+\mu)^{\alpha+\beta}}\,d\lambda}$$

$$= \frac{\mu^{\alpha}}{B(\alpha,\beta) \int\limits_{0}^{\infty} \lambda^{\beta-r-1} \, (\lambda+\mu)^{\alpha+\beta}} \, d\lambda$$

Let,

$$\lambda = \mu t \quad \therefore \quad d\lambda = \mu dt$$

$$E\left[\frac{1}{\Lambda^r}\right] = \frac{\mu^{\alpha} \, \mu^{\beta-r-1}}{B(\alpha,\beta)} \int\limits_{0}^{\infty} t^{\beta-r-1} \, (\mu t+\mu)^{\alpha+\beta} \, \mu dt$$

$$= \frac{1}{\mu^r} \, \frac{B(\alpha+r,\beta-r)}{B(\alpha,\beta)}$$

$$\therefore \quad E(X^r) = \frac{r!}{\mu^r} \, \frac{B(\alpha+r,\beta-r)}{B(\alpha,\beta)} \tag{2.203}$$

and,

$$E(X) = \frac{1}{\mu} \, \frac{B(\alpha+1,\beta-1)}{B(\alpha,\beta)}$$

$$= \frac{\alpha}{\beta-1} \, \frac{1}{\mu} \tag{2.204}$$

## 2.6 Conclusion

In the category of type I exponential mixtures, seven (7) out of twenty six (26) mixing distributions considered had the corresponding hazard functions in explicit form, thus exponential, gamma I, gamma II, shifted gamma, Lindley, generalized Lindley and half-logistic distributions.

The hazard functions for the first three mixing distributions take the pattern

$$\boldsymbol{h(t) = \frac{p}{(1+ct)^a}} \quad \text{for } \boldsymbol{p > 0,\ c > 0} \text{ and } \boldsymbol{a \geq 0}$$

Corresponding to shifted gamma and reciprocal inverse Gaussian, the hazard functions of type I exponential mixtures are sums of two functions with the pattern above. Corresponding to Lindley and Generalized Lindley, the hazard functions are differences of functions with this pattern.

So far, no pattern has been determined for the hazard function of type I exponential-Half logistic distribution. For the other mixing distributions the corresponding hazard functions are either ratios of modified Bessel functions of the third kind or ratios of confluent hyper-geometric functions.

Laplace transform of a probability density function characterizes type I exponential mixture and the rth moment of the type I exponential mixture is easily obtained using the the conditional expectation approach.

Shifted gamma mixture of type I exponential distribution has been expressed both explicitly and in terms of confluent hyper-geometric functions.
Further research is recommended to obtain posterior distributions and their moments, estimate parameters, fit data to the models and test hypothesis.

# Chapter 3

# TYPE II EXPONENTIAL MIXTURES AND THEIR MOMENTS

## 3.1 Introduction

From literature, type II moments for exponential mixtures for the exponential and scaled beta mixing distributions were obtained directly using Mellin transform technique and the mixtures were expressed in terms of modified Bessel function of the third kind and Tricomi confluent hyper-geometric function respectively.

Other type II exponential mixtures were constructed and expressed in terms of the modified Bessel function of the third kind, Tricomi confluent hyper-geometric function, Appell function, truncated gamma function and generalized hyper-geometric function. However, their moments were not obtained.

In this chapter, type II exponential mixtures are constructed using mixing distributions used to construct mixed Poisson distributions and they include inverse gamma, Lindley, Reciprocal inverse Gaussian, Generalized Lindley, Generalized Pareto (also known as Generalized Beta) and Generalized inverse Gaussian distribution. Moments for these mixtures have been obtained using both the Mellin transform technique and the conditional expectation approach, especially for cases where the Mellin transform technique fails. The conditional expectation approach has also been used to verify the results obtained using the Mellin transform technique.

## 3.2 The problem in mathematical form

Let

$$f\,(x\,|\,\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad x \geq 0 \quad \text{for} \quad \lambda > 0 \tag{3.1}$$

be the conditional type II exponential distribution, whose mean is the parameter $\lambda$.

Then,

$$f\,(x) = \int_0^\infty \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}\, g\,(\lambda)\, d\lambda \tag{3.2}$$

is type II exponential mixture, with $g(\lambda)$ being the mixing distribution.

The survival function is given by

$$\begin{aligned} S\,(x) &= \int_0^\infty S\,(x\,|\,\lambda)\;\; g\,(\lambda)\, d\lambda \\ &= \int_0^\infty e^{-\frac{x}{\lambda}}\, g\,(\lambda)\, d\lambda \end{aligned}$$

The hazard function is,

$$h(x) = \frac{f(x)}{S(x)}$$

The problem is to find the functions $f(x)$, $S(x)$ and $h(x)$ for the various mixing distributions and obtain moments directly and indirectly, in which case the following are useful formulas

$$\int_0^\infty \omega^{s-1}\, K_v(\omega)\; d\omega = 2^{s-2}\, \Gamma\left(\frac{s+v}{2}\right)\, \Gamma\left(\frac{s-v}{2}\right)$$

$$\int_0^\infty \omega^{s-1}\, \Psi(a;\; c;\; x) dx = \frac{\Gamma(s)\, \Gamma(a-s)\, \Gamma(s-c+1)}{\Gamma(a)\, \Gamma(a-c+1)}$$

### 3.2.1 Conditional expectation approach

The conditional expectation approach is given by:

**Lemma 3.1.**

$$\boldsymbol{E}(\boldsymbol{X}^r) = \boldsymbol{r}!\ \boldsymbol{E}[\boldsymbol{\Lambda}^r] \tag{3.3}$$

*where* $\boldsymbol{E}(\boldsymbol{X}^r)$ *is the rth moment of the mixture and* $\boldsymbol{E}[\boldsymbol{\Lambda}^r]$ *is the rth moment of the mixing distribution*

Proof

$$\boldsymbol{E}(\boldsymbol{X}^r) = \boldsymbol{E}\ \boldsymbol{E}\ (\boldsymbol{X}^r|\boldsymbol{\Lambda})$$

where

$$\begin{aligned}
\boldsymbol{E}\ (\boldsymbol{X}^r|\boldsymbol{\Lambda}) &= \int_0^\infty \boldsymbol{x}^r\ \boldsymbol{f}(\boldsymbol{x}|\boldsymbol{\lambda})\ \boldsymbol{dx}\ \boldsymbol{d\lambda} \\
&= \int_0^\infty \boldsymbol{x}^r\ \frac{\boldsymbol{1}}{\boldsymbol{\lambda}}\ \boldsymbol{e}^{-\frac{1}{\lambda}\boldsymbol{x}}\ \boldsymbol{dx} \\
&= \frac{\boldsymbol{1}}{\boldsymbol{\lambda}} \int_0^\infty \boldsymbol{x}^r\ \boldsymbol{e}^{-\frac{1}{\lambda}\boldsymbol{x}}\ \boldsymbol{dx} \\
&= \frac{\boldsymbol{1}}{\boldsymbol{\lambda}}\ \frac{\boldsymbol{\Gamma}(\boldsymbol{r}+\boldsymbol{1})}{\left(\frac{1}{\lambda}\right)^{\boldsymbol{r}+\boldsymbol{1}}} \\
&= \boldsymbol{r}!\ \boldsymbol{E}\,(\boldsymbol{\Lambda}^r)
\end{aligned}$$

## 3.3 Mixtures in explicit form

### 3.3.1 Inverse gamma mixing distribution

The inverse gamma is the only mixing distribution whose corresponding exponential mixture is in explicit form. Indeed all the functions $\boldsymbol{f(x)}$, $\boldsymbol{s(x)}$, $\boldsymbol{h(x)}$ and $\boldsymbol{E(X)}$ are in explicit form.

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-\frac{\beta}{\lambda}} \frac{1}{\lambda^{\alpha+1}} \qquad \lambda > 0 \quad \beta > 0, \quad \alpha > 0 \tag{3.4}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \; \frac{1}{\lambda^{\alpha+1}} \, \frac{\beta^\alpha}{\Gamma(\alpha)} \; e^{-\frac{\beta}{\lambda}} \, d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\lambda^{\alpha+2}} \; e^{-(x+\beta)\,\frac{1}{\lambda}} \, d\lambda \end{aligned}$$

Let

$$\frac{1}{\lambda} = t \quad \Longrightarrow \lambda = \frac{1}{t} \quad d\lambda = -\frac{dt}{t^2} \tag{3.5}$$

$$\begin{aligned} f\,(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha+2}\; e^{-(x+\beta)\;t} \, \frac{dt}{t^2} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^\alpha \; e^{-(x+\beta)\;t} \, dt \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{(\alpha+1)-1} \; e^{-(x+\beta)\;t} \, dt \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \, \frac{\Gamma(\alpha+1)}{(x+\beta)^{\alpha+1}} \\ &= \frac{\alpha\,\beta^\alpha}{(x+\beta)^{\alpha+1}} \qquad x > 0;, \quad \alpha,\, \beta > 0 \end{aligned} \tag{3.6}$$

which is Pareto II (Lomax) distribution with parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$\begin{aligned} S(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\frac{x}{\lambda}} \; \frac{1}{\lambda^{\alpha+1}} \; e^{-\frac{\beta}{\lambda}} \, d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\lambda^{\alpha+1}} \; e^{-(x+\beta)\,\frac{1}{\lambda}} \, d\lambda \end{aligned}$$

Using the substitution (3.5)

$$S\,(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha+1}\; e^{-(x+\beta)\;t} \, \frac{dt}{t^2}$$

$$= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha} \, e^{-(x+\beta)\, t} \, dt$$
$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \, \frac{\Gamma(\alpha)}{(x+\beta)^{\alpha}}$$
$$= \frac{\beta^{\alpha}}{(x+\beta)^{\alpha}} \quad x > 0; , \quad \alpha, \, \beta > 0$$

Therefore,

$$h(x) = \frac{\alpha}{x+\beta} \tag{3.7}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \alpha \, \beta^{\alpha} \int_0^\infty \frac{x^r}{(x+\beta)^{\alpha+1}}$$
$$= \alpha \, \beta^{r} \int_0^\infty \frac{t^r}{(t+1)^{\alpha+1}}$$
$$= \alpha \, \beta^{r} \, B(r+1, \alpha-1)$$
$$= r! \, \beta^{r} \, \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)}$$

Using conditional expectation approach, we have

$$E\left[\Lambda^r\right] = \int_0^\infty \lambda^r \, \frac{1}{\lambda^{\alpha+1}} \, \frac{\beta^{\alpha}}{\Gamma(\alpha)} \, e^{-\frac{\beta}{\lambda}} \, d\lambda$$
$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^\infty \lambda^{r-\alpha-1} \, e^{-\frac{\beta}{\lambda}} \, d\lambda$$

Using the substitution (3.5),

$$E\left[\Lambda^r\right] = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-r+1} \, e^{-\beta t} \, \frac{dt}{t^2}$$
$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-r-1} \, e^{-\beta t} \, dt$$
$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \, \frac{\Gamma(\alpha - r)}{\beta^{\alpha - r}}$$
$$= \frac{\Gamma(\alpha - r)}{\beta^{-r} \, \Gamma(\alpha)}$$
$$\therefore \quad E(X^r) = r! \, E(\lambda^r)$$
$$= \frac{r! \, \beta^{r} \, \Gamma(\alpha - r)}{\Gamma(\alpha)} \tag{3.8}$$

and,

$$\boldsymbol{E}(\boldsymbol{X}) = \frac{\boldsymbol{\beta}\,\boldsymbol{\Gamma}(\boldsymbol{\alpha}-\mathbf{1})}{\boldsymbol{\Gamma}(\boldsymbol{\alpha})}$$
$$= \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}-\mathbf{1}}, \quad \boldsymbol{\alpha} \neq \mathbf{1} \tag{3.9}$$

## 3.4 Mixtures in terms of modified Bessel function of the third kind

The following mixing distributions yield exponential mixtures that are in terms of the Bessel function of the third kind: exponential, gamma I, gamma II, half logistic, Lindley, generalized Lindley, inverse Gaussian and generalized inverse Gaussian distributions.

### 3.4.1 Exponential mixing distribution

In the case of exponential mixing distribution, the corresponding exponential mixture is constructed as follows:

$$g(\lambda) = \beta\, e^{-\beta\lambda} \quad \lambda > 0,\ \beta > 0 \tag{3.10}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}\, (\beta\, e^{-\beta\lambda})\, d\lambda \\ &= \beta \int_0^\infty \frac{1}{\lambda}\, e^{-\beta\left(\lambda+\frac{x}{\beta}\frac{1}{\lambda}\right)}\, d\lambda \end{aligned}$$

Let,

$$\lambda = \sqrt{\frac{x}{\beta}}\, z \quad \Longrightarrow \quad d\lambda = \sqrt{\frac{x}{\beta}}\, dz \tag{3.11}$$

$$\begin{aligned} f(x) &= \beta \int_0^\infty z^{-1}\, e^{-\frac{2}{2}\sqrt{x\beta}\left(z+\frac{1}{z}\right)}\, dz \\ &= 2\beta\, K_0\left(2\sqrt{\beta x}\right) \end{aligned} \tag{3.12}$$

(2.) The survival function is

$$\begin{aligned} S(x) &= \int_0^\infty e^{-\frac{x}{\lambda}}\, (\beta\, e^{-\beta\lambda})\, d\lambda \\ &= \beta \int_0^\infty \lambda^{1-1}; e^{-\beta\left(\lambda+\frac{x}{\beta}\frac{1}{\lambda}\right)}\, d\lambda \end{aligned}$$

Using the substitution (3.11)

$$\begin{aligned} S(x) &= \beta \int_0^\infty \sqrt{\frac{x}{\beta}}\, z^{1-1}\, e^{-\frac{2}{2}\sqrt{x\beta}\left(z+\frac{1}{z}\right)}\, dz \\ &= 2\sqrt{\beta\, x}\, K_1\left(2\sqrt{\beta x}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} h(x) &= \frac{\beta}{\sqrt{\beta x}} \frac{K_0\left(2\sqrt{\beta x}\right)}{K_1\left(2\sqrt{\beta x}\right)} \\ &= \sqrt{\frac{\beta}{x}} \frac{K_0\left(2\sqrt{\beta x}\right)}{K_1\left(2\sqrt{\beta x}\right)} \end{aligned} \tag{3.13}$$

**Theorem 3.1** (Bhattacharya, 1966)**.**

The hazard function of type II exponential-exponential mixture is a decreasing function.

*Proof.* □

Let

$$\beta = \frac{1}{\mu}$$

Then the hazard function (3.13) becomes

$$h(x) = \frac{1}{\sqrt{\mu x}} \frac{K_0\left(2\sqrt{\beta x}\right)}{K_1\left(2\sqrt{\beta x}\right)}$$

$$\frac{d\,h(x)}{dx} = \frac{\sqrt{\mu x}K_1\left(2\sqrt{\frac{x}{\mu}}\right)\frac{d}{dx}\left(K_0\left(2\sqrt{\frac{x}{\mu}}\right)\right) - K_0\left(2\sqrt{\frac{x}{\mu}}\right)\frac{d}{dx}\left(\sqrt{\mu x}K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right)}{\left[\sqrt{\mu x}K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right]^2}$$

Consider the first term in the numerator

$$\begin{aligned} &\sqrt{\mu x}\, K_1\left(2\sqrt{\frac{x}{\mu}}\right)\frac{d}{dx}\left(K_0\left(2\sqrt{\frac{x}{\mu}}\right)\right) \\ &= \sqrt{\mu x}\, K_1\left(2\sqrt{\frac{x}{\mu}}\right)\left\{-\frac{1}{2}\left\{K_1\left(2\sqrt{\frac{x}{\mu}}\right) + K_{-1}\left(2\sqrt{\frac{x}{\mu}}\right)\right\}\right\}\frac{1}{\sqrt{\mu x}} \\ &= \sqrt{\mu x}\, K_1\left(2\sqrt{\frac{x}{\mu}}\right)\left\{-\left[K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right]\left[2.\frac{1}{2}\left(\frac{x}{\mu}\right)^{-\frac{1}{2}}\frac{1}{\mu}\right]\right\}\frac{1}{\sqrt{\mu x}} \\ &= \sqrt{\mu x}\, K_1\left(2\sqrt{\frac{x}{\mu}}\right)\left\{-\left[K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right]\frac{1}{\sqrt{\mu x}}\right\} \\ &= -\left[K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right]^2 \end{aligned} \tag{3.14}$$

Next, consider the second term in the numerator, that is ,

$$-K_0\left(2\sqrt{\frac{x}{\mu}}\right)\frac{d}{dx}\left(\sqrt{\mu x}K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right)$$

$$= -K_0\left(2\sqrt{\frac{x}{\mu}}\right)\left\{\sqrt{\mu x}\,\left[K_1'\left(2\sqrt{\frac{x}{\mu}}\right)\right]\left[\frac{1}{\sqrt{\mu x}}\right]+\left[K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right]\left[\frac{\mu}{2}\frac{1}{\sqrt{\mu x}}\right]\right\}$$
$$= -K_0\left(2\sqrt{\frac{x}{\mu}}\right)\left\{K_1'\left(2\sqrt{\frac{x}{\mu}}\right)+\frac{1}{2}\sqrt{\frac{\mu}{x}}\,K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right\}$$
$$= -K_0\left(2\sqrt{\frac{x}{\mu}}\right)\left\{-\frac{1}{2}\left[K_2\left(2\sqrt{\frac{x}{\mu}}\right)+K_0\left(2\sqrt{\frac{x}{\mu}}\right)\right]+\frac{1}{2}\sqrt{\frac{\mu}{x}}\,K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right\}$$
$$= -K_0\left(2\sqrt{\frac{x}{\mu}}\right)\left\{-\frac{1}{2}\left[\frac{2}{2\sqrt{\frac{x}{\mu}}}K_1\left(2\sqrt{\frac{x}{\mu}}\right)+2\,K_0\left(2\sqrt{\frac{x}{\mu}}\right)\right]+\frac{1}{2}\sqrt{\frac{\mu}{x}}\,K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right\}$$
$$= -K_0\left(2\sqrt{\frac{x}{\mu}}\right)\left\{-\frac{1}{2}\frac{\mu}{x}\,K_1\left(2\sqrt{\frac{x}{\mu}}\right)-K_0\left(2\sqrt{\frac{x}{\mu}}\right)+\frac{1}{2}\sqrt{\frac{\mu}{x}}\,K_1\left(2\sqrt{\frac{x}{\mu}}\right)\right\}$$
$$= \left[K_0\left(2\sqrt{\frac{x}{\mu}}\right)\right]^2$$

Therefore,

$$\frac{d\,h(x)}{dx} = \left\{\left[\frac{K_0\left(2\sqrt{\frac{x}{\mu}}\right)}{K_1\left(2\sqrt{\frac{x}{\mu}}\right)}\right]^2 - 1\right\}\frac{1}{\mu x} < 0$$

Since $K_{v+\varepsilon} > K_v$ for all $\varepsilon > 0$ and $v > 0$. Hence $h(x)$ is a decreasing function.

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r\, f(x)\, dx$$
$$= \int_0^\infty x^r\, 2\beta K_0\left(2\sqrt{x\beta}\right)\, dx$$
$$= 2\beta \int_0^\infty x^r\, K_0\left(2\sqrt{x\beta}\right)\, dx$$

Let,

$$\omega = 2\sqrt{x\beta} \implies \omega^2 = 4x\beta \implies 2\omega d\omega = 4\beta dx$$
$$E(X^r) = 2\beta \int_0^\infty (\frac{\omega^2}{4\beta})^r\, K_0(\omega)\, \frac{\omega\, d\omega}{2\beta}$$
$$= \frac{1}{\beta^r\, 2^{2r}} \int_0^\infty \omega^{2r+1}\, K_0(\omega) d\omega$$
$$= \frac{\mu^r}{2^{2r}}\, 2^{2r+2-2}\,\Gamma(\frac{2r+2}{2})\,\Gamma(\frac{2r+2}{2})$$

$$= \frac{1}{\beta^r \, 2^{2r}} \, 2^{2r} \, \Gamma(r+1) \, \Gamma(r+1)$$

$$E(X^r) = \frac{(r!)^2}{\beta^r} \tag{3.15}$$

Using conditional expectation approach,

$$\begin{aligned} E(X^r) &= r! \, E[\Lambda^r] \\ &= \beta \, r! \int_{\lambda=0}^{\infty} \lambda^{r+1-1} \, e^{-\beta \lambda} \, d\lambda \\ &= \beta \, r! \, \frac{\Gamma(r+1)}{(\beta)^{r+1}} \\ &= \frac{(r!)^2}{\beta^r} \end{aligned} \tag{3.16}$$

Therefore,

$$E(X) = \frac{1}{\beta} \tag{3.17}$$

Using properties of modified Bessel function of the third kind, Bhattacharya (1966) proved the following theorem

**Theorem 3.2.** *The hazard function of type II exponential-exponential mixture is a decreasing function such that* $h'(x) < 0$.

Proof

$$\begin{aligned} h(x) &= \frac{f(x)}{S(x)} \\ \therefore \quad h'(x) &= \frac{S(x)f'(x) - f(x)S'(x)}{[S(x)]^2} \\ &= \frac{S(x)f'(x) + [f(x)]^2}{[S(x)]^2} \\ &= \frac{f'(x)}{S(x)} + \left[\frac{f(x)}{S(x)}\right]^2 \\ &= \frac{f'(x)}{S(x)} + [h(x)]^2 \end{aligned}$$

In this case,

$$f'(x) = \frac{d}{dx} \, 2\beta K_0(2\sqrt{\beta x}) = 2\beta \frac{d}{dx} \left[K_0(2\sqrt{\beta x})\right]$$

But

$$\frac{d}{d\omega} \, K_v(\omega) = -\frac{1}{2} \left[K_{v+1}(\omega) + K_{v-1}(\omega)\right]$$

and

$$K_v(\omega) = K_{-v}(\omega)$$

Therefore

$$\begin{aligned}
f'(x) &= 2\beta\left\{-\frac{1}{2}\left[K_1(2\sqrt{\beta x}) + K_{-1}(2\sqrt{\beta x})\right]\frac{d}{dx}\sqrt{\beta x}\right\} \\
&= 2\beta\left\{-\frac{1}{2}\left[2\,K_1(2\sqrt{\beta x})\right]\sqrt{\frac{\beta}{x}}\right\} \\
&= -2\beta\sqrt{\frac{\beta}{x}}\left[K_1(2\sqrt{\beta x})\right]
\end{aligned}$$

Therefore

$$\begin{aligned}
h(x) &= \frac{-2\beta\sqrt{\frac{\beta}{x}}\left[K_1(2\sqrt{\beta x})\right]}{2\sqrt{\beta\,x}\,K_1\left(2\sqrt{\beta x}\right)} + \left[\sqrt{\frac{\beta}{x}}\,\frac{K_0\left(2\sqrt{\beta x}\right)}{K_1\left(2\sqrt{\beta x}\right)}\right]^2 \\
&= -\left(\sqrt{\frac{\beta}{x}}\right)^2 + \left[\sqrt{\frac{\beta}{x}}\right]^2\left[\frac{K_0\left(2\sqrt{\beta x}\right)}{K_1\left(2\sqrt{\beta x}\right)}\right]^2 \\
&= \left(\sqrt{\frac{\beta}{x}}\right)^2\left\{\left[\frac{K_0\left(2\sqrt{\beta x}\right)}{K_1\left(2\sqrt{\beta x}\right)}\right]^2 - 1\right\}
\end{aligned} \tag{3.18}$$

Since $K_{v+\epsilon}(\omega) > K_v(\omega) \quad \forall \quad \epsilon > 0 \; and \; v > 0$

Then

$$h'(x) < 0$$

implying that $h(x)$ is a decreasing function

### 3.4.2 Gamma I mixing distribution

In this case the mixing distribution is gamma I and the corresponding exponential mixture is constructed below:

$$g(\lambda) = \frac{\beta^\alpha\,e^{-\beta\,\lambda}\,\lambda^{\alpha-1}}{\Gamma(\alpha)}, \quad \lambda > 0 \quad \alpha, \beta > 0 \tag{3.19}$$

The pdf of the mixture is,

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}\,\frac{\beta^\alpha\,e^{-\beta\,\lambda}\,\lambda^{\alpha-1}}{\Gamma(\alpha)}\,d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \lambda^{(\alpha-1)-1}\,e^{-\beta[\lambda+\frac{x}{\beta}\,\frac{1}{\lambda})}\,d\lambda
\end{aligned}$$

Let,

$$\boldsymbol{\lambda = \sqrt{\frac{x}{\beta}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{x}{\beta}}\, dz} \tag{3.20}$$

$$\boldsymbol{f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, (\sqrt{\frac{x}{\beta}})^{\alpha-1} \int_0^{\infty} z^{\alpha-1-1}\, e^{-2\,\frac{\sqrt{x\,\beta}}{2}\,(z+\frac{1}{z})}\, dz}$$

$$\boldsymbol{= 2\, \frac{\left(\sqrt{\beta x}\right)^{\alpha}}{\Gamma(\alpha)}\, \sqrt{\frac{\beta}{x}}\, K_{\alpha-1}(2\sqrt{\beta\, x})} \tag{3.21}$$

(2.) The survival function is

$$\boldsymbol{S(x) = \int_0^{\infty} e^{-\frac{x}{\lambda}}\, \frac{\beta^{\alpha}\, e^{-\beta\,\lambda}\, \lambda^{\alpha-1}}{\Gamma(\alpha)}\, d\lambda}$$

$$\boldsymbol{= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{\alpha-1}\, e^{-\beta[\lambda+\frac{x}{\beta}\,\frac{1}{\lambda})}\, d\lambda}$$

Using the substitution (3.20)

$$\boldsymbol{S(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, (\sqrt{\frac{x}{\beta}})^{\alpha} \int_0^{\infty} z^{\alpha-1}\, e^{-\sqrt{x\,\beta}\,(z+\frac{1}{z})}\, dz}$$

$$\boldsymbol{= 2\, \frac{\left(\sqrt{\beta x}\right)^{\alpha}}{\Gamma(\alpha)}\, K_{\alpha}(2\sqrt{\beta\, x})} \tag{3.22}$$

and the hazard function is

$$\boldsymbol{h(x) = \sqrt{\frac{\beta}{x}}\, \frac{K_{\alpha-1}(2\sqrt{\beta\, x})}{K_{\alpha}(2\sqrt{\beta\, x})}} \tag{3.23}$$

The $\boldsymbol{r^{th}}$ moment about zero is,

$$\boldsymbol{E(X^r) = \int_0^{\infty} x^r \left\{2\, \frac{\left(\sqrt{\beta x}\right)^{\alpha}}{\Gamma(\alpha)}\, \sqrt{\frac{\beta}{x}}\, K_{\alpha-1}(2\sqrt{\beta\, x})\right\} dx}$$

$$\boldsymbol{= \frac{2\, \beta^{\frac{\alpha}{2}+\frac{1}{2}}}{\Gamma(\alpha)} \int_0^{\infty} x^{r+\frac{\alpha}{2}-\frac{1}{2}}\, K_{\alpha-1}(2\sqrt{\beta\, x})\, dx}$$

Let,

$$\boldsymbol{\omega = 2\sqrt{\beta\, x}, \quad \omega^2 = 4x\beta \quad \text{and} \quad dx = \frac{\omega}{2\beta}\, d\omega}$$

so that

$$\boldsymbol{E(X^r) = \frac{2\beta^{\frac{\alpha+1}{2}}}{\Gamma(\alpha)}\, \frac{1}{(4\beta)^{r+\frac{\alpha}{2}-\frac{1}{2}}} \int_0^{\infty} (\omega^2)^{r+\frac{\alpha}{2}-\frac{1}{2}}\, K_{\alpha-1}(\omega)\, d\omega}$$

$$= \frac{\beta^{-r}}{\Gamma(\alpha)\; 2^{2r+\alpha-1}} \int_0^\infty \omega^{2r+\alpha}\; K_{\alpha-1}(\omega)\; d\omega$$
$$= \frac{1}{\beta^r\; 2^{2r+\alpha-1}\; \Gamma(\alpha)}\; M[K_{\alpha-1}(\omega)\; 2r+\alpha+1]$$

Applying ((1.18)), we have

Let,

$$s = 2r+\alpha+1 \quad and \quad v = \alpha - 1$$
$$E(X^r) = \frac{\Gamma(r+\alpha)\,\Gamma(r+1)}{\beta^r\;\Gamma(\alpha)}$$
$$= \frac{r!\,\Gamma(r+\alpha)}{\beta^r\;\Gamma(\alpha)} \tag{3.24}$$

Using conditional expectation approach, we have

$$E(X^r) = \; r!\; E\,[\Lambda^r]$$
$$= r!\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{r+\alpha-1}\; e^{-\beta\,\alpha}\; d\lambda$$
$$= r!\frac{\beta^\alpha}{\Gamma(\alpha)}\; \frac{\Gamma(r+\alpha)}{\beta^r+\alpha}$$
$$= \frac{r!\,\Gamma(r+\alpha)}{\beta^r\;\Gamma(\alpha)} \tag{3.25}$$

and,

$$E(X) = \frac{\alpha}{\beta} \tag{3.26}$$

### 3.4.3 Gamma II mixing distribution

The exponential-gamma II distribution is constructed and moments obtained as follows:

$$g(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)}\; e^{-\frac{\lambda}{\beta}} \lambda^{\alpha-1}, \quad \lambda > 0 \quad \beta > 0 \tag{3.27}$$

The pdf of the mixture is,

$$f(x) = \int_0^\infty \frac{1}{\lambda}\; e^{-\frac{x}{\lambda}} \frac{1}{\beta^\alpha \Gamma\alpha}\; e^{-\frac{\lambda}{\beta}} \lambda^{\alpha-1} d\lambda$$
$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty \lambda^{(\alpha-1)-1}\; e^{-\frac{1}{\beta}(\lambda+\beta\, x \frac{1}{\lambda})}\; d\lambda$$

Let,

$$\boldsymbol{\lambda = \sqrt{\beta x}\, z \quad \text{and therefore} \quad d\lambda = \sqrt{\beta x}\, dz} \tag{3.28}$$

$$\begin{aligned}
\boldsymbol{f(x)} &= \boldsymbol{\frac{1}{\beta^{\alpha}\Gamma(\alpha)}(\sqrt{\beta x})^{\alpha-1}\int_{0}^{\infty} z^{(\alpha-1)-1} e^{-\frac{1}{2}\left(2\sqrt{\frac{x}{\beta}}\right)(z+\frac{1}{z})} dz} \\
&= \boldsymbol{\frac{2}{\beta^{\alpha}\Gamma(\alpha)}(\sqrt{\beta x})^{\alpha-1} K_{\alpha-1}\left(2\sqrt{\frac{x}{\beta}}\right)} \\
&= \boldsymbol{(\sqrt{\frac{x}{\beta}})^{\alpha}\, \frac{2}{\Gamma(\alpha)\, \sqrt{\beta\, x}}\, K_{\alpha-1}\left(2\, \sqrt{\frac{x}{\beta}}\right)} \\
&= \boldsymbol{\frac{2}{\Gamma(\alpha)\, \beta^{\frac{\alpha+1}{2}}}\, x^{\frac{\alpha-1}{2}}\, K_{\alpha-1}\left(2\, \sqrt{\frac{x}{\beta}}\right)}
\end{aligned} \tag{3.29}$$

as obtained by Nadarajah and Kotz (2006)

(2.) The survival function is

$$\begin{aligned}
\boldsymbol{S(x)} &= \boldsymbol{\int_{0}^{\infty} S(x|\lambda)\, g(\lambda)\, d\lambda} \\
&= \boldsymbol{\int_{0}^{\infty} e^{-\frac{x}{\lambda}}\, \frac{1}{\beta^{\alpha}\, \Gamma(\alpha)}\, e^{-\frac{\lambda}{\beta}}\, \lambda^{\alpha-1}\, d\lambda} \\
&= \boldsymbol{\frac{1}{\beta^{\alpha}\Gamma\alpha}\, \int_{0}^{\infty} \lambda^{(\alpha-1)}\, e^{-\frac{1}{\beta}(\lambda+\beta\, x\frac{1}{\lambda})}\, d\lambda}
\end{aligned}$$

Using the substitution (3.28)

$$\boldsymbol{\lambda = \sqrt{\beta\, x}\, z \quad \text{and} \quad d\lambda = \sqrt{\beta\, x}\, dz}$$

$$\begin{aligned}
\boldsymbol{S(x)} &= \boldsymbol{\frac{\left(\sqrt{\beta\, x}\right)^{\alpha}}{\beta^{\alpha}\, \Gamma(\alpha)}\, \int_{0}^{\infty} z^{(\alpha-1)}\, e^{-\frac{2}{2}\, \sqrt{\frac{x}{\beta}}\, (z+\frac{1}{z})}\, dz} \\
&= \boldsymbol{(\frac{x}{\beta})^{\frac{\alpha}{2}}\, \frac{2}{\Gamma(\alpha)}\, K_{\alpha}\left(2\sqrt{\frac{x}{\beta}}\right)}
\end{aligned} \tag{3.30}$$

(3.) The hazard function is

$$\begin{aligned}
\boldsymbol{h(x)} &= \boldsymbol{\frac{(\frac{x}{\beta})^{\frac{\alpha}{2}}\, \frac{2}{\Gamma(\alpha)}\, \frac{1}{\sqrt{\beta\, x}}\, K_{\alpha-1}\left(2\sqrt{\frac{x}{\beta}}\right)}{(\frac{x}{\beta})^{\frac{\alpha}{2}}\, \frac{2}{\Gamma(\alpha)}\, K_{\alpha}\left(2\sqrt{\frac{x}{\beta}}\right)}} \\
&= \boldsymbol{\frac{1}{\sqrt{\beta\, x}}\, \frac{K_{\alpha-1}\left(2\sqrt{\frac{x}{\beta}}\right)}{K_{\alpha}\left(2\sqrt{\frac{x}{\beta}}\right)}}
\end{aligned} \tag{3.31}$$

The $r^{th}$ moment about zero is,

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r \, \frac{2x^{\frac{\alpha-1}{2}}}{\Gamma(\alpha)\,\beta^{\frac{\alpha+1}{2}}} \, K_{\alpha-1}(2\sqrt{\frac{x}{\beta}})\,dx \\
&= \frac{2}{\Gamma(\alpha)\,\beta^{\frac{\alpha+1}{2}}} \int_0^\infty x^{r+\frac{\alpha}{2}-\frac{1}{2}} \; K_{\alpha-1}(2\sqrt{\frac{x}{\beta}})\,dx \\
&= \frac{2}{\Gamma(\alpha)\,\beta^{\frac{\alpha+1}{2}}} \int_0^\infty x^{r+\frac{\alpha}{2}+\frac{1}{2}-1} \; K_{\alpha-1}(2\sqrt{\frac{x}{\beta}})\,dx \\
&= \frac{2}{\Gamma(\alpha)\,\beta^{\frac{\alpha+1}{2}}} \int_0^\infty x^{\frac{2r+\alpha+1}{2}-1} \; K_{\alpha-1}(2\sqrt{\frac{x}{\beta}})\,dx
\end{aligned}$$

Let

$$\omega = 2\sqrt{\frac{x}{\beta}} \quad \therefore \quad \omega^2 = 4\frac{x}{\beta}$$

$$\therefore \quad x = \frac{\beta}{4}\,\omega^2 \quad and \quad dx = \frac{\beta}{2}\omega d\omega$$

Therefore

$$\begin{aligned}
E(X^r) &= \frac{2}{\Gamma(\alpha)\,\beta^{\frac{\alpha+1}{2}}} \int_0^\infty \left[\frac{\beta}{4}\,\omega^2\right]^{\frac{2r+\alpha+1}{2}-1} \left(\frac{\beta}{2}\omega\right) K_{\alpha-1}\left(2\sqrt{\frac{x}{\beta}}\right) d\omega \\
&= \frac{\beta^r}{\Gamma(\alpha)\,2^{(2r+\alpha+1)-2}} \int_0^\infty (\omega)^{(2r+\alpha+1)-1} \; K_{\alpha-1}(\omega) d\omega
\end{aligned}$$

By equation (1.18)

$$\begin{aligned}
E(X^r) &= \frac{\beta^r}{\Gamma(\alpha) 2^{(2r+\alpha+1)-2}} \, 2^{(2r+\alpha+1)-2} \Gamma\left(\frac{2r+\alpha+1+\alpha-1}{2}\right) \Gamma\left(\frac{2r+\alpha+1-\alpha+1}{2}\right) \\
&= \frac{\beta^r}{\Gamma(\alpha)} \, \Gamma(\alpha+r)\,\Gamma(r+1) \\
&= r!\,\beta^r \, \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \qquad (3.32)
\end{aligned}$$

Using conditional expectation approach

$$\begin{aligned}
E(X^r) &= r!\,E(\Lambda^r) \\
&= r! \int_0^\infty \lambda^r \, \frac{1}{\beta^\alpha \Gamma(\alpha)} \, e^{-\frac{\lambda}{\beta}} \, \lambda^{\alpha-1}\,d\lambda \\
&= r! \, \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \lambda^{\alpha+r-1} \, e^{-\frac{\lambda}{\beta}}\,d\lambda
\end{aligned}$$

$$= r! \, \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \, \frac{\Gamma(\alpha+r)}{\left(\frac{1}{\beta}\right)^{\alpha+r}}$$

$$= r! \, \frac{\beta^{\alpha+r}}{\beta^{\alpha}\Gamma(\alpha)} \, \Gamma(\alpha+r) \tag{3.33}$$

$$= r! \, \beta^{r} \, \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{3.34}$$

### 3.4.4 Half logistic mixing distribution

Using half logistic mixing distribution, the exponential-half logistic distribution is constructed and moments are obtained.

$$g(\lambda) = \frac{2\,\mu\, e^{-\mu\,\lambda}}{(1+e^{-\mu\,\lambda})^2} \quad \lambda > 0 \quad \mu > 0 \tag{3.35}$$

The pdf of the mixture is,

$$f(x) = \int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \, \frac{2\,\mu\, e^{-\mu\,\lambda}}{(1+e^{-\mu\,\lambda})^2} \, d\lambda$$

$$= 2\mu \int_0^{\infty} \lambda^{-1} \, e^{-\frac{x}{\lambda}-\mu\lambda} \, \sum_{k=0}^{\infty} \binom{-2}{k} e^{-\mu\lambda k} d\lambda$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \lambda^{-1} \, e^{-\mu(1+k)\left(\lambda+\frac{x}{\mu(k+1)}\,\frac{1}{\lambda}\right)} \, d\lambda$$

Using the substitution

$$\lambda = \sqrt{\frac{x}{\mu(k+1)}}\, z \quad \text{so that} \quad d\lambda = \sqrt{\frac{x}{\mu(k+1)}}\, dz \tag{3.36}$$

$$f(x) = 2\,\mu \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \left(\sqrt{\frac{x}{\mu(k+1)}}\, z\right)^{-1} e^{-\sqrt{\mu(k+1)\,x}\,(z+\frac{1}{z})} \sqrt{\frac{x}{\mu(k+1)}}\, dz$$

$$= 4\,\mu \sum_{k=0}^{\infty} \binom{-2}{k} K_0\left(2\sqrt{\mu(k+1)\,x}\right) \tag{3.37}$$

The survival function is,

$$S(x) = \int_0^{\infty} e^{-\frac{x}{\lambda}} \, \frac{2\,\mu\, e^{-\mu\,\lambda}}{(1+e^{-\mu\,\lambda})^2} \, d\lambda$$

$$= 2\mu \int_0^{\infty} e^{-\frac{x}{\lambda}-\mu\lambda} \, \sum_{k=0}^{\infty} \binom{-2}{k} \, e^{-\mu\lambda k} d\lambda$$

$$= 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \lambda^{0} \, e^{-\mu(1+k)\left[\lambda+\frac{x}{\mu(k+1)}\,\frac{1}{\lambda}\right]} \, d\lambda \tag{3.38}$$

Using the substitution (3.36)

$$\begin{aligned} S(x) &= 2\,\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sqrt{\frac{x}{\mu(k+1)}} \int_0^{\infty} z^{1-1}\, e^{-\sqrt{\mu(k+1)\,x}\,(z+\frac{1}{z})}\, dz \\ &= 4\,\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sqrt{\frac{x}{\mu(k+1)}}\, K_1\,(2\,\sqrt{\mu(k+1)\,x}) \end{aligned} \tag{3.39}$$

$$h(x) = \frac{\sum_{k=0}^{\infty} \binom{-2}{k}\, K_0\,(2\,\sqrt{\mu(k+1)\,x})}{\sum_{k=0}^{\infty} \binom{-2}{k}\, \sqrt{\frac{x}{\mu(k+1)}}\,;K_1\,(2\,\sqrt{\mu(k+1)\,x})} \tag{3.40}$$

The $r^{th}$ moment about zero is,

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r\, 4\,\mu \sum_{k=0}^{\infty} \binom{-2}{k} [K_0\,(2\,\sqrt{\mu(k+1)\,x}]\, dx \\ &= \ 4\,\mu \sum_{k=0}^{\infty} \int_0^{\infty} x^r\, (K_0\,(2\,\sqrt{\mu(k+1)\,x})\, dx \end{aligned}$$

Let,

$$\begin{aligned} \omega &= 2\,\sqrt{\mu(k+1)\,x} \quad \omega^2 \ = \ 4\,\mu(k+1)\,x \\ x &= \frac{\omega^2}{4\mu(k+1)} \quad dx = \frac{\omega}{2\mu(k+1)}\, d\omega \\ E(X^r) &= \ 4\,\mu \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} (\frac{\omega^2}{4\mu(k+1)})^r\, K_0\,(\omega)\, \frac{\omega}{2\mu(k+1)}\, d\omega \\ &= \ \frac{2}{(4\mu)^r} \sum_{k=0}^{\infty} \binom{-2}{k} \int_0^{\infty} \frac{\omega^{2r+1}}{(k+1)^{r+1}} K_0\,(\omega)\, d\omega \\ &= \ \frac{2}{(4\mu)^r} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{(k+1)^{r+1}} \int_0^{\infty} \ \omega^{(2r+2-1)} K_0\,(\omega)\, d\omega \end{aligned}$$

Using equation (1.18)

$$\begin{aligned} E(X^r) &= \ \frac{2\,.\,2^{2r}}{(4\mu)^r} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{(k+1)^{r+1}}\, (\Gamma(r+1))^2 \\ &= \ \frac{2}{(4\mu)^r} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{(k+1)^{r+1}}\, 2^{2r+2-2} (\Gamma(\frac{2r+2}{2}))^2 \\ &= \frac{2}{\mu^r}\, (r!)^2 \sum_{k=0}^{\infty} \binom{-2}{k}\, \frac{1}{[\,(1+k)]^{r+1}} \end{aligned} \tag{3.41}$$

Using conditional expectation approach, we have

$$E(X^r) = \ r!\, E\,[\Lambda^r]$$

$$= r! \int_{\lambda=0}^{\infty} \lambda^{r+1-1} \frac{2\,\mu\, e^{-\mu\,\lambda}}{(1+e^{-\mu\,\lambda})^2}\, d\lambda$$

$$= 2\, r!\, \mu \int_{\lambda=0}^{\infty} \lambda^r\, e^{-\mu\,\lambda} (1+e^{-\mu\,\lambda})^{-2}\, d\lambda$$

$$= 2\, r!\, \mu \int_{\lambda=0}^{\infty} \lambda^r\, e^{-\mu\,\lambda} \sum_{k=0}^{\infty} \binom{-2}{k}\, e^{-\mu\lambda k} d\lambda$$

$$= 2\, r!\, \mu \sum_{k=0}^{\infty} \binom{-2}{k} \int_{\lambda=0}^{\infty} \lambda^r\, e^{-\mu\,(1+k)\lambda}\, d\lambda$$

$$= 2\, r!\, \mu \sum_{k=0}^{\infty} \binom{-2}{k} \frac{\Gamma(r+1)}{[\mu\,(1+k)]^{r+1}}$$

$$= \frac{2}{\mu^r}\, (r!)^2 \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{[\,(1+k)]^{r+1}}$$

and,

$$E(X) = \frac{2}{\mu} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{1}{(k+1)^2}$$

$$= \frac{2}{\mu} \sum_{k=0}^{\infty} (-1)^k \binom{2+k-1}{k} \frac{1}{(k+1)^2}$$

$$= \frac{2}{\mu} \sum_{k=0}^{\infty} (-1)^k \binom{k+1}{k} \frac{1}{(k+1)^2}$$

$$= \frac{2}{\mu} \sum_{k=0}^{\infty} (-1)^k \binom{k+1}{1} \frac{1}{(k+1)^2}$$

$$= \frac{2}{\mu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

$$= \frac{2}{\mu} \left\{ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \ldots \right\} \tag{3.42}$$

But

$$-\log(1-\theta) = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \ldots$$

$$\therefore \quad -\log(1+\theta) = -\theta + \frac{\theta^2}{2} - \frac{\theta^3}{3} \pm \ldots$$

$$\therefore \quad \log(1+\theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \frac{\theta^5}{5} \pm \ldots$$

$$\therefore \quad \log(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \pm \ldots$$

$$= \log 2 \tag{3.43}$$

Therefore,

$$E(X) = \frac{2}{\mu}\ \log 2 \tag{3.44}$$

### 3.4.5 Lindley mixing distribution

In the case of Lindley mixing distribution, the corresponding exponential mixture is constructed as follows:

$$g(\lambda) = \frac{\theta^2}{\theta+1}\ (\lambda+1)\ e^{-\theta\,\lambda} \quad \lambda > 0 \quad \theta > 0 \tag{3.45}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{\lambda}\ e^{-\frac{x}{\lambda}}\ \frac{\theta^2}{\theta+1}\ (\lambda+1)\ e^{-\theta\,\lambda}\ d\lambda \\ &= \frac{\theta^2}{\theta+1} \int_0^\infty \left(1+\frac{1}{\lambda}\right)\ e^{-\theta\,\lambda-\frac{x}{\lambda}}\ d\lambda \\ &= \frac{\theta^2}{\theta+1} \int_0^\infty \left(1+\frac{1}{\lambda}\right)\ e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\ d\lambda \\ &= \frac{\theta^2}{\theta+1}\ \left\{\int_0^\infty \lambda^{1-1}\ e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\ d\lambda + \int_0^\infty \lambda^{0-1}\ e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\ d\lambda\right\} \end{aligned}$$

Let,

$$\lambda = \sqrt{\frac{x}{\theta}}\ z \quad \therefore \quad \lambda = \sqrt{\frac{x}{\theta}}\ dz \tag{3.46}$$

$$\begin{aligned} f(x) &= \frac{\theta^2}{\theta+1}\ \left\{\sqrt{\frac{x}{\theta}} \int_0^\infty z^{1-1}\ e^{-\sqrt{\theta\,x}\,\left(z+\frac{1}{z}\right)}\ dz + \int_0^\infty z^{0-1}\ e^{-\sqrt{\theta\,x}\,\left(z+\frac{1}{z}\right)}\ dz\right\} \\ &= \frac{2\,\theta^2}{\theta+1}\ \left\{\sqrt{\frac{x}{\theta}}\ K_1(2\,\sqrt{\theta\,x}) + \ K_0(2\,\sqrt{\theta\,x})\right\} \end{aligned} \tag{3.47}$$

The survival function is,

$$\begin{aligned} S(x) &= \int_0^\infty e^{-\frac{x}{\lambda}}\ \frac{\theta^2}{\theta+1}\ (\lambda+1)\ e^{-\theta\,\lambda}\ d\lambda \\ &= \frac{\theta^2}{\theta+1} \int_0^\infty \left\{\lambda^{2-1}\ e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)} + \lambda^{1-1}\ e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\right\}\ d\lambda \end{aligned}$$

Using the substitution (3.46)

$$S(x) = \frac{\theta^2}{\theta+1}\ \left\{\left(\sqrt{\frac{x}{\theta}}\right)^2 \int_0^\infty z^{2-1}\ e^{-\sqrt{\theta\,x}\,\left(z+\frac{1}{z}\right)}\ dz + \sqrt{\frac{x}{\theta}} \int_0^\infty z^{1-1}\ e^{-\sqrt{\theta\,x}\,\left(z+\frac{1}{z}\right)}\ dz\right\}$$

$$= \frac{2\,\theta^2}{\theta+1}\left\{\left(\sqrt{\frac{x}{\theta}}\right)^2 K_2(2\sqrt{\theta\,x}) + \sqrt{\frac{x}{\theta}}\,K_1(2\sqrt{\theta\,x})\right\} \tag{3.48}$$

Therefore,

$$h(x) = \frac{\left\{\sqrt{\frac{x}{\theta}}\,K_1(2\sqrt{\theta\,x}) + K_0(2\sqrt{\theta\,x})\right\}}{\sqrt{\frac{x}{\theta}}\left\{\sqrt{\frac{x}{\theta}}\,K_2(2\sqrt{\theta\,x}) + K_1(2\sqrt{\theta\,x})\right\}} \tag{3.49}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r\,\frac{2\,\theta^2}{\theta+1}\left\{\sqrt{\frac{x}{\theta}}\,K_1(2\sqrt{\theta\,x}) + K_0(2\sqrt{\theta\,x})\right\}dx$$

$$= \frac{2\,\theta^2}{(\theta+1)\sqrt{\theta}}\int_0^\infty x^{r+\frac{1}{2}}\,K_1(2\sqrt{\theta\,x})dx + \frac{2\,\theta^2}{\theta+1}\int_0^\infty x^r\,K_0(2\sqrt{\theta\,x})dx$$

Let,

$$\omega = 2\sqrt{\theta x} \quad \therefore \quad x = \frac{\omega^2}{4\theta} \quad and \quad dx = \frac{\omega}{2\theta}d\omega$$

$$E(X^r) = \frac{2\,\theta^2}{(\theta+1)\sqrt{\theta}}\int_0^\infty\left(\frac{\omega^2}{4\theta}\right)^{r+\frac{1}{2}} K_1(\omega)\,\frac{\omega}{2\theta}d\omega + \frac{2\,\theta^2}{\theta+1}\int_0^\infty\left(\frac{\omega^2}{4\theta}\right)^r K_0(\omega)\,\frac{\omega}{2\theta}d\omega$$

$$= \frac{1}{2^{2r+1}}\,\frac{1}{\theta+1}\,\frac{1}{\theta^r}\int_0^\infty \omega^{2r+2}\,K_1(\omega)\,d\omega + \frac{1}{2^{2r}}\,\frac{1}{\theta+1}\,\frac{1}{\theta^{r+1}}\int_0^\infty \omega^{2r+1}\,K_0(\omega)\,d\omega$$

$$= \frac{1}{2^{2r+1}}\,\frac{1}{\theta+1}\,\frac{1}{\theta^r}\left[2^{2r+1}\Gamma(r+2)\Gamma(r+1)\right] + \frac{1}{2^{2r}}\,\frac{1}{\theta+1}\,\frac{1}{\theta^{r+1}}\left[2^{2r}\Gamma(r+1)\Gamma(\right.$$

$$= \frac{1}{2^{2r}}\frac{2^{2r}}{\theta+1}\frac{1}{\theta^r}\left[\frac{1}{2}2(r+1)(r!)^2 + \theta(r!)^2\right]$$

$$= \frac{(r!)^2\,(r+\theta+1)}{\theta^r\,(\theta+1)} \tag{3.50}$$

Using conditional expectation approach, we have

$$E(X^r) = r!\,E\left[\Lambda^r\right]$$

$$E(\Lambda^r) = \frac{\theta^2}{\theta+1}\left\{\int_0^\infty \lambda^{r+1}\,e^{-\theta\lambda} + \lambda^r\,e^{-\theta\lambda}\right\}$$

$$= \frac{\theta^2}{\theta+1}\left\{\frac{\Gamma(r+2)}{\theta^{r+2}} + \frac{\Gamma(r+1)}{\theta^{r+1}}\right\}$$

$$= \frac{r!}{\theta+1}\left\{\frac{r+1}{\theta^r} + \frac{\theta}{\theta^r}\right\}$$

$$= \frac{r!}{\theta^r\,(\theta+1)}\left\{r+\theta+1\right\}$$

$$\therefore \quad E(X^r) = \frac{(r!)^2\,(r+\theta+1)}{\theta^r\,(\theta+1)} \tag{3.51}$$

and,

$$E(X) = \frac{\theta+2}{\theta\,(\theta+1)} \tag{3.52}$$

### 3.4.6 Generalized Lindley mixing distribution

Using generalized Lindley mixing distribution, the exponential-generalized Lindley distribution is constructed and moments are obtained.

$$g(\lambda) = \frac{\theta^2\,(\theta\lambda)^{\alpha-1}\,(\alpha+\lambda)\,e^{-\theta\,\lambda}}{(\theta+1)\,\Gamma(\alpha+1)} \quad \lambda > 0 \quad \theta, \alpha > 0 \tag{3.53}$$

The pdf of the mixture is,

$$f(x) = \int_0^\infty \frac{1}{\lambda}\,e^{-\frac{x}{\lambda}}\,\frac{\theta^2\,(\theta\lambda)^{\alpha-1}\,(\alpha+\lambda)\,e^{-\theta\,\lambda}}{(\theta+1)\,\Gamma(\alpha+1)}\,d\lambda$$

$$= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left\{\alpha\int_0^\infty \lambda^{(\alpha-1)-1}\,e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\,d\lambda + \int_0^\infty \lambda^{\alpha-1}\,e^{-\theta\,\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\,d\lambda\right\}$$

Let,

$$\lambda = \sqrt{\frac{x}{\theta}}\,z \quad \therefore \quad \lambda = \sqrt{\frac{x}{\theta}}\,dz \tag{3.54}$$

$$f(x) = \frac{\alpha\,\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left(\sqrt{\frac{x}{\theta}}\right)^{\alpha-1}\int_0^\infty z^{(\alpha-1)-1}\,e^{-\frac{2\sqrt{\theta\,x}}{2}\,\left(z+\frac{1}{z}\right)}\,dz +$$

$$\frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left(\sqrt{\frac{x}{\theta}}\right)^{\alpha}\int_0^\infty z^{\alpha-1}\,e^{-\frac{2\sqrt{\theta\,x}}{2}\,\left(z+\frac{1}{z}\right)}\,dz$$

$$= \frac{\alpha\,\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left(\sqrt{\frac{x}{\theta}}\right)^{\alpha-1}\,2\,K_{(\alpha-1)}(2\,\sqrt{\theta\,x}) +$$

$$\frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left(\sqrt{\frac{x}{\theta}}\right)^{\alpha}\,2\,K_{(\alpha)}(2\,\sqrt{\theta\,x})$$

$$= \frac{2\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left(\sqrt{\frac{x}{\theta}}\right)^{\alpha-1}\left\{\alpha\,K_{(\alpha-1)}(2\,\sqrt{\theta\,x}) + \sqrt{\frac{x}{\theta}}\,K_{(\alpha)}(2\,\sqrt{\theta\,x})\right\} \tag{3.55}$$

The survival function is,

$$S(x) = \int_0^\infty e^{-\frac{x}{\lambda}}\,\frac{\theta^2\,(\theta\lambda)^{\alpha-1}\,(\alpha+\lambda)\,e^{-\theta\,\lambda}}{(\theta+1)\,\Gamma(\alpha+1)}\,d\lambda$$

$$= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)} \left\{ \alpha \int_0^\infty \lambda^{\alpha-1}\, e^{-\theta\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\, d\lambda + \int_0^\infty \lambda^{(\alpha+1)-1}\, e^{-\theta\left(\lambda+\frac{x}{\theta}\frac{1}{\lambda}\right)}\, d\lambda \right\}$$

Using the substitution (3.54)

$$S(x) = \frac{2\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)} \left(\sqrt{\frac{x}{\theta}}\right)^{\alpha} \left\{\alpha\, K_{(\alpha)}(2\sqrt{\theta\, x}) + \sqrt{\frac{x}{\theta}}\, K_{(\alpha+1)}(2\sqrt{\theta\, x})\right\} \tag{3.56}$$

Therefore,

$$h(x) = \frac{\alpha\, K_{\alpha-1}(2\sqrt{\theta\, x}) + \sqrt{\frac{x}{\theta}}\, K_{\alpha}(2\sqrt{\theta\, x})}{\sqrt{\frac{x}{\theta}}\left\{\alpha\, K_{\alpha}(2\sqrt{\theta\, x}) + \sqrt{\frac{x}{\theta}}\, K_{\alpha+1}(2\sqrt{\theta\, x})\right\}} \tag{3.57}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r\, \frac{2\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)} \left(\sqrt{\frac{x}{\theta}}\right)^{\alpha-1} \left\{\alpha\, K_{(\alpha-1)}(2\sqrt{\theta\, x}) + \sqrt{\frac{x}{\theta}}\, K_{(\alpha)}(2\sqrt{\theta\, x})\right\}$$

$$= \frac{2\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\, \frac{\alpha}{\theta^{\frac{\alpha}{2}-\frac{1}{2}}} \int_0^\infty x^{r+\frac{\alpha}{2}+\frac{1}{2}}\, K_{\alpha-1}(2\sqrt{\theta\, x})dx$$

$$+ \quad \frac{2\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\, \frac{1}{\theta^{\frac{\alpha}{2}}} \int_0^\infty x^{r+\frac{\alpha}{2}}\, K_{\alpha}(2\sqrt{\theta\, x})dx$$

Let,

$$\omega = 2\sqrt{\theta x} \quad \therefore \quad x = \frac{\omega^2}{4\theta} \quad and \quad dx = \frac{\omega}{2\theta} d\omega$$

$$E(X^r) = \frac{2\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\, \frac{\alpha}{\theta^{\frac{\alpha}{2}-\frac{1}{2}}} \int_0^\infty \left(\frac{\omega^2}{4\theta}\right)^{r+\frac{\alpha}{2}+\frac{1}{2}} K_{\alpha-1}(\omega)\frac{\omega}{2\theta}d\omega$$

$$+ \frac{2\theta^{\alpha+1}}{(\theta+1)\Gamma(\alpha+1)}\, \frac{1}{\theta^{\frac{\alpha}{2}}} \int_0^\infty \left(\frac{\omega^2}{4\theta}\right)^{r+\frac{\alpha}{2}} K_{\alpha}(\omega)\frac{\omega}{2\theta}d\omega$$

$$= \frac{\alpha\theta^{1-r}}{2^{2r+\alpha-1}(\theta+1)\Gamma(\alpha+1)} \int_0^\infty \omega^{2r+\alpha}\, K_{\alpha-1}(\omega)\, d\omega$$

$$+ \frac{\theta^{-r}}{2^{2r+\alpha}(\theta+1)\Gamma(\alpha+1)} \int_0^\infty \omega^{2r+\alpha+1}\, K_{\alpha}(\omega)\, d\omega$$

$$= \frac{\alpha\theta^{1-r}}{2^{2r+\alpha-1}(\theta+1)\Gamma(\alpha+1)} \left[2^{2r+\alpha-1}\Gamma(r+\alpha)\Gamma(r+1)\right]$$

$$+ \frac{\theta^{-r}}{2^{2r+\alpha}(\theta+1)\Gamma(\alpha+1)} \left[2^{2r+\alpha}\Gamma(r+\alpha+1)\Gamma(r+1)\right]$$

$$= \frac{r!\,\alpha\theta}{\theta^r(\theta+1)\Gamma(\alpha+1)}\,\Gamma(r+\alpha) + \frac{r!}{\theta(\theta+1)\Gamma(\alpha+1)}\,\Gamma(r+\alpha+1)$$

$$= \frac{r!\,\Gamma(\alpha+r)}{\theta^r\,(\theta+1)\Gamma(\alpha+1)}\;[\alpha\theta+r+\alpha] \tag{3.58}$$

Using conditional expectation approach, we have

$$\begin{aligned}
E(X^r) &= r!\,E[\Lambda^r] \\
E(\Lambda^r) &= \frac{\theta^2\,(\theta)^{\alpha-1}}{(\theta+1)\,\Gamma(\alpha+1)}\left\{\int_0^\infty \alpha\lambda^{\alpha+r-1}e^{-\theta\lambda}d\lambda + \int_0^\infty \lambda^{\alpha+r+1-1}e^{-\theta\lambda}\,d\lambda\right\} \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\left\{\frac{\alpha\,\Gamma(\alpha+r)}{\theta^{\alpha+r}}+\frac{\Gamma(\alpha+r+1)}{\theta^{\alpha+r+1}}\right\} \\
&= \frac{\theta^{\alpha+1}}{(\theta+1)\,\Gamma(\alpha+1)}\,\frac{\Gamma(\alpha+r)}{\theta^{\alpha+r}}\left\{\alpha+\frac{\alpha+r}{\theta}\right\} \\
&= \frac{1}{\theta^r(\theta+1)}\,\frac{\Gamma(\alpha+r)}{\Gamma(\alpha+1)}\{\alpha\theta+\alpha+r\} \\
E(X^r) &= r!\,\frac{1}{\theta^r(\theta+1)}\,\frac{\Gamma(\alpha+r)}{\Gamma(\alpha+1)}\{\alpha\theta+\alpha+r\} \qquad (3.59)
\end{aligned}$$

and,

$$E(X) = \frac{\alpha\theta+\alpha+1}{\theta(\theta+1)} \tag{3.60}$$

### 3.4.7 Inverse Gaussian mixing distribution

The inverse Gaussian mixing distribution yields exponential-inverse Gaussian distribution and the construction is as follows:

$$g(\lambda) = (\frac{\Phi}{2\,\pi\,\lambda^3})^{\frac{1}{2}}\;\exp\left\{-\frac{\Phi(\lambda-\mu)^2}{2\,\lambda\,\mu^2})\right\} \quad \text{for} \quad \lambda>0 \quad \text{and} \quad \infty<\mu<\infty$$

Let

$$\Psi = \frac{\Phi}{\mu^2} \quad \therefore \quad \mu^2 = \frac{\Phi}{\Psi} \quad \text{and} \quad \mu = \sqrt{\frac{\Phi}{\Psi}}$$

$$\begin{aligned}
\therefore \quad g(\lambda) &= \left(\frac{\Phi}{2\,\pi\,\lambda^3}\right)^{\frac{1}{2}}\;\exp\left\{-\frac{\Psi(\lambda-\mu)^2}{2\,\lambda})\right\} \\
&= \left(\frac{\Phi}{2\,\pi\,\lambda^3}\right)^{\frac{1}{2}}\;\exp\left\{-\frac{\Psi(\lambda^2-2\mu\lambda+\mu^2)}{2\,\lambda})\right\} \\
&= \left(\frac{\Phi}{2\,\pi\,\lambda^3}\right)^{\frac{1}{2}}\;\exp\left\{-\frac{\Psi\lambda}{2}+\sqrt{\Psi\Phi}-\frac{\Phi}{2\lambda}\right\} \\
&= e^{\sqrt{\Psi\Phi}}\left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}}\;\lambda^{-\frac{3}{2}}\;\exp\left\{-\frac{\Psi\lambda}{2}-\frac{\Phi}{2\lambda}\right\} \quad \text{for} \quad x>0 \qquad (3.61)
\end{aligned}$$

The pdf of the mixture is,

$$f(x) = e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}\, \lambda^{-\frac{3}{2}}\, \exp\left\{-\frac{\Psi\lambda}{2} - \frac{\Phi}{2\lambda}\right\} d\lambda$$

$$= e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{3}{2}-1}\, e^{-\frac{\Psi\lambda}{2}-(x+\frac{\Phi}{2})\frac{1}{\lambda}} d\lambda$$

$$= e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{3}{2}-1}\, e^{-\frac{\Psi}{2}[\lambda+\frac{2x+\Phi}{\Psi}\,\frac{1}{\lambda}]} d\lambda$$

Let

$$\lambda = \sqrt{\frac{2x+\Phi}{\Psi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{2x+\Phi}{\Psi}}\, dz \tag{3.62}$$

$$f(x) = e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} \int_0^\infty z^{-\frac{3}{2}-1}\, e^{-\sqrt{\frac{\Psi(2x+\Phi)}{2}}(z+\frac{1}{z})} dz$$

$$= 2\, e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} K_{-\frac{3}{2}}(\sqrt{\Psi(2x+\Phi)}) dz$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} K_{-\frac{3}{2}}(\sqrt{\Psi(2x+\Phi)}) \tag{3.63}$$

(2.) The survival function is

$$S(x) = e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{x}{\lambda}}\, \lambda^{-\frac{3}{2}}\, \exp\left\{-\frac{\Psi\lambda}{2} - \frac{\Phi}{2\lambda}\right\} d\lambda$$

$$= e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}-1}\, e^{-\frac{\Psi\lambda}{2}-(x+\frac{\Phi}{2})\frac{1}{\lambda}} d\lambda$$

$$= e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}-1}\, e^{-\frac{\Psi}{2}[\lambda+\frac{2x+\Phi}{\Psi}\,\frac{1}{\lambda}]} d\lambda$$

Using the substitution (3.62)

$$\lambda = \sqrt{\frac{2x+\Phi}{\Psi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{2x+\Phi}{\Psi}}\, dz$$

$$S(x) = e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} \int_0^\infty z^{-\frac{1}{2}-1}\, e^{-\sqrt{\frac{\Psi(2x+\Phi)}{2}}(z+\frac{1}{z})} dz$$

$$= 2\, e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Psi(2x+\Phi)})$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Psi(2x+\Phi)}) \tag{3.64}$$

Therefore

$$h(x) = \frac{\left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} K_{-\frac{3}{2}}(\sqrt{\Psi(2x+\Phi)})}{\left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Psi(2x+\Phi)})}$$

$$= \frac{1}{\sqrt{\frac{2x+\Phi}{\Psi}}} \left[1 + \frac{1}{\sqrt{\Psi(2x+\Phi)}}\right]$$

$$= \sqrt{\frac{\Psi}{2x+\Phi}} + \frac{1}{2x+\Phi} \tag{3.65}$$

The limits of integration therefore change from $(0,\infty)$ **to** $(\sqrt{\Psi\,\Phi},\infty)$. Since Mellin transform of the Bessel function of the third kind applies for the limits $(0,\infty)$, the technique fails in this case.

However, using the conditional expectation technique

$$E(X^r) = r!\; E(\Lambda^r)$$

$$= r!\; e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{r-\frac{1}{2}-1}\, e^{-\frac{\Psi}{2}\left[\lambda+\frac{\Psi}{\Phi}\,\frac{1}{\lambda}\right]} d\lambda$$

Let

$$\lambda = \sqrt{\frac{\Psi}{\Phi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{\Psi}{\Phi}}\, dz$$

$$E(X^r) = r!\; e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r-\frac{1}{2}} \int_0^\infty z^{r-\frac{1}{2}-1}\, e^{-\frac{\sqrt{\Psi\Phi}}{2}(z+\frac{1}{z})} dz$$

$$= 2r!\; e^{\sqrt{\Psi\Phi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r-\frac{1}{2}} K_{r-\frac{1}{2}}(\sqrt{\Psi\Phi})$$

$$= r!\; e^{\sqrt{\Psi\Phi}} \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r-\frac{1}{2}} K_{r-\frac{1}{2}}(\sqrt{\Psi\Phi}) \tag{3.66}$$

$$\therefore \quad E(X) = e^{\sqrt{\Psi\Phi}} \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Psi\Phi}) \tag{3.67}$$

Using the equation (1.7), we get

$$E(X) = \sqrt{\frac{\Phi}{\Psi}} \tag{3.68}$$

### 3.4.8 Reciprocal inverse Gaussian mixing distribution

The reciprocal inverse Gaussian mixing distribution yields exponential-reciprocal inverse Gaussian distribution and the construction is as follows:

$$g(\lambda) = \left(\frac{\Phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\Phi}{\mu}} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\Phi}{2}\lambda + \frac{\Phi}{2\mu^2}\frac{1}{\lambda}\right\} \text{ for } \lambda > 0 \text{ and } -\infty < \mu < \infty$$

Let

$$\Phi = \frac{\Psi}{\mu^2} \quad \therefore \quad \mu^2 = \frac{\Psi}{\Phi} \quad and \quad \mu = \sqrt{\frac{\Psi}{\Phi}}$$

$$\therefore \quad g(\lambda) = e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\Phi\lambda}{2} - \frac{\Psi}{2\lambda}\right\} \quad \text{for} \quad \lambda > 0 \tag{3.69}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\Phi\lambda}{2} - \frac{\Psi}{2\lambda}\right\} d\lambda \\ &= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}-1} e^{-\frac{\Phi\lambda}{2} - (x+\frac{\Psi}{2})\frac{1}{\lambda}} d\lambda \\ &= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}-1} e^{-\frac{\Phi}{2}[\lambda + \frac{2x+\Psi}{\Phi}\frac{1}{\lambda}]} d\lambda \end{aligned}$$

Let

$$\lambda = \sqrt{\frac{2x+\Psi}{\Phi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{2x+\Psi}{\Phi}}\, dz \tag{3.70}$$

$$\begin{aligned} f(x) &= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Psi}{\Phi}}\right)^{-\frac{1}{2}} \int_0^\infty z^{-\frac{1}{2}-1} e^{-\sqrt{\frac{\Phi(2x+\Psi)}{2}}(z+\frac{1}{z})} dz \\ &= 2\, e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Psi}{\Phi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Phi(2x+\Psi)}) dz \\ &= \left(\frac{2\Psi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{2x+\Psi}{\Phi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Phi(2x+\Psi)}) \end{aligned} \tag{3.71}$$

(2.) The survival function is

$$\begin{aligned} S(x) &= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{x}{\lambda}} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\Phi\lambda}{2} - \frac{\Psi}{2\lambda}\right\} d\lambda \\ &= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{\frac{1}{2}-1} e^{-\frac{\Phi\lambda}{2} - (x+\frac{\Psi}{2})\frac{1}{\lambda}} d\lambda \end{aligned}$$

$$= e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{\frac{1}{2}-1}\, e^{-\frac{\Phi}{2}\left[\lambda + \frac{2x+\Psi}{\Phi}\,\frac{1}{\lambda}\right]} d\lambda$$

Using the substitution (3.70)

$$\lambda = \sqrt{\frac{2x+\Psi}{\Phi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{2x+\Psi}{\Phi}}\, dz$$

$$S(x) = e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Psi}{\Phi}}\right)^{\frac{1}{2}} \int_0^\infty z^{\frac{1}{2}-1}\, e^{-\sqrt{\frac{\Phi(2x+\Psi)}{2}}\left(z+\frac{1}{z}\right)} dz$$

$$= 2\, e^{\sqrt{\Phi\Psi}} \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Psi}{\Phi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(\sqrt{\Phi(2x+\Psi)})$$

$$= \left(\frac{2\Psi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{2x+\Psi}{\Phi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(\sqrt{\Phi(2x+\Psi)}) \tag{3.72}$$

Therefore

$$h(x) = \sqrt{\frac{\Phi}{2x+\Psi}} \tag{3.73}$$

Since Mellin transform of the Bessel function of the third kind applies for the limits $(0,\infty)$, the Mellin transform technique fails in this case.

However, using the conditional expectation technique

$$E(X^r) = r! \int_0^\infty \lambda^r \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}}\, \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\Phi\lambda}{2} - \frac{\Psi}{2\lambda}\right\} d\lambda$$

$$= r! \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \int_0^\infty \lambda^{r+\frac{1}{2}-1} \exp\left\{-\frac{\Phi}{2}\left(\lambda + \frac{\Psi}{\Phi}\,\frac{1}{\lambda}\right)\right\} d\lambda$$

Let

$$\lambda = \sqrt{\frac{\Psi}{\Phi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{\Psi}{\Phi}}\, dz$$

$$E(X^r) = r! \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r+\frac{1}{2}} \int_0^\infty z^{(r+\frac{1}{2})-1}\, e^{-\frac{\sqrt{\Phi\Psi}}{2}\left(z+\frac{1}{z}\right)} dz$$

$$= r!\, 2 \left(\frac{\Phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r+\frac{1}{2}} K_{r+\frac{1}{2}}(\sqrt{\Phi\Psi})$$

$$= r! \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r+\frac{1}{2}} K_{r+\frac{1}{2}}(\sqrt{\Phi\Psi})$$

$$\therefore \quad E(X) = \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{3}{2}} K_{\frac{3}{2}}(\sqrt{\Phi\Psi})$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Phi\Psi}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{3}{2}} \left(1+\frac{1}{\sqrt{\Phi\Psi}}\right)\left(\frac{\pi}{2\sqrt{\Phi\Psi}}\right)^{\frac{1}{2}} e^{-\sqrt{\Phi\Psi}} \tag{3.74}$$

$$\therefore \quad E(X) = \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{3}{2}} \left(1+\frac{1}{\sqrt{\Phi\Psi}}\right)\left(\frac{\pi}{2\Phi}\sqrt{\frac{\Phi}{\Psi}}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{\Psi}{\Phi}}\left(1+\frac{1}{\sqrt{\Phi\Psi}}\right)$$

$$= \sqrt{\frac{\Psi}{\Phi}+\frac{1}{\Phi}} \tag{3.75}$$

### 3.4.9 Generalized Inverse Gaussian mixing distribution

The generalized inverse Gaussian mixing distribution yields exponential-generalized inverse Gaussian distribution and the construction is as follows:

When

$$g(\lambda) = \frac{\left(\frac{\Psi}{\Phi}\right)^{\frac{v}{2}}}{2K_v\left(\sqrt{\Phi\Psi}\right)} \lambda^{v-1} \exp\left\{-\frac{1}{2}\left(\Psi\lambda+\Phi\,\frac{1}{\lambda}\right)\right\} \tag{3.76}$$

with parameters taking values in one of the following ranges:

1. $\Phi \geq 0 \quad \Psi \; 0 \quad if \quad v < 0$
2. $\Phi > 0 \quad \Psi > 0 \quad if \quad v = 0$
3. $\Phi \geq 0 \quad \Psi = 0 \quad if \quad v > 0$

The pdf of the mixture is,

$$f(x) = \int_0^\infty \frac{1}{\lambda} \quad e^{-\frac{x}{\lambda}} \quad \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}\lambda^{v-1} \quad e^{-\frac{1}{2}(\frac{\Phi}{\lambda}+\Psi\lambda)} \quad d\lambda$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}\int_0^\infty \lambda^{v-2} \quad e^{-\frac{\Psi}{2}(\lambda+\frac{2}{\Psi}(\frac{\Phi}{2}+x)\frac{1}{\lambda})} \quad d\lambda$$

Let,

$$\lambda = \quad \sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)} \quad z \quad \text{and} \quad d\lambda = \sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)} \quad dz \tag{3.77}$$

Therefore,

$$
\begin{aligned}
f(x) &= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty (\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}\ z)^{v-2}\ e^{-\frac{\Psi}{2}(\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}(z+\frac{1}{z})}\ \sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}\ dz \\
&= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}\ (\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)})^{v-1} \int_0^\infty z^{(v-1)-1}\ e^{-\frac{\Psi}{2}(\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}(z+\frac{1}{z})}\ dz \\
&= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} (\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)})^{v-1} \int_0^\infty z^{(v-1)-1}\ e^{-\frac{\Psi}{2}(\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}(z+\frac{1}{z})}\ dz \\
&= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} (\sqrt{\frac{2}{\Psi} \quad (\frac{\Phi}{2}+x)}\ )^{v-1} \quad 2\ K_{v-1}(\sqrt{\Psi(\Phi+2x)}) \\
&= (\frac{\Psi}{\Phi})^{\frac{v}{2}} (\frac{1}{\Psi})^{\frac{v-1}{2}} \quad (\Phi+2x)^{\frac{v-1}{2}} \quad \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\
&= (\frac{\Psi}{\Phi})^{\frac{v}{2}} (\frac{1}{\Psi})^{\frac{v-1}{2}} \quad (\Phi+2x)^{\frac{v-1}{2}} \quad \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\
&= \frac{(\frac{1}{\Phi})^{\frac{v}{2}}}{(\frac{1}{\Psi})^{\frac{1}{2}}} \quad (\Phi+2x)^{\frac{v-1}{2}} \quad \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\
&= \quad \frac{\Psi^{\frac{1}{2}}\Phi^{-\frac{1}{2}}}{\Phi^{\frac{v}{2}}\Phi^{-\frac{1}{2}}} \quad (\Phi+2x)^{\frac{v-1}{2}} \quad \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\
&= \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{v} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{v-1} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \qquad (3.78)
\end{aligned}
$$

The survival function is,

$$
\begin{aligned}
S(x) &= \int_0^\infty e^{-\frac{x}{\lambda}} \quad \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \lambda^{v-1} \quad e^{-\frac{1}{2}(\frac{\Phi}{\lambda}+\Psi\lambda)} \quad d\lambda \\
&= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty \lambda^{v-1} \quad e^{-\frac{\Psi}{2}(\lambda+\frac{2}{\Psi}(\frac{\Phi}{2}+x)\frac{1}{\lambda})} \quad d\lambda
\end{aligned}
$$

Using the substitution (3.77)

$$
\begin{aligned}
S(x) &= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty (\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}\ z)^{v-1}\ e^{-\frac{\Psi}{2}(\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}(z+\frac{1}{z})}\ \sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}\ dz \\
&= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}\ (\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)})^{v} \int_0^\infty z^{(v-1)-1}\ e^{-\frac{\Psi}{2}(\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}(z+\frac{1}{z})}\ dz \\
&= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} (\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)})^{v} \int_0^\infty z^{(v-1)}\ e^{-\frac{\Psi}{2}(\sqrt{\frac{2}{\Psi}(\frac{\Phi}{2}+x)}(z+\frac{1}{z})}\ dz
\end{aligned}
$$

$$= \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})}(\sqrt{\frac{2}{\Psi} \quad (\frac{\Phi}{2}+x)}\ )^v \quad 2\ K_v(\sqrt{\Psi(\Phi+2x)})$$

$$= \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^v \frac{K_v\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \tag{3.79}$$

Hence,

$$h(x) = (\sqrt{\frac{\Psi}{2x+\Phi}} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v\sqrt{\Psi(\Phi+2x)}} \tag{3.80}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = r!\ E(\Lambda^r)$$

$$= r!\ \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty \lambda^r\ \lambda^{v-1}\ e^{-\frac{1}{2}(\Psi\lambda+\Phi\frac{1}{\lambda})} d\lambda$$

$$= r!\ \frac{(\frac{\Psi}{\Phi})^{\frac{v}{2}}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty \lambda^{v+r-1}\ e^{-\frac{\Psi}{2}(\lambda+\frac{\Phi}{\Psi}\frac{1}{\lambda})}\ d\lambda$$

Let,

$$\lambda = \sqrt{\frac{\Phi}{\Psi}}\, z \quad \therefore \quad d\lambda = \sqrt{\frac{\Phi}{\Psi}}\, dz$$

Therefore,

$$E(X^r) = r!\ \frac{\left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{v+r}}{2K_v(\sqrt{\Phi\Psi})} \int_0^\infty z^{v+r-1}\ e^{-\frac{\Psi\Phi}{2}(z+\frac{1}{z})}\ d\lambda$$

$$= r!\ \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{v+r} \frac{K_{v+r}(\sqrt{\Phi\Psi})}{K_v(\sqrt{\Phi\Psi})}$$

$$= r!\ \left(\sqrt{\frac{\Psi}{\Phi}}\right)^r \frac{K_{v+r}(\sqrt{\Phi\Psi})}{K_v(\sqrt{\Phi\Psi})} \tag{3.81}$$

$$E(X) = \sqrt{\frac{\Psi}{\Phi}}\ \frac{K_{v+1}(\sqrt{\Phi\Psi})}{K_v(\sqrt{\Phi\Psi})} \tag{3.82}$$

### 3.4.10 Special cases of type II exponential-GIG distribution

The exponential-inverse Gaussian, the exponential-reciprocal inverse Gaussian, gamma I and Pareto II distributions are special cases of the exponential-generalized inverse Gaussian when $v = -\frac{1}{2}$, $v = \frac{1}{2}$, $\Phi \geq 0 \quad \Psi = 0$ if $v > 0$ and $\Psi = 0$, $v < 0$ respectively.

a) When $v = -\frac{1}{2}$

Then

$$f(x) = \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} \frac{K_{-\frac{3}{2}}\sqrt{\Psi(\Phi+2x)}}{K_{-\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

$$= \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} \frac{K_{\frac{3}{2}}\sqrt{\Psi(\Phi+2x)}}{K_{\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

$$= \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} \frac{K_{\frac{3}{2}}\sqrt{\Psi(\Phi+2x)}}{\left(\frac{\pi}{2\sqrt{\Psi\Phi}}\right)^{\frac{1}{2}} e^{-\sqrt{\Psi\Phi}}}$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{3}{2}} K_{\frac{3}{2}}\sqrt{\Psi(\Phi+2x)} \tag{3.83}$$

$$S(x) = \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{-\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} \frac{K_{-\frac{1}{2}}\sqrt{\Psi(\Phi+2x)}}{K_{-\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

$$= \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{\Psi(\Phi+2x)}) \tag{3.84}$$

$$h(x) = (\sqrt{\frac{\Psi}{2x+\Phi}} \frac{K_{-\frac{3}{2}}\sqrt{\Psi(\Phi+2x)}}{K_{-\frac{1}{2}}(\sqrt{\Psi(\Phi+2x)})}$$

$$= (\sqrt{\frac{\Psi}{2x+\Phi}} \left(1+\frac{1}{\sqrt{\Psi(\Phi+2x)}}\right)$$

$$= (\sqrt{\frac{\Psi}{2x+\Phi}} + \frac{1}{2x+\Phi} \tag{3.85}$$

$$E(X^r) = r! \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r} \frac{K_{r-\frac{1}{2}}(\sqrt{\Phi\Psi})}{K_{-\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

$$= r! \left(\frac{2\Phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\Psi\Phi}} \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{r-\frac{1}{2}} K_{r-\frac{1}{2}}(\sqrt{\Psi\Phi}) \tag{3.86}$$

$$E(X) = \sqrt{\frac{\Phi}{\Psi}} \tag{3.87}$$

These are results for the inverse Gaussian mixture of type II exponential distribution.

b) When $v = \frac{1}{2}$

$$f(x) = \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\frac{1}{2}} \frac{K_{-\frac{1}{2}}\sqrt{\Psi(\Phi+2x)}}{K_{\frac{1}{2}}(\sqrt{\Phi\Psi})}$$

**Remark 3.1.** *: The Bessel function for the type II exponential Reciprocal inverse Gaussian mixture is in the form*

$$\boldsymbol{K_{-\frac{1}{2}}\sqrt{\Phi(\Psi+2x)}}$$

*and yet the result above is in the form*

$$\boldsymbol{K_{-\frac{1}{2}}\sqrt{\Psi(\Phi+2x)}}$$

*There is therefore need to interchange the parameters* $\boldsymbol{\Psi}$ *and* $\boldsymbol{\Phi}$ *in the general formula.*

So we now have

$$\begin{aligned}
\boldsymbol{f(x)} &= \left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{\Psi}}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\boldsymbol{2x+\Psi}}{\boldsymbol{\Phi}}}\right)^{-\frac{1}{2}} \frac{\boldsymbol{K_{-\frac{1}{2}}\sqrt{\Phi(\Psi+2x)}}}{\boldsymbol{K_{\frac{1}{2}}(\sqrt{\Psi\Phi})}} \\
&= \left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{\Psi}}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\boldsymbol{2x+\Psi}}{\boldsymbol{\Phi}}}\right)^{-\frac{1}{2}} \frac{\boldsymbol{K_{\frac{1}{2}}\sqrt{\Phi(\Psi+2x)}}}{\left(\frac{\pi}{2\sqrt{\boldsymbol{\Phi\Psi}}}\right)^{\frac{1}{2}} \boldsymbol{e^{-\sqrt{\Phi\Psi}}}} \\
&= \left(\frac{\boldsymbol{2\Phi}}{\pi}\right)^{\frac{1}{2}} \boldsymbol{e^{\sqrt{\Phi\Psi}}} \left(\sqrt{\frac{\boldsymbol{2x+\Psi}}{\boldsymbol{\Phi}}}\right)^{-\frac{1}{2}} \boldsymbol{K_{\frac{1}{2}}\sqrt{\Phi(\Psi+2x)}} \qquad (3.88)
\end{aligned}$$

Similarly in the survival function $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are interchanged

$$\begin{aligned}
\therefore \quad \boldsymbol{S(x)} &= \left(\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{\Psi}}}\right)^{-\frac{1}{2}} \left(\sqrt{\frac{\boldsymbol{2x+\Psi}}{\boldsymbol{\Phi}}}\right)^{\frac{1}{2}} \frac{\boldsymbol{K_{\frac{1}{2}}\sqrt{\Phi(\Psi+2x)}}}{\boldsymbol{K_{\frac{1}{2}}(\sqrt{\Psi\Phi})}} \\
&= \left(\frac{\boldsymbol{2\Phi}}{\pi}\right)^{\frac{1}{2}} \boldsymbol{e^{\sqrt{\Phi\Psi}}} \left(\sqrt{\frac{\boldsymbol{2x+\Psi}}{\boldsymbol{\Phi}}}\right)^{\frac{1}{2}} \boldsymbol{K_{\frac{1}{2}}(\sqrt{\Phi(\Psi+2x)})} \qquad (3.89)
\end{aligned}$$

$$\boldsymbol{h(x)} = (\sqrt{\frac{\boldsymbol{\Phi}}{\boldsymbol{2x+\Psi}}} \qquad (3.90)$$

$$\begin{aligned}
\boldsymbol{E(X^r)} &= \boldsymbol{r!} \left(\sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{\Phi}}}\right)^{\boldsymbol{r}} \frac{\boldsymbol{K_{r+\frac{1}{2}}(\sqrt{\Psi\Phi})}}{\boldsymbol{K_{\frac{1}{2}}(\sqrt{\Psi\Phi})}} \\
&= \boldsymbol{r!} \left(\frac{\boldsymbol{2\Phi}}{\pi}\right)^{\frac{1}{2}} \boldsymbol{e^{\sqrt{\Phi\Psi}}} \left(\sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{\Phi}}}\right)^{\boldsymbol{r+\frac{1}{2}}} \boldsymbol{K_{r+\frac{1}{2}}(\sqrt{\Phi\Psi})} \qquad (3.91)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{E(X)} &= \sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{\Phi}}} \frac{\boldsymbol{K_{\frac{3}{2}}(\sqrt{\Psi\Phi})}}{\boldsymbol{K_{\frac{1}{2}}(\sqrt{\Psi\Phi})}} \\
&= \sqrt{\frac{\boldsymbol{\Psi}}{\boldsymbol{\Phi}}} \left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{\Psi\Phi}}}\right)
\end{aligned}$$

$$= \sqrt{\frac{\Psi}{\Phi} + \frac{1}{\Phi}} \tag{3.92}$$

c) One of the ranges for the parameters is:

$$\begin{array}{lllll} & \Phi \geq 0 & \Psi = 0 & if & v > 0 \\ \therefore & \Phi \geq 0 & \Psi = 0 & if & v > 0 \end{array}$$

and

$$\lim_{\Phi \to 0} \frac{\Psi}{\Phi} = 1 \quad \text{for} \quad v > 0$$

Therefore

$$\begin{aligned} \lim_{\Phi \to 0} f(x) &= \lim_{\Phi \to 0} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{v} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{v-1} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\ &= \left(\sqrt{\frac{2x}{\Psi}}\right)^{v-1} \frac{K_{v-1}\sqrt{\Psi(2x)}}{\lim_{\Phi \to 0} K_v(\sqrt{\Phi\Psi})} \end{aligned}$$

Since

$$\int_0^\infty g(\lambda)\, d\lambda = 1$$

i.e.

$$\int_0^\infty \frac{\left(\frac{\Psi}{\Phi}\right)^{\frac{v}{2}}}{2K_v\left(\sqrt{\Phi\Psi}\right)} \lambda^{v-1} \exp\left\{-\frac{1}{2}\left(\Psi\lambda + \Phi\,\frac{1}{\lambda}\right)\right\} = 1$$

$$\begin{aligned} K_v(\sqrt{\Phi\Psi}) &= \frac{1}{2}\left(\frac{\Psi}{\Phi}\right)^{\frac{v}{2}} \int_0^\infty \lambda^{v-1} \exp\left\{-\frac{1}{2}\left(\Psi\lambda + \Phi\,\frac{1}{\lambda}\right)\right\} d\lambda \\ \lim_{\Phi \to 0} K_v(\sqrt{\Phi\Psi}) &= \frac{1}{2}\int_0^\infty \lambda^{v-1}\, e^{-\frac{\Psi}{2}} d\lambda \\ &= \frac{1}{2}\,\frac{\Gamma v}{\left(\frac{\Psi}{2}\right)^{v}} \\ \lim_{\Phi \to 0} f(x) &= \left(\sqrt{\frac{2x}{\Psi}}\right)^{v-1} \left(\frac{\Psi}{2}\right)^{v} \frac{K_{v-1}\sqrt{\Psi(2x)}}{\frac{1}{2}\,\Gamma v} \\ &= \frac{2}{\Gamma v}\left(\frac{\Psi}{2}\right)^{v} \left(\sqrt{\frac{2x}{\Psi}}\right)^{v-1} K_{v-1}\sqrt{2\Psi\, x)} \end{aligned}$$

Put

$$\boldsymbol{\Psi = 2\beta \quad and \quad v = \alpha}$$

$$\begin{aligned}
\therefore \quad f(x) &= \frac{2}{\Gamma(\alpha)}\, \beta^{\alpha} \left(\sqrt{\frac{x}{\beta}}\right)^{\alpha-1} K_{\alpha-1}\sqrt{2\beta\, x)} \\
&= \frac{2}{\Gamma(\alpha)} \left(\beta\sqrt{\frac{x}{\beta}}\right)^{\alpha} \left(\sqrt{\frac{x}{\beta}}\right)^{-1} K_{\alpha-1}\sqrt{2\beta\, x)} \\
&= \frac{2}{\Gamma(\alpha)} \left(\sqrt{\beta x}\right)^{\alpha} \sqrt{\frac{\beta}{x}}\, K_{\alpha-1}\sqrt{2\beta\, x)} \\
&= \frac{2\beta}{\Gamma(\alpha)} \left(\sqrt{\beta x}\right)^{\alpha-1} K_{\alpha-1}\sqrt{2\beta\, x)} \qquad (3.93)
\end{aligned}$$

Survival function is

$$\begin{aligned}
\lim_{\Phi\to 0} S(x) &= \lim_{\Phi\to 0}\left(\sqrt{\frac{\Psi}{\Phi}}\right)^{v}\left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{v}\frac{K_v\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\
&= \left(\sqrt{\frac{2x}{\Psi}}\right)^{v}\frac{K_v\sqrt{2\Psi\, x)}}{\lim_{\Phi\to 0} K_v(\sqrt{\Phi\Psi})} \\
&= \left(\sqrt{\frac{2x}{\Psi}}\right)^{v}\frac{K_v\sqrt{2\Psi\, x)}}{\frac{1}{2}\,\frac{\Gamma v}{\left(\frac{\Psi}{2}\right)^{v}}} \\
&= 2\left(\frac{\Psi}{2}\sqrt{\frac{2x}{\Psi}}\right)^{v}\frac{K_v\sqrt{2\Psi\, x)}}{\Gamma v} \\
&= 2\left(\sqrt{\frac{\Psi x}{2}}\right)^{v}\frac{K_v\sqrt{2\Psi\, x)}}{\Gamma v}
\end{aligned}$$

Put

$$\boldsymbol{\Psi = 2\beta \quad and \quad v = \alpha}$$

$$\begin{aligned}
\lim_{\Phi\to 0} S(x) &= 2\left(\sqrt{\beta x}\right)^{\alpha}\frac{K_{\alpha}(2\sqrt{\beta x})}{\Gamma(\alpha)} \\
&= \frac{2\left(\sqrt{\beta x}\right)^{\alpha}}{\Gamma(\alpha)} K_{\alpha}(2\sqrt{\beta x}) \qquad (3.94)
\end{aligned}$$

$$\begin{aligned}
\lim_{\Phi\to 0} h(x) &= \lim_{\Phi\to 0}\sqrt{\frac{\Psi}{2x+\Phi}}\,\frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v\sqrt{\Psi(\Phi+2x)}} \\
&= \sqrt{\frac{\Psi}{2x}}\,\frac{K_{v-1}\sqrt{2\Psi x}}{K_v\sqrt{2\Psi x}} \\
&= \sqrt{\frac{\beta}{x}}\,\frac{K_{\alpha-1}\sqrt{\beta x}}{K_v\sqrt{\beta x}} \qquad (3.95)
\end{aligned}$$

$$\lim_{\Phi \to 0} E(X^r) = \lim_{\Phi \to 0} r! \left(\sqrt{\frac{\Psi}{\Phi}}\right)^r \frac{K_{v+r}(\sqrt{\Phi\Psi})}{K_v(\sqrt{\Phi\Psi})}$$

$$= r! \frac{\Gamma(v+r)}{\frac{1}{2}\left(\frac{\Psi}{2}\right)^{v+r}} \frac{\frac{1}{2}\left(\frac{\Psi}{2}\right)^v}{\Gamma v}$$

$$= r! \frac{\Gamma(v+r)}{\Gamma v} \frac{1}{\left(\frac{\Psi}{2}\right)^r}$$

$$= \frac{r!}{\beta^r} \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{3.96}$$

These are results of gamma I mixture of type II exponential distribution

$$E(X) = \frac{1}{\beta} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \tag{3.97}$$

d) When $\Psi = 0$, $v < 0$

Let

$$\lim_{\Psi \to 0} \frac{\Phi}{\Psi} = 1$$

and

$$v = -\alpha \quad \text{where} \quad \alpha > 0$$

Therefore

$$\lim_{\Psi \to 0, v=-\alpha} f(x) = \lim_{\Psi \to 0} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{v-1} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})}$$

$$= \lim_{\Psi \to 0} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^v \left(\sqrt{\frac{2x+\Phi}{\Phi}\frac{\Phi}{\Psi}}\right)^{v-1} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})}$$

$$= \left(\sqrt{\frac{2x+\Phi}{\Phi}}\right)^{-\alpha-1} \lim_{\Psi \to 0, v=-\alpha} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})}$$

$$= \left(\sqrt{\frac{2x+\Phi}{\Phi}}\right)^{-\alpha-1} \lim_{\Psi \to 0} \frac{K_{\alpha-1}\sqrt{\Psi(\Phi+2x)}}{K_\alpha(\sqrt{\Phi\Psi})}$$

Consider,

$$K_{-\alpha-1}(\sqrt{\Psi(\Phi+2x)}) = \frac{1}{2}\int_0^\infty t^{(-\alpha-1)-1} e^{-\frac{\sqrt{\Psi(\Phi+2x)}}{2}(t+\frac{1}{t})} dt$$

$$= \frac{1}{2}\int_0^\infty \frac{1}{t^{(\alpha+1)+1}} e^{-\frac{\Psi}{2}(\sqrt{\frac{2x+\Phi}{\Psi}}t+\sqrt{\frac{2x+\Phi}{\Psi}}\frac{1}{t})} dt$$

Let

$$t = \frac{1}{\sqrt{\frac{2x+\Phi}{\Psi}}}\, z \quad \therefore \quad dt = \frac{dz}{\sqrt{\frac{2x+\Phi}{\Psi}}}$$

$$K_{-\alpha-1}(\sqrt{\Psi(\Phi+2x)}) = \frac{1}{2}\int_0^\infty \left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \frac{e^{-\frac{\Psi}{2}\left[z+\frac{2x+\Phi}{\Psi}\frac{1}{z}\right]}}{z^{(\alpha+1)+1}} dz$$

$$= \frac{1}{2}\int_0^\infty \left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \frac{e^{-\frac{1}{2}\left[\Psi z+\frac{2x+\Phi}{z}\right]}}{z^{(\alpha+1)+1}} dz$$

$$\lim_{\Psi\to 0} K_{-\alpha-1}(\sqrt{\Psi(\Phi+2x)}) = \frac{1}{2}\int_0^\infty \left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \frac{e^{-\frac{2x+\Phi}{2z}}}{z^{(\alpha+1)+1}} dz$$

$$= \frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \int_0^\infty \frac{e^{-\frac{2x+\Phi}{2}\frac{1}{z}}}{z^{(\alpha+1)+1}} dz$$

Let

$$y = \frac{1}{z} \quad \therefore \quad z = \frac{1}{y} \quad and \quad dz = -\frac{dy}{y^2}$$

Therefore,

$$\lim_{\Psi\to 0} K_{-\alpha-1}(\sqrt{\Psi(\Phi+2x)}) = \frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \int_0^\infty y^{\alpha+2}\, e^{-\frac{2x+\Phi}{2}\,y}\, \frac{dy}{y^2}$$

$$= \frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \int_0^\infty y^{(\alpha+1)-1}\, e^{-\frac{2x+\Phi}{2}\,y}\, dy$$

$$= \frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \frac{\Gamma(\alpha+1)}{\left(\frac{2x+\Phi}{2}\right)^{\alpha+1}}$$

Let

$$\Phi = 2\beta$$

Therefore,

$$\lim_{\Psi\to 0} K_{-\alpha-1}(\sqrt{\Psi(\Phi+2x)}) = \frac{1}{2}\left[\sqrt{\frac{x+\beta}{\beta}}\right]^{\alpha+1} \frac{\Gamma(\alpha+1)}{(x+\beta)^{\alpha+1}}$$

Next, consider

$$K_{-\alpha}(\sqrt{\Phi\Psi}) = \frac{1}{2}\int_0^\infty t^{-\alpha-1}\, e^{-\frac{\sqrt{\Phi\Psi}}{2}(t+\frac{1}{t})} dt$$

$$= \frac{1}{2}\int_0^\infty t^{-\alpha-1}\, e^{-\frac{\Psi}{2}\left(\sqrt{\frac{\Phi}{\Psi}}\, t + \frac{\Phi}{\Psi}\,\frac{1}{t}\right)} dt$$

Let,

$$t = \frac{z}{\sqrt{\frac{\Phi}{\Psi}}} \quad \therefore \quad dt = \frac{dz}{\sqrt{\frac{\Phi}{\Psi}}}$$

Therefore

$$K_{-\alpha}(\sqrt{\Psi\Phi}) = \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\Phi}{\Psi}}\right)^{\alpha+1} z^{-\alpha-1}\, e^{-\frac{\Psi}{2}\left(z+\frac{\Phi}{\Psi}\,\frac{1}{z}\right)} \frac{dz}{\sqrt{\frac{\Phi}{\Psi}}}$$

$$= \frac{1}{2}\left(\sqrt{\frac{\Phi}{\Psi}}\right)^{\alpha} \int_0^\infty z^{-\alpha-1}\, e^{-\frac{1}{2}\left(\Psi z+\frac{\Phi}{z}\right)} dz$$

Therefore,

$$\lim_{\Psi \to 0} K_{-\alpha}(\sqrt{\Psi\Phi}) = \frac{1}{2}\int_0^\infty z^{-\alpha-1}\, e^{-\frac{\Phi}{2}\,\frac{1}{z}} dz$$

Let

$$y = \frac{1}{z} \quad \therefore \quad z = \frac{1}{y} \quad and \quad dz = -\frac{dy}{y^2}$$

$$\lim_{\Psi \to 0} K_{-\alpha}(\sqrt{\Psi\Phi}) = \frac{1}{2}\int_0^\infty y^{\alpha-1}\, e^{-\frac{\Phi}{2}\, y} dy$$
$$= \frac{1}{2}\,\frac{\Gamma(\alpha)}{\left(\frac{\Phi}{2}\right)^{\alpha}}$$

Putting $\Phi = 2\beta$, we have

$$\lim_{\Psi \to 0} K_{-\alpha}(\sqrt{\Psi\Phi}) = \frac{1}{2}\,\frac{\Gamma(\alpha)}{\beta^{\alpha}}$$

Therefore

$$\lim_{\Psi \to 0, v=-\alpha} f(x) = \left(\sqrt{\frac{2x+\Phi}{\Phi}}\right)^{-\alpha-1} \frac{\frac{1}{2}\left[\sqrt{\frac{x+\beta}{\beta}}\right]^{\alpha+1} \frac{\Gamma(\alpha+1)}{(x+\beta)^{\alpha+1}}}{\frac{1}{2}\,\frac{\Gamma(\alpha)}{\beta^{\alpha}}}$$
$$= \left(\sqrt{\frac{2+\beta}{\beta}}\right)^{-\alpha-1} \left(\sqrt{\frac{x+\beta}{\beta}}\right)^{\alpha+1} \frac{\alpha\beta^{\alpha}}{(x+\beta)^{\alpha+1}}$$
$$= \frac{\alpha\beta^{\alpha}}{(x+\beta)^{\alpha+1}} \tag{3.98}$$

which is a Pareto II (Lomax) distribution

$$\begin{aligned}
\lim_{\Psi \to 0, v=-\alpha} S(x) &= \lim_{\Psi \to 0, v=-\alpha} \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{v} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{v} \frac{K_v\sqrt{\Psi(\Phi+2x)}}{K_v(\sqrt{\Phi\Psi})} \\
&= \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\alpha} \lim_{\Psi \to 0} \frac{K_{-\alpha}\sqrt{\Psi(\Phi+2x)}}{K_{-\alpha}(\sqrt{\Phi\Psi})} \\
&= \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\alpha} \frac{\frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha} \frac{\Gamma(\alpha)}{(\frac{2x+\Phi}{2})^{\alpha}}}{\frac{1}{2}\frac{\Gamma(\alpha)}{\beta^{\alpha}}} \\
&= \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{-\alpha} \left(\sqrt{\frac{2x+\Phi}{\Psi}}\right)^{\alpha} \frac{\left(\frac{\Phi}{2}\right)^{\alpha}}{\left(\frac{2x+\Phi}{2}\right)^{\alpha}} \\
&= \left(\frac{\Phi}{2x+\Phi}\right)^{\alpha}
\end{aligned}$$

Let

$$\begin{aligned}
\Phi &= 2\beta \\
\therefore \quad S(x) &= \left(\frac{\beta}{x+\beta}\right)^{\alpha}
\end{aligned}$$

which is a survival function of Pareto II distribution

$$\begin{aligned}
\lim_{\Psi \to 0, v=-\alpha} h(x) &= \lim_{\Psi \to 0, v=-\alpha} (\sqrt{\frac{\Psi}{2x+\Phi}} \frac{K_{v-1}\sqrt{\Psi(\Phi+2x)}}{K_v\sqrt{\Psi(\Phi+2x)}} \\
&= \frac{1}{\sqrt{\frac{2x+\Phi}{\Psi}}} \lim_{\Psi \to 0} \frac{K_{-\alpha-1}\sqrt{\Psi(\Phi+2x)}}{K_{-\alpha}\sqrt{\Psi(\Phi+2x)}} \\
&= \frac{1}{\sqrt{\frac{2x+\Phi}{\Psi}}} \frac{\frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha+1} \frac{\Gamma(\alpha+1)}{(\frac{2x+\Phi}{2})^{\alpha+1}}}{\frac{1}{2}\left[\sqrt{\frac{2x+\Phi}{\Psi}}\right]^{\alpha} \frac{\Gamma(\alpha)}{(\frac{2x+\Phi}{2})^{\alpha+1}}} \\
&= \frac{1}{\sqrt{\frac{2x+\Phi}{\Psi}}} \sqrt{\frac{2x+\Phi}{\Psi}} \frac{\alpha}{\frac{2x+\Phi}{2}} \\
&= \frac{\alpha}{\frac{2x+\Phi}{2}} \\
&= \frac{\alpha}{x+\beta} \quad \text{where} \quad \Phi = 2\beta
\end{aligned} \tag{3.99}$$

$$\begin{aligned}
\lim_{\Psi \to 0, v=-\alpha} E(X^r) &= \lim_{\Psi \to 0, v=-\alpha} r! \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r} \frac{K_{v+r}(\sqrt{\Phi\Psi})}{K_v(\sqrt{\Phi\Psi})} \\
&= \lim_{\Psi \to 0} r! \left(\sqrt{\frac{\Psi}{\Phi}}\right)^{r} \frac{K_{r-\alpha}(\sqrt{\Phi\Psi})}{K_{-\alpha}(\sqrt{\Phi\Psi})}
\end{aligned}$$

$$
\begin{aligned}
&= r! \lim_{\Psi \to 0} \frac{K_{r-\alpha}(\sqrt{\Phi\Psi})}{K_{-\alpha}(\sqrt{\Phi\Psi})} \\
&= r! \lim_{\Psi \to 0} \frac{K_{-(\alpha-r)}(\sqrt{\Phi\Psi})}{K_{-\alpha}(\sqrt{\Phi\Psi})} \\
&= r! \frac{\frac{1}{2}\Gamma(\alpha - r)}{\frac{1}{2}\frac{\Gamma(\alpha)}{\left(\frac{\Phi}{2}\right)^{\alpha}} \left(\frac{\Phi}{2}\right)^{\alpha - r}} \\
&= r! \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)} \left(\frac{\Phi}{2}\right)^{r} \\
&= r! \, \beta^r \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)}
\end{aligned}
\tag{3.100}
$$

## 3.5 Mixtures in terms of confluent hyper-geometric function

The following mixing distributions yield mixtures that are in terms of confluent hyper-geometric function: beta I, beta II, scaled beta, full beta, uniform, Pareto I, Pareto II and generalized Pareto distributions:

### 3.5.1 Beta I mixing distribution

Using beta I as the mixing distribution, the exponential-beta I mixture is constructed below and the moments have been obtained.

$$g(\lambda) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} \; 0 < \lambda < 1, \; \alpha,\beta > 0 \tag{3.101}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^1 \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} d\lambda \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 \lambda^{\alpha-2}(1-\lambda)^{\beta-1} e^{-\frac{1}{\lambda}x} d\lambda \end{aligned}$$

Let,

$$\lambda = \frac{u}{1+u} \quad \therefore \quad u = \frac{\lambda}{1-\lambda} \quad \text{and} \quad d\lambda = \frac{du}{(1+u)^2}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{B(\alpha,\beta)} \int_0^\infty (\frac{u}{1+u})^{\alpha-2} (\frac{1}{1+u})^{\beta-1} e^{-\frac{1+u}{u}x} \frac{du}{(1+u)^2} \\ &= \frac{1}{B(\alpha,\beta)} \int_0^\infty (\frac{u^{\alpha-2}}{(1+u)^{(\alpha-2)+(\beta-1)+2}} e^{-\frac{1+u}{u}x} du \\ &= \frac{e^{-x}}{B(\alpha,\beta)} \int_0^\infty u^{\alpha-2}(1+u)^{-\alpha-\beta+1} e^{-\frac{x}{u}} du \end{aligned}$$

Put,

$$t = \frac{1}{u} \;\therefore\; u = \frac{1}{t} \quad \text{and} \quad du = -\frac{dt}{t^2}$$

so that

$$f(x) = \frac{e^{-x}}{B(\alpha,\beta)} \int_0^\infty t^{-\alpha+2}(1+\frac{1}{t})^{-\alpha-\beta+1} e^{-xt} \frac{dt}{t^2}$$

$$= \frac{e^{-x}}{B(\alpha,\beta)} \int_0^{\infty} t^{\beta-1}(1+t)^{-\alpha-\beta+1} e^{-xt}\, dt$$

But

$$\Psi(a;c;x) = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1}(1+t)^{(c-a)-1}\, e^{-xt}\, dt \quad for \quad a,b,>0$$

Hence,

$$\begin{aligned} f(x) &= \frac{e^{-x}\Gamma(\beta)}{B(\alpha,\beta)} \int_{t=0}^{\infty} \frac{t^{\beta-1}}{\Gamma(\beta)}(1+t)^{2-\alpha-\beta-1} e^{-xt}\, dt \\ &= \frac{e^{-x}\Gamma(\beta)}{B(\alpha,\beta)}\, \Psi(\beta;2-\alpha;x) \\ &= e^{-x}\, \frac{\Gamma\alpha+\beta}{\Gamma(\alpha)}\, \Psi(\beta;2-\alpha;x) \end{aligned} \tag{3.102}$$

(2.) The survival function is

$$\begin{aligned} S(x) &= \int_0^1 e^{-\frac{1}{\lambda}x}\, \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}\, d\lambda \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}\, e^{-\frac{1}{\lambda}x}\, d\lambda \end{aligned}$$

Let,

$$\begin{aligned} \lambda &= \frac{u}{1+u} \quad \therefore \quad u = \frac{\lambda}{1-\lambda} \quad and \quad d\lambda = \frac{du}{(1+u)^2} \\ \therefore \quad S(x) &= \frac{1}{B(\alpha,\beta)} \int_0^{\infty} (\frac{u}{1+u})^{\alpha-1}(\frac{1}{1+u})^{\beta-1}\, e^{-\frac{1+u}{u}x}\, \frac{du}{(1+u)^2} \\ &= \frac{1}{B(\alpha,\beta)} \int_0^{\infty} (\frac{u^{\alpha-1}}{(1+u)^{(\alpha-1)+(\beta-1)+2}}\, e^{-\frac{1+u}{u}x}\, du \\ &= \frac{e^{-x}}{B(\alpha,\beta)} \int_0^{\infty} u^{\alpha-1}(1+u)^{-\alpha-\beta} e^{-\frac{x}{u}}\, du \end{aligned}$$

Put,

$$\begin{aligned} t &= \frac{1}{u} \;\therefore\; u = \frac{1}{t} \quad \text{and} \quad du = -\frac{dt}{t^2} \\ \therefore \quad S(x) &= \frac{e^{-x}}{B(\alpha,\beta)} \int_0^{\infty} t^{-\alpha+2}(1+\frac{1}{t})^{-\alpha-\beta} e^{-xt}\, \frac{dt}{t^2} \end{aligned}$$

$$= \frac{e^{-x}}{B(\alpha,\beta)} \int_0^\infty t^{\beta-1}(1+t)^{-\alpha-\beta} e^{-xt}\, dt$$

$$= \frac{e^{-x}\Gamma(\beta)}{B(\alpha,\beta)} \int_0^\infty \frac{t^{\beta-1}}{\Gamma(\beta)}(1+t)^{1-\alpha-\beta-1}\, e^{-xt}\, dt$$

$$= e^{-x}\, \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\, \Psi(\beta;1-\alpha;x) \tag{3.103}$$

Hence,

$$h(x) = \frac{\Psi(\beta;2-\alpha;x)}{\Psi(\beta;1-\alpha;x)} \tag{3.104}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r\, \frac{e^{-x}\Gamma q}{B(p,q)}\, \Psi(q;2-p;x)\, dx$$

$$= \frac{\Gamma q}{B(p,q)} \int_0^\infty x^{(r+1)-1}\, e^{-x}\, \Psi(q;2-p;x)\, dx$$

$$= \frac{\Gamma q}{B(p,q)}\, M(e^{-x}\, \Psi(q;2-p;x);r+1)$$

Applying equation (1.20)

$$M(e^{-x}\Psi(a;c;x),s) = \frac{\Gamma s \Gamma(s-c+1)}{\Gamma(a+s-c+1)}$$

$$E(X^r) = \frac{\Gamma(\beta)}{B(\alpha,\ \beta)}\, \frac{\Gamma(r+1)\Gamma((r+1)-(2-\alpha)+1)}{\Gamma(\beta+(r+1)-(2-\alpha)+1)}$$

$$= \frac{\Gamma(\beta)}{B(\alpha,\ \beta)}\, \frac{\Gamma(\alpha+r)\Gamma(r+1)}{\Gamma(\alpha+\beta+r)}$$

$$= r!\, \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+r)}\, \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{3.105}$$

Using conditional expectation approach,

$$E(X^r) = r!\, E(\Lambda^r)$$

$$= \int_0^1 \frac{\lambda^{\alpha+r-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} d\lambda$$

$$= \frac{B(\alpha+r,\beta)}{B(\alpha,\beta)}$$

$$E(X^r) = r!\, \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+r)}\, \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{3.106}$$

and,

$$\begin{aligned} E(X) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \\ &= \frac{\alpha}{(\alpha+\beta)} \end{aligned} \tag{3.107}$$

### 3.5.2 Beta II mixing distribution

Using beta II as the mixing distribution, the exponential-beta II mixture is constructed below and the moments have been obtained.

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{B(\alpha,\ \beta)\ (1+\lambda)^{\alpha+\beta}} \quad \lambda > 0 \quad \alpha,\ \beta > 0 \tag{3.108}$$

The pdf of the mixture is,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{\lambda}\, e^{\frac{1}{\lambda}x}\ \frac{\lambda^{\alpha-1}}{B(\alpha,\ \beta)\ (1+\lambda)^{\alpha+\beta}}\, d\lambda \\ &= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty \frac{1}{\lambda}\, e^{\frac{1}{\lambda}x}\ \frac{\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}}\, d\lambda \end{aligned}$$

Let,

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{-\alpha+2}(1+\frac{1}{t})^{-\alpha-\beta}\ e^{-xt}\ \frac{dt}{t^2} \\ &= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{-\alpha}(1+t)^{-\alpha-\beta}\ t^{\alpha+\beta}\ e^{-xt}\ dt \\ &= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{\beta}(1+t)^{-\alpha-\beta}\ e^{-xt}\ dt \\ &= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{(\beta+1)-1}(1+t)^{2-\alpha-\beta-2}\ e^{-xt}\ dt \\ &= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)} \int_0^\infty \frac{t^{(\beta+1)-1}}{\Gamma(\beta+1)}(1+t)^{2-\alpha-\beta-2}\ e^{-xt}\ dt \\ &= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\ \Psi(\beta+1; 2-\alpha; x) \\ &= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\ x^{1-(2-\alpha)}\ \Psi(\beta+1-(2-\alpha)+1; 2-(2-\alpha); x) \end{aligned}$$

$$= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\, x^{\alpha-1}\, \Psi(\alpha+\beta;\alpha;x) \quad \text{for} \quad x>0;\ \alpha,\ \beta>0 \tag{3.109}$$

as obtained by Bhattacharya and Holla (1965)

(2.) The survival function is given by

$$S(x) = \int_0^\infty e^{\frac{1}{\lambda}x}\, \frac{\lambda^{\alpha-1}}{B(\alpha,\ \beta)\,(1+\lambda)^{\alpha+\beta}}\, d\lambda$$

$$= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty e^{\frac{1}{\lambda}x}\, \frac{\lambda^{\alpha-1}}{(1+\lambda)^{\alpha+\beta}}\, d\lambda$$

Let,

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2}$$

Therefore,

$$S(x) = \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{-\alpha+1}(1+\frac{1}{t})^{-\alpha-\beta}\, e^{-xt}\, \frac{dt}{t^2}$$

$$= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{\beta-1}(1+t)^{-\alpha-\beta}\, e^{-xt}\, dt$$

$$= \frac{1}{B(\alpha,\ \beta)} \int_0^\infty t^{\beta-1}(1+t)^{1-\alpha-\beta-1}\, e^{-xt}\, dt$$

$$= \frac{\Gamma(\beta)}{B(\alpha,\ \beta)} \int_0^\infty \frac{t^{\beta-1}}{\Gamma(\beta)}(1+t)^{1-\alpha-\beta-1}\, e^{-xt}\, dt$$

$$= \frac{\Gamma(\beta)}{B(\alpha,\ \beta)}\, \Psi(\beta;1-\alpha;x) \tag{3.110}$$

Hence,

$$h(x) = \beta\, \frac{\Psi(\beta+1;2-\alpha;x)}{\Psi(\beta;1-\alpha;x)} \tag{3.111}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_0^\infty x^r\, \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\, \Psi(\beta+1;2-\alpha;x)dx$$

Using the equation (1.12)

$$E(X^r) = \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)} \int_0^\infty x^{p+r-1)}\Psi(\alpha+\beta;\alpha;x)dx$$

$$= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\ M[\Psi(\alpha+\beta,p,x;p+r)]$$

$$= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\ \frac{(\Gamma(\alpha+r)\Gamma(\alpha+\beta-(\alpha+r)\Gamma(\alpha+r-\alpha+1)}{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-\alpha+1)}$$

$$= \frac{\Gamma(\beta+1)}{B(\alpha,\ \beta)}\ \frac{(\Gamma(\alpha+r)\Gamma(\beta-r)\Gamma(r+1)}{\Gamma(\alpha+\beta)\Gamma(\beta+1)}$$

$$= \frac{r!\ (\Gamma(\alpha+r)\Gamma(\beta-r)}{\Gamma(\alpha)\Gamma(\beta)} \tag{3.112}$$

Using conditional expectation approach, we have

$$E(\Lambda^r) = \int_0^\infty \frac{\lambda^{\alpha+r-1}}{B(\alpha,\ \beta)\ (1+\lambda)^{\alpha+\beta}}$$

$$= \frac{B(\alpha+r,\beta-r)}{B(\alpha,\beta)}$$

$$\therefore \quad E(X^r) = \frac{r!\ (\Gamma(\alpha+r)\Gamma(\beta-r)}{\Gamma(\alpha)\Gamma(\beta)} \quad \beta > r \tag{3.113}$$

and

$$E(X) = \frac{\alpha}{\beta-1} \tag{3.114}$$

### 3.5.3 Scaled beta mixing distribution

Using scaled beta as the mixing distribution, the exponential-scaled beta mixture is constructed below and the moments have been obtained.

$$g(\lambda) = \frac{1}{\alpha\ B(p,q)} \left(\frac{\lambda}{\alpha}\right)^{p-1} \left(1-\frac{\lambda}{\alpha}\right)^{q-1}, \quad 0 \le \lambda \le \alpha; \quad p,q,\alpha > 0 \tag{3.115}$$

The pdf of the mixture is,

$$f(x) = \frac{1}{\alpha\ B(p,q)} \int_0^\alpha \frac{1}{\lambda}\ e^{-\frac{x}{\lambda}} \left(\frac{\lambda}{\alpha}\right)^{p-1} \left(1-\frac{\lambda}{\alpha}\right)^{q-1} d\lambda$$

$$= \frac{1}{\alpha\ B(p,q)} \int_0^\alpha \frac{1}{\lambda}\ e^{-\frac{x}{\lambda}}\ \frac{\lambda^{p-1}\ (\alpha-\lambda)^{q-1}}{\alpha^{p+q-2}} d\lambda$$

$$= \frac{1}{\alpha^{p+q-1}\ B(p,q)} \int_0^\alpha \lambda^{p-2}\ (\alpha-\lambda)^{q-1}\ e^{-\frac{x}{\lambda}}\ d\lambda$$

Let,

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2} \quad and \quad t = \frac{1}{\lambda} \tag{3.116}$$

Therefore,

$$f(x) = \frac{1}{\alpha^{p+q-1}\,B(p,q)} \int_{\frac{1}{\alpha}}^{\infty} \frac{1}{t^{p-2}} \frac{(\alpha t-1)^{q-1}}{t^{q-1}}\, e^{-xt}\, \frac{dt}{t^2}$$

$$= \frac{1}{\alpha^{p+q-1}\,B(p,q)} \int_{\frac{1}{\alpha}}^{\infty} \frac{(\alpha t-1)^{q-1}}{t^{p+q-1}}\, e^{-xt}\, dt$$

Next, let

$$\alpha t = y \quad \therefore \quad t = \frac{y}{\alpha} \quad \text{and} \quad dt = \frac{dy}{\alpha}$$

Therefore,

$$f(x) = \frac{1}{\alpha^{p+q-1}\,B(p,q)} \int_{1}^{\infty} \frac{(y-1)^{q-1}}{\left(\frac{y}{\alpha}\right)^{p+q-1}}\, e^{-\frac{x}{\alpha}y}\, \frac{dy}{\alpha}$$

$$= \frac{1}{\alpha\,B(p,q)} \int_{1}^{\infty} \frac{(y-1)^{q-1}}{y^{p+q-1}}\, e^{-\frac{x}{\alpha}y}\, dy$$

Put,

$$z = y-1 \quad \therefore \quad y = 1+z \quad \text{and} \quad dy = dz$$

Therefore,

$$f(x) = \frac{1}{\alpha\,B(p,q)} \int_{0}^{\infty} \frac{z^{q-1}}{(1+z)^{p+q-1}}\, e^{-\frac{x}{\alpha}(1+z)}\, dz$$

$$= \frac{1}{\alpha\,B(p,q)} \int_{0}^{\infty} \frac{z^{q-1}}{(1+z)^{p+q-1}}\, e^{-\frac{x}{\alpha}(1+z)}\, dz$$

$$= \frac{e^{-\frac{x}{\alpha}}}{\alpha\,B(p,q)} \int_{0}^{\infty} z^{q-1}(1+z)^{2-p-q-1}\, e^{-\frac{x}{\alpha}z}\, dz$$

$$= \frac{e^{-\frac{x}{\alpha}}\,\Gamma(p+q)}{\alpha\,\Gamma(p)}\, \Psi(q; 2-p; \frac{x}{\alpha})$$

Using the equation (1.12)

$$f(x) = \frac{e^{-\frac{x}{\alpha}}\,\Gamma(p+q)}{\alpha\,\Gamma(p)} \left(\frac{x}{\alpha}\right)^{1-(2-p)} \Psi(1+q-(2-p), 2-(2-p); \frac{x}{\alpha})$$

$$= \frac{e^{-\frac{x}{\alpha}}\,\Gamma(p+q)}{\alpha\,\Gamma(p)} \left(\frac{x}{\alpha}\right)^{p-1} \Psi(p+q-1, p; \frac{x}{\alpha})$$

$$= \frac{\Gamma(p+q)}{\alpha\,\Gamma(p)} \left(\frac{x}{\alpha}\right)^{p-1} e^{-\frac{x}{\alpha}}\, \Psi(p+q-1, p; \frac{x}{\alpha})$$

a result obtained by Bhattacharya (1966) using Riemann-Loiuville integral.

The survival function is,

$$\begin{aligned} S(x) &= \frac{1}{\alpha\, B(p,q)} \int_{0}^{\alpha} e^{-\frac{x}{\lambda}} \left(\frac{\lambda}{\alpha}\right)^{p-1} \left(1-\frac{\lambda}{\alpha}\right)^{q-1} \\ &= \frac{1}{\alpha\, B(p,q)} \int_{0}^{\alpha} e^{-\frac{x}{\lambda}} \frac{\lambda^{p-1}\, (\alpha-\lambda)^{q-1}}{\alpha^{p+q-2}} d\lambda \\ &= \frac{1}{\alpha^{p+q-1}\, B(p,q)} \int_{0}^{\alpha} \lambda^{p-1}\, (\alpha-\lambda)^{q-1}\, e^{-\frac{x}{\lambda}}\, d\lambda \end{aligned}$$

Using the substitution (3.116),

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2} \quad and \quad t = \frac{1}{\lambda}$$

Therefore,

$$\begin{aligned} S(x) &= \frac{1}{\alpha^{p+q-1}\, B(p,q)} \int_{\frac{1}{\alpha}}^{\infty} \frac{1}{t^{p-1}} \frac{(\alpha t-1)^{q-1}}{t^{q-1}}\, e^{-xt}\, \frac{dt}{t^2} \\ &= \frac{1}{\alpha^{p+q-1}\, B(p,q)} \int_{\frac{1}{\alpha}}^{\infty} \frac{(\alpha t-1)^{q-1}}{t^{p+q}}\, e^{-xt}\, dt \end{aligned}$$

Next, let

$$\alpha t = y \quad \therefore \quad t = \frac{y}{\alpha} \quad and \quad dt = \frac{dt}{\alpha}$$

Therefore,

$$\begin{aligned} S(x) &= \frac{1}{\alpha^{p+q-1}\, B(p,q)} \int_{1}^{\infty} \frac{(y-1)^{q-1}}{\left(\frac{y}{\alpha}\right)^{p+q}}\, e^{-\frac{x}{\alpha}y}\, \frac{dy}{\alpha} \\ &= \frac{1}{\alpha\, B(p,q)} \int_{1}^{\infty} \frac{(y-1)^{q-1}}{y^{p+q}}\, e^{-\frac{x}{\alpha}y}\, dy \end{aligned}$$

Put,

$$z = y-1 \quad \therefore \quad y = 1+z \quad and \quad dy = dz$$

Therefore,

$$S(x) = \frac{1}{B(p,q)} \int_{0}^{\infty} \frac{z^{q-1}}{(1+z)^{p+q+1}}\, e^{-\frac{x}{\alpha}(1+z)}\, dz$$

$$= \frac{1}{B(p,q)} \int_0^\infty \frac{z^{q-1}}{(1+z)^{p+q}} e^{-\frac{x}{\alpha}(1+z)} dz$$

$$= \frac{e^{-\frac{x}{\alpha}}}{\alpha B(p,q)} \int_0^\infty z^{q-1}(1+z)^{1-p-q-1} e^{-\frac{x}{\alpha}z} dz$$

$$= \frac{e^{-\frac{x}{\alpha}}}{B(p,q)} \Gamma q \, \Psi(q, 1-p; \frac{x}{\alpha})$$

Therefore,

$$S(x) = \frac{\Gamma(p+q)}{\Gamma(p)} e^{-\frac{x}{\alpha}} \Psi(q, 1-p; \frac{x}{\alpha})$$

Using the equation (1.12)

$$S(x) = \frac{\Gamma(p+q)}{\Gamma(p)} e^{-\frac{x}{\alpha}} \left(\frac{x}{\alpha}\right)^{1-(1-p)} \Psi(1+q-(1-p), 2-(1-p); \frac{x}{\alpha})$$

$$= \frac{\Gamma(p+q)}{\Gamma(p)} e^{-\frac{x}{\alpha}} \left(\frac{x}{\alpha}\right)^{p} \Psi(p+q, 1+p; \frac{x}{\alpha}) \tag{3.117}$$

Therefore,

$$h(x) = \frac{\Psi(q, 2-p; \frac{x}{\alpha})}{\alpha \, \Psi(q, 1-p; \frac{x}{\alpha})}$$

or

$$h(x) = \frac{1}{\alpha} \frac{\alpha}{x} \frac{\Psi(p+q-1, p; \frac{x}{\alpha})}{\Psi(p+q, p+1; \frac{x}{\alpha})}$$

$$= \frac{1}{x} \frac{\Psi(p+q-1, p; \frac{x}{\alpha})}{\Psi(p+q, p+1; \frac{x}{\alpha})} \tag{3.118}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{\Gamma(p+q)}{\alpha \, \Gamma(p)} \int_0^\infty x^r \left(\frac{x}{\alpha}\right)^{p-1} e^{-\frac{x}{\alpha}} \Psi(p+q-1, p; \frac{x}{\alpha}) dx$$

Let,

$$x = \alpha t \quad \therefore \quad dx = \alpha \, dt$$

$$= \frac{\alpha^r \, \Gamma(p+q)}{\Gamma(p)} \int_0^\infty t^{p+r-1} e^{-t} \Psi(p+q-1, p; t) dt$$

$$= \frac{\alpha^r \, \Gamma(p+q)}{\Gamma(p)} M[e^{-t} \Psi(p+q-1, p; t), p+r]$$

Applying equation (1.20)

$$\begin{aligned} E(X^r) &= \frac{\alpha^r\,\Gamma(p+q)}{\Gamma(p)}\,\frac{\Gamma(p+r)\,\Gamma(r+1)}{\Gamma(p+q+r)} \\ &= r!\,\frac{\alpha^r\,\Gamma(p+q)}{\Gamma(p)}\,\frac{\Gamma(p+r)}{\Gamma(p+q+r)} \end{aligned} \tag{3.119}$$

Using conditional expectation approach,

$$\begin{aligned} E(\Lambda^r) &= \frac{1}{\alpha\,B(p,\ q)}\int_0^\alpha \lambda^r\left(\frac{\lambda}{\alpha}\right)^{p-1}\left(1-\frac{\lambda}{\alpha}\right)^{q-1} \\ &= \frac{\alpha^r}{\alpha\,B(p,\ q)}\int_0^\alpha \left(\frac{\lambda}{\alpha}\right)^{p+r-1}\left(1-\frac{\lambda}{\alpha}\right)^{q-1} d\lambda \\ &= \frac{\alpha^{r+1}}{\alpha\,B(p,\ q)}\int_0^1 t^{p+r-1}\,(1-t)^{q-1}\,dt \\ &= \frac{\alpha^r\Gamma(p+r)}{\Gamma(p)}\,\frac{\Gamma(p+q)}{\Gamma(p+q+r)} \end{aligned}$$

$$\therefore\quad E(X^r) = r!\,\alpha^r\,\frac{\Gamma(p+q)}{\Gamma(p)}\,\frac{\Gamma(p+r)}{\Gamma(p+q+r)} \tag{3.120}$$

and,

$$E(X) = \alpha\,\frac{p}{p+q} \tag{3.121}$$

### 3.5.4 Full beta mixing distribution

Kempton (1975) mixed gamma I distribution with gamma II distribution to obtain what he called Full beta model given by

$$\begin{aligned} g(\lambda) &= \int_0^\infty \frac{a^p}{\Gamma(p)}\,e^{-a\lambda}\,\lambda^{p-1}.\frac{1}{b^q\,\Gamma q}\,e^{-\frac{1}{b}}\,a^{q-1}\,da \\ &= \frac{\lambda^{p-1}}{\Gamma(p)\,\Gamma q\,b^q}\,\frac{\Gamma(p+q)}{(\lambda+\frac{1}{b})^{p+q}} \\ &= \frac{b^{p+q}\,\lambda^{p-1}}{\Gamma(p)\,\Gamma q\,b^q}\,\frac{\Gamma(p+q)}{(1+\lambda b)^{p+q}} \\ &= \frac{b^p}{B(p,q)}\,\frac{\lambda^{p-1}}{(1+\lambda b)^{p+q}} \qquad \lambda > 0;\ p,q,b > 0 \end{aligned} \tag{3.122}$$

The pdf of the mixture is,

$$f(x) = \frac{b^p}{B(p,\ q)}\int_0^\infty \frac{\lambda^{p-2}\,e^{-\frac{1}{\lambda}\,x}}{(1+b\lambda)^{p+q}}\,d\lambda$$

$$= \frac{b^p}{B(p,\ q)} \int_0^\infty \frac{\lambda^{p-2}\ e^{-\frac{1}{\lambda}x}}{(1+b\lambda)^{p+q}}\ d\lambda$$

Let,

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2}$$

$$f(x) = \frac{b^p}{B(p,\ q)} \int_0^\infty \frac{t^q\ e^{-xt}}{(t+b)^{p+q}} dt$$

Put,

$$t = bz \quad \therefore \quad dt = bdz$$

$$f(x) = \frac{b}{B(p,\ q)} \int_0^\infty \frac{z^q\ e^{-bxz}}{(1+z)^{p+q}} dz$$

$$= \frac{b\Gamma(q+1)}{B(p,\ q)}\ \Psi(q+1; 2-p; bx)$$

Using the equation (1.12)

$$\Psi(a, c; x) = x^{1-c}\ \Psi(1+a-c, 2-c; x)$$

$$f(x) = \frac{b^p\ \Gamma(q+1)}{B(p,\ q)}\ x^{p-1}\Psi(p+q; p; bx) \tag{3.123}$$

The survival function is,

$$S(x) = \frac{b^p}{B(p,\ q)} \int_0^\infty \frac{\lambda^{p-1}\ e^{-\frac{1}{\lambda}x}}{(1+b\lambda)^{p+q}}\ d\lambda$$

$$= \frac{b^p}{B(p,\ q)} \int_0^\infty \frac{\lambda^{p-1}\ e^{-\frac{1}{\lambda}x}}{(1+b\lambda)^{p+q}}\ d\lambda$$

Let,

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2}$$

$$S(x) = \frac{b^p}{B(p,\ q)} \int_0^\infty \frac{t^{q-1}\ e^{-xt}}{(t+b)^{p+q}} dt$$

Put,

$$t = bz \quad \therefore \quad dt = bdz$$

$$S(x) = \frac{1}{B(p,\ q)} \int_0^\infty \frac{z^{q-1}\ e^{-bxz}}{(1+z)^{p+q}} dz$$

$$= \frac{\Gamma q}{B(p,\ q)}\ \Psi(q;1-p;bx)$$

Using the equation (1.12)

$$S(x) = \frac{b^p\ \Gamma(p+q)}{\Gamma(p)}\ x^p \Psi(p+q;p+1;bx) \tag{3.124}$$

$$\therefore \quad h(x) = bq \frac{\Psi(q+1,2-p;bx)}{\Psi(q,1-p;bx)} \tag{3.125}$$

or

$$h(x) = \frac{q}{x}\ \frac{\Psi(p+q;p;bx)}{\Psi(p+q;p+1;bx)} \tag{3.126}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{b^p\ \Gamma(q+1)}{B(p,\ q)} \int_0^\infty x^{r+p-1}\Psi(p+q;p;bx)$$

Let,

$$y = bx \quad \therefore \quad x = \frac{y}{b} \quad and \quad dx = \frac{dy}{b}$$

$$E(X^r) = \frac{b^p\ \Gamma(q+1)}{B(p,q)} \int_0^\infty \left(\frac{y}{b}\right)^{p+r-1}\ \Psi(p+q,\ p;\ y)\ \frac{dy}{b}$$

$$= \frac{b^p\ \Gamma(q+1)}{B(p,q)}\ \frac{1}{b^{p+r}} \int_0^\infty y^{p+r-1}\ \Psi(p+q,\ p;\ y)\ dy$$

$$= \frac{\Gamma(q+1)}{b^r\ B(p,q)} \int_0^\infty y^{p+r-1}\ \Psi(p+q,\ p;\ y)\ dy$$

Applying equation (1.19)

$$E(X^r) = \frac{\Gamma(q+1)}{b^r\ B(p,q)}\ \frac{\Gamma(p+r)}{\Gamma(p+q)}\ \frac{\Gamma(p+q-p-r)\ \Gamma(p+r-p+1)}{\Gamma(p+q-p+1)}$$

$$= \frac{r!}{b^r}\ \frac{\Gamma(p+r)\ \Gamma(q-r)}{B(p,q)\ \Gamma(p+q)}$$

$$= \frac{r!}{b^r}\ \frac{B(p+r,q-r)}{B(p,q)} \tag{3.127}$$

and,

$$E(X) = \frac{1}{b}\ \frac{B(p+1,q-1)}{B(p,q)} \tag{3.128}$$

### 3.5.5 Uniform mixing distribution

In this section we shall use the following Lemma

**Lemma 3.2.**

$$\int_{x}^{\infty} y^{-1}\, e^{-y}\, dy = e^{-x}\, \Psi(1,1;x)$$

**Proof 3.1.** .

From (1.11),

$$\Psi(1,1;x) = \int_{0}^{\infty} (1+t)^{-1}\, e^{-xt}\, dt$$

Let

$$z = xt \quad \therefore \quad t = \frac{z}{x} \quad and \quad dt = \frac{dz}{x}$$

$$\Psi(1,1;x) = \int_{0}^{\infty} (1+\frac{z}{x})^{-1}\, e^{-z}\, \frac{dz}{x}$$

$$= \int_{0}^{\infty} (x+z)^{-1}\, e^{-z}\, dz$$

Let

$$y = x+z \quad \therefore \quad z = y-x \quad and \quad dz = dy$$

$$\Psi(1,1;x) = \int_{x}^{\infty} y^{-1}\, e^{-(y-x)}\, dy$$

$$= e^{x} \int_{x}^{\infty} y^{-1}\, e^{-y}\, dy$$

$$\therefore \quad e^{-x}\Psi(1,1;x) = \int_{x}^{\infty} y^{-1}\, e^{-y}\, dy$$

The uniform mixing distribution is given by

$$g(\lambda) = \frac{1}{(\beta-\alpha)} \quad \alpha < \lambda < \beta$$

Therefore,

$$f(x) = \int_{\alpha}^{\beta} \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}\, \frac{1}{(\beta-\alpha)} d\lambda$$

$$= \frac{1}{(\beta - \alpha)} \int_{\alpha}^{\beta} \frac{1}{\lambda} \, e^{-\frac{x}{\lambda}} d\lambda$$

Let,

$$t = \frac{1}{\lambda} \quad \lambda = \frac{1}{t} \Rightarrow d\lambda = -\frac{dt}{t^2}$$

$$\therefore \quad f(x) = \frac{1}{(\beta - \alpha)} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} t \, e^{-xt} \, (-\frac{dt}{t^2})$$

$$= \frac{1}{(\beta - \alpha)} \int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} t \, e^{-xt} \, \frac{dt}{t^2}$$

$$= \frac{1}{(\beta - \alpha)} \int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} t^{-1} \, e^{-xt} \, dt$$

Let,

$$y = x \, t \quad t = \frac{y}{x} \Rightarrow dt = -\frac{dy}{x}$$

$$\therefore \quad f(x) = \frac{1}{(\beta - \alpha)} \int_{\frac{x}{\beta}}^{\frac{x}{\alpha}} (\frac{y}{x})^{-1} \, e^{-y} \, \frac{dy}{x}$$

$$= \frac{1}{(\beta - \alpha)} \int_{\frac{x}{\beta}}^{\frac{x}{\alpha}} y^{-1} \, e^{-y} \, dy$$

$$= \frac{1}{\beta - \alpha} \left\{ \int_{\frac{x}{\beta}}^{\infty} y^{-1} \, e^{-y} \, dy - \int_{\frac{x}{\alpha}}^{\infty} y^{-1} \, e^{-y} \, dy \right\} \tag{3.129}$$

By Lemma

$$f(x) = \frac{1}{(\beta - \alpha)} \left\{ e^{-\frac{x}{\beta}} \, \Psi(1; 1; \frac{x}{\beta}) - e^{-\frac{x}{\alpha}} \, \Psi(1; 1; \frac{x}{\alpha}) \right\} \tag{3.130}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \frac{1}{(\beta - \alpha)} \left\{ \int_{0}^{\infty} x^r \, e^{-\frac{x}{\beta}} \, \Psi(1; 1; \frac{x}{\beta}) - \int_{0}^{\infty} x^r \, e^{-\frac{x}{\alpha}} \, \Psi(1; 1; \frac{x}{\alpha}) \right\}$$

Let

$$I_1 = \int_{0}^{\infty} x^r \, e^{-\frac{x}{\beta}} \, \Psi(1; 1; \frac{x}{\beta}) dx$$

Put

$$y = \frac{x}{\beta} \therefore \quad x = \beta y \quad and \quad dx = \beta dy$$

$$\therefore \quad I_1 = \int_0^\infty (\beta y)^r \, e^{-y} \, \Psi(1;1;y)\beta dy$$

$$= \beta^{r+1} \int_0^\infty y^r \, e^{-y} \, \Psi(1;1;y)dy$$

$$= \beta^{r+1} \int_0^\infty y^{(r+1)-1} \, e^{-y} \, \Psi(1;1;y)dy$$

Applying equation (1.19),

$$I_1 = \beta^{r+1} \frac{\Gamma(r+1)\,\Gamma(r+1-1+1)}{\Gamma(1+r+1-1+1)}$$

$$= \beta^{r+1} \frac{\Gamma(r+1)\,\Gamma(r+1)}{\Gamma(r+2)}$$

$$= \frac{r!\,\beta^{r+1}}{r+1}$$

Similarly

$$I_2 = \int_0^\infty x^r \, e^{-\frac{x}{\alpha}} \, \Psi(1;1;\frac{x}{\alpha})dx$$

$$= \frac{r!\,\alpha^{r+1}}{r+1}$$

$$E(X^r) = \frac{1}{(\beta-\alpha)} \, \{I_1 - I_2\}$$

$$= \frac{1}{(\beta-\alpha)} \left\{ \frac{r!\,\beta^{r+1}}{r+1} - \frac{r!\,\alpha^{r+1}}{r+1} \right\}$$

$$= \frac{r!}{r+1} \quad \frac{\beta^{r+1}-\alpha^{r+1}}{\beta-\alpha} \tag{3.131}$$

Using conditional expectation approach, we have

$$E(X^r) = r!\,E(\Lambda^r)$$

$$= \frac{r!}{\beta-\alpha} \int_\alpha^\beta \lambda^r \, d\lambda$$

$$= \frac{r!}{\beta-\alpha} \, \frac{\lambda^{r+1}}{r+1} \Big|_\alpha^\beta$$

$$= \frac{r!}{r+1} \quad \frac{\beta^{r+1}-\alpha^{r+1}}{\beta-\alpha}$$

$$\therefore \quad E(X) = \frac{1}{2}\beta\,[1+\frac{\alpha}{\beta}]$$
$$= \frac{1}{2}\,(\beta+\alpha)$$

Putting,

$$\alpha = \theta - \delta \quad \text{and} \quad \beta = \theta + \delta$$
$$E(X^r) = \frac{r!}{r+1}\ \frac{(\theta+\delta)^{r+1} - (\theta-\delta)^{r+1}}{(\theta+\delta)-(\theta-\delta)}$$
$$= \frac{r!}{r+1}\ \frac{(\theta+\delta)^{r+1} - (\theta-\delta)^{r+1}}{2\delta} \tag{3.132}$$

and

$$\therefore \quad E(X) = \frac{(\theta+\delta)^2 - (\theta-\delta)^2}{2\delta} \tag{3.133}$$

These results for the special case of $\beta$ and $\alpha$ were obtained by Bhattacharya and Holla (1965)

### 3.5.6 Pareto I mixing distribution

Using Pareto I as the mixing distribution, the exponential-Pareto I mixture is constructed below and the moments have been obtained.

$$g(\lambda) = \frac{c\,b^c}{\lambda^{c+1}}; \quad \lambda > b > 0 \quad c > 0 \tag{3.134}$$

which Willmot (1993) calls the shifted Pareto distribution.

The pdf of the mixture is,

$$f(x) = c\,b^c \int_b^\infty \lambda^{-c-1-1}\,\frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,d\lambda$$

Let,

$$\lambda = \frac{1}{t} \ \therefore \quad d\lambda = -\frac{dt}{t^2}$$
$$f(x) = c\,b^c \int_0^{\frac{1}{b}} t^{c+2}\,e^{-xt}\,\frac{dt}{t^2}$$
$$= c\,b^c \int_0^{\frac{1}{b}} t^c\,e^{-xt}\,dt$$

$$= c\,b^c \int_0^{\frac{1}{b}} t^{(c+1)-1}\, e^{-xt}\, dt$$

$$= c\,b^c\,\frac{\gamma(c+1, \frac{x}{b})}{x^{c+1}}$$

incomplete gamma in terms of hyper-geometric function gives

$$f(x) = \frac{cb^c}{x^{c+1}} \left(\frac{x}{b}\right)^{c+1} \frac{1}{c+1}\, {}_1F_1(c+1; c+2; -\frac{x}{b})$$

$$= \frac{c}{b}\,\frac{1}{c+1}\, {}_1F_1(c+1; c+2; -\frac{x}{b})$$

$$S(x) = \int_x^{\infty} f(t)dt \tag{3.135}$$

$$= \frac{c}{b(c+1)} \int_x^{\infty} {}_1F_1(c+1; c+2; -\frac{t}{b})dt$$

$$= \frac{c}{b(c+1)} \int_x^{\infty} \left\{ \int_0^1 \frac{Z^{(c+1)-1}\,(1-Z)^{(c+2)-(c+1)-1}\, e^{-\frac{t}{b}Z}}{B(c+1, (c+2)-(c+1)-1)} dZ \right\} dt$$

$$= \frac{c}{b(c+1)B(c+1,1)} \int_x^{\infty} \left\{ \int_0^1 Z^c\,(1-Z)^0\, e^{-\frac{t}{b}Z}\, dZ \right\} dt$$

$$= \frac{c\Gamma(c+2)}{b(c+1)\Gamma(c+1)} \int_x^{\infty} \left\{ \int_0^1 Z^c\, e^{-\frac{t}{b}Z}\, dZ \right\} dt$$

$$= \frac{c}{b} \int_0^1 \left\{ \int_x^{\infty} e^{-\frac{t}{b}Z}\, dt \right\} Z^c\, dZ$$

$$= \frac{c}{b} \int_0^1 \left[ -\frac{b}{Z}\, e^{-\frac{Z}{b}t} \right]_x^{\infty} Z^c\, dZ$$

$$= \frac{c}{b} \int_0^1 \frac{b}{Z}\, e^{-\frac{Z}{b}x}\, Z^c\, dZ$$

$$= c \int_0^1 Z^{c-1}\, e^{-\frac{x}{b}Z}\, dZ$$

$$= c \int_0^1 Z^{c-1}\,(1-Z)^{(c+1)-c-1}\, e^{-\frac{x}{b}Z}\, dZ$$

$$= c \int_0^1 Z^{c-1}\,(1-Z)^{c-c+1-1}\, e^{-\frac{x}{b}Z}\, dZ$$

$$\therefore \quad S(x) = c\, B(c,(c+1)-c) \int_0^1 \frac{Z^{c-1}}{B(c,(c+1)-c)} (1-Z)^{(c+1)-c-1}\, e^{-\frac{x}{b}Z}\, dZ$$

$$= c\, B(c,1)\ ;_1F_1(c;c+1;-\frac{x}{b})$$

$$= \frac{c\Gamma(c)}{\Gamma(c+1)}\ {}_1F_1(c;c+1;-\frac{x}{b})$$

$$= {}_1F_1(c;c+1;-\frac{x}{b})$$

$$\therefore \quad h(x) = \frac{c}{b(c+1)} \frac{{}_1F_1(c+1;c+2;-\frac{x}{b})}{{}_1F_1(c;c+1;-\frac{x}{b})} \tag{3.136}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_{x=0}^{\infty} x^r\, f(x)\, dx$$

$$= \int_{x=0}^{\infty} x^r \frac{c}{b(c+1)}\ {}_1F_1(c+1;c+2;-\frac{x}{b})\, dx$$

$$= \frac{c}{b(c+1)} \int_{x=0}^{\infty} x^r\ {}_1F_1(c+1;c+2;-\frac{x}{b})\, dx$$

$$= \frac{c}{b(c+1)} \int_0^{\infty} x^r \left\{ \int_0^1 \frac{t^{(c+1)-1}\, (1-t)^{(c+2)-(c+1)-1}\, e^{-\frac{x}{b}t}}{B(c+1,1)} dt \right\} dx$$

$$= \frac{c}{b} \int_0^{\infty} \int_0^1 x^r\, t^c\, e^{-\frac{x}{b}t}\, dt\, dx$$

$$= \frac{c}{b} \int_0^1 \left[ \int_0^{\infty} x^r\, e^{-\frac{t}{b}x}\, dx \right] t^c\, dt$$

$$= \frac{c}{b} \int_0^1 \frac{\Gamma(r+1)}{\left(\frac{t}{b}\right)^{r+1}}\, t^c\, dt$$

$$= cb^r \int_0^1 \Gamma(r+1)\, t^{c-r-1}\, dt$$

$$= cb^r\, \frac{\Gamma(r+1)}{c-r}$$

$$= r!\, b^r\, \frac{c}{c-r} \tag{3.137}$$

Using conditional expectation approach

$$E(X^r) = r!\, E(\Lambda^r)$$

$$= r! \int_b^{\infty} \lambda^r\, \frac{c\, b^c}{\lambda^{c+1}}\, d\lambda$$

$$= r!\, c\, b^c \int_b^\infty \lambda^{r-c-1}\, d\lambda$$

$$= r!\, c\, b^c \left[\frac{\lambda^{r-c}}{r-c}\right]_b^\infty$$

$$= r!\, \frac{c\, b^c}{c-r} \left[-\frac{1}{\lambda^{c-r}}\right]_b^\infty$$

$$= r!\, \frac{c\, b^c}{c-r}\, \frac{1}{b^{c-r}}$$

$$= r!\, b^r\, \frac{c}{c-r} \tag{3.138}$$

### 3.5.7 Pareto II mixing distribution

Using Pareto II as the mixing distribution, the exponential-Pareto II mixture is constructed below and the moments have been obtained.

$$g(\lambda) = \frac{\alpha\, \beta^\alpha}{(\lambda+\beta)^{\alpha+1}}; \quad \lambda > 0 \quad \alpha, \beta > 0 \tag{3.139}$$

The pdf of the mixture is,

$$f(x) = \alpha\, \beta^\alpha \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\, (\lambda+\beta)^{-\alpha-1}\, d\lambda$$

$$= \alpha\, \beta^\alpha \int_0^\infty \frac{1}{\lambda}\, \frac{1}{(\lambda+\beta)^{\alpha+1}}\, e^{-\frac{x}{\lambda}}\, d\lambda$$

Let,

$$t = \frac{1}{\lambda} \quad \therefore \quad \lambda = \frac{1}{t} \quad and \quad d\lambda = -\frac{dt}{t^2} \tag{3.140}$$

$$f(x) = \alpha\, \beta^\alpha \int_0^\infty t\, e^{-xt}\, (\frac{1}{t}+\beta)^{-\alpha-1}\, \frac{dt}{t^2}$$

$$= \alpha\, \beta^\alpha \int_0^\infty \frac{t^\alpha\, e^{-xt}}{(1+\beta t)^{\alpha+1}}\, dt$$

Let,

$$y = \beta t \quad \therefore \quad t = \frac{y}{\beta} \quad and \quad dt = -\frac{dy}{\beta} \tag{3.141}$$

$$\therefore \quad f(x) = \alpha\, \beta^\alpha \int_0^\infty \frac{\left(\frac{y}{\beta}\right)^\alpha\, e^{-\frac{x}{\beta}y}}{(1+y)^{\alpha+1}}\, \frac{dy}{\beta}$$

$$= \frac{\alpha}{\beta} \int_0^\infty \frac{y^\alpha\, e^{-\frac{x}{\beta}y}}{(1+y)^{\alpha+1}}\, dy$$

$$= \frac{\alpha}{\beta} \int_0^\infty y^\alpha \, (1+y)^{-\alpha-1} \, e^{-\frac{x}{\beta}y} \, dy$$

$$= \frac{\alpha}{\beta} \, \Gamma(\alpha+1) \int_0^\infty \frac{y^{(\alpha+1)-1}}{\Gamma(\alpha+1)} \, (1+y)^{1-(\alpha+1)-1} \, e^{-\frac{x}{\beta}y} \, dy$$

$$= \frac{\alpha}{\beta} \, \Gamma(\alpha+1) \, \Psi(\alpha+1, 1; \frac{x}{\beta}) \quad x > 0; \; \alpha > 0; \; \beta > 0 \tag{3.142}$$

(2.) The survival function is

$$S(x) = \alpha \, \beta^\alpha \int_0^\infty e^{-\frac{x}{\lambda}} \, (\lambda+\beta)^{-\alpha-1} \, d\lambda$$

$$= \alpha \, \beta^\alpha \int_0^\infty \frac{1}{\lambda} \, \frac{1}{(\lambda+\beta)^{\alpha+1}} \, e^{-\frac{x}{\lambda}} \, d\lambda$$

Using substitution (3.140),

$$S(x) = \alpha \, \beta^\alpha \int_0^\infty e^{-xt} \, (\frac{1}{t} + \beta)^{-\alpha-1} \, \frac{dt}{t^2}$$

$$= \alpha \, \beta^\alpha \int_0^\infty \frac{t^{\alpha-1} \, e^{-xt}}{(1+\beta t)^{\alpha+1}} \, dt$$

Using substitution (3.141),

$$\therefore \quad S(x) = \alpha \, \beta^\alpha \int_0^\infty \frac{\left(\frac{y}{\beta}\right)^{\alpha-1} \, e^{-\frac{x}{\beta}y}}{(1+y)^{\alpha+1}} \, \frac{dy}{\beta}$$

$$= \alpha \, \Gamma(\alpha) \int_0^\infty \frac{y^{\alpha-1}}{\Gamma(\alpha)} \, (1+y)^{0-\alpha-1} \, e^{-\frac{x}{\beta}y} \, dy$$

$$= \alpha \, \Gamma(\alpha) \, \Psi(\alpha, 0; \frac{x}{\beta})$$

$$= \Gamma(\alpha+1) \, \Psi(\alpha, 0; \frac{x}{\beta})$$

$$\therefore \quad h(x) = \frac{\alpha}{\beta} \Gamma(\alpha+1) \, \frac{\Psi(\alpha+1, 1; \frac{x}{\beta})}{\alpha \, \Gamma(\alpha) \, \Psi(\alpha, 0; \frac{x}{\beta})}$$

$$= \frac{\alpha}{\beta} \, \frac{\Psi(\alpha+1, 1; \frac{x}{\beta})}{\Psi(\alpha, 0; \frac{x}{\beta})} \tag{3.143}$$

The $r^{th}$ moment about zero is,

$$E(X^r) = \int_{x=0}^\infty x^r \, \frac{\alpha}{\beta} \, \Gamma(\alpha+1) \, \Psi(\alpha+1; 1; \frac{x}{\beta}) \, dx$$

$$= \frac{\alpha}{\beta}\,\Gamma(\alpha+1) \int_{0}^{\infty} x^r \;\; \Psi(\alpha+1;1;\frac{x}{\beta})\, dx$$

Let,

$$u = \frac{x}{\beta} \quad \Rightarrow \quad x = \beta u \quad and \quad dx = \beta du$$

$$\begin{aligned}
E(X^r) = &\ \frac{\alpha}{\beta}\,\Gamma(\alpha+1) \int_{0}^{\infty} (\beta u)^r \;\; \Psi(\alpha+1;1;u)\, \beta du \\
&= \alpha\,\Gamma(\alpha+1)\,\beta^r \int_{0}^{\infty} u^r \;\; \Psi(\alpha+1;1;u)\, du \\
&= \ \alpha\,\Gamma(\alpha+1)\,\beta^r \int_{u=0}^{\infty} u^{(r+1)-1} \;\; \Psi(\alpha+1;1;u)\, du \\
&= \ \alpha\,\Gamma(\alpha+1)\,\beta^r\, M\,(\Psi(\alpha+1;1;u);(r+1))
\end{aligned}$$

Applying equation (1.19)

$$M[\Psi(a;c;x),s] = \frac{\Gamma(s)\Gamma(a-s)\Gamma(s-c+1)}{\Gamma(a)\Gamma(a-c+1)}$$

$$\therefore \quad M\,(\Psi(\alpha+1;1;u);(r+1)) = \frac{\Gamma(r+1)\;\Gamma(\alpha-r)\;\Gamma(r+1)}{\Gamma(\alpha+1)\;\Gamma(\alpha+1)}$$

and

$$\begin{aligned}
E(X^r) = &\ \alpha\,\beta^r\,\frac{\Gamma(r+1)\;\Gamma(\alpha-r)\;\Gamma(r+1)}{\Gamma(\alpha+1)} \\
&= \ \alpha\,\beta^r\,(r!)^2\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \\
&= \ (r!)^2\,\beta^r\,\frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \quad \alpha > r
\end{aligned}$$

hence,

$$E(X) = \frac{\beta}{\alpha-1} \tag{3.144}$$

Using conditional expectation approach,

$$\begin{aligned}
E(X^r) &= r!\; E(\Lambda^r) \\
&= r!\;\alpha\,\beta^{\alpha} \int_{0}^{\infty} \frac{\lambda^r}{(\lambda+\beta)^{\alpha+1}}\, d\lambda
\end{aligned}$$

Let

$$\lambda = \beta u \quad \therefore \quad d\lambda = \beta du$$

$$\begin{aligned}
\therefore \quad E(X^r) &= r!\, \alpha\, \beta^{\alpha} \int_0^{\infty} \frac{(\beta u)^r}{\beta^{\alpha+1}\,(1+u)^{\alpha+1}}\, \beta du \\
&= r!\, \alpha\, \beta^{r} \int_0^{\infty} \frac{u^r}{(1+u)^{\alpha+1}}\, du \\
&= r!\, \alpha\, \beta^{r} \int_0^{\infty} \frac{u^{(r+1)-1}}{(1+u)^{(r+1)-(r+1)+\alpha+1}}\, du \\
&= r!\, \alpha\, \beta^{r} \int_0^{\infty} \frac{u^{(r+1)-1}}{(1+u)^{(r+1)+(\alpha-r)}}\, du \\
&= \alpha\, \beta^r\, r!\, B(r+1, \alpha-r) \\
&= \alpha\, \beta^r\, r!\, \frac{\Gamma(r+1)\,\Gamma(\alpha-r)}{\Gamma(\alpha+1)} \\
&= \alpha\, \beta^r\, (r!)^2\, \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \qquad (3.145)
\end{aligned}$$

### 3.5.8 Generalized Pareto mixing distribution

The generalized Pareto distribution is a gamma-gamma mixture and it can be obtained as follows:
Let us consider a gamma-gamma mixture, where the first gamma distribution is given by

$$g(\lambda \mid \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta\lambda}\, \lambda^{\alpha-1}, \quad \lambda > 0; \alpha, \beta > 0$$

The second gamma distribution is

$$f(\beta) = \frac{\delta^{v}}{\Gamma v}\, e^{-\delta\beta}\, \beta^{v-1}, \quad \beta > 0; \delta, v > 0$$

Therefore the gamma-gamma mixture is

$$\begin{aligned}
g(\lambda) &= \int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, e^{-\beta\lambda}\, \lambda^{\alpha-1}\, \frac{\delta^{v}}{\Gamma v}\, e^{\delta\beta}\, \beta^{v-1}\, d\beta \\
&= \frac{\delta^{v}\, \lambda^{\alpha-1}}{\Gamma(\alpha)\,\Gamma v} \int_0^{\infty} \beta^{\alpha+v-1}\, e^{-\beta(\lambda+\delta)}\, d\beta \\
&= \frac{\delta^{v}\, \lambda^{\alpha-1}}{\Gamma(\alpha)\,\Gamma v}\, \frac{\Gamma(\alpha+v)}{(\lambda+\delta)^{\alpha+v}} \quad \lambda > 0\ \alpha, \delta > 0 \\
&= \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v}\, \delta^{v}\, \frac{\lambda^{\alpha-1}}{(\lambda+\delta)^{\alpha+v}} \quad \lambda > 0;\ \alpha, \delta, v > 0 \qquad (3.146)
\end{aligned}$$

(2.) The pdf of the mixture is,

$$f(x) = \int_0^\infty \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}\, \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v}\, \frac{\delta^v\, \lambda^{\alpha-1}}{(\lambda+\delta)^{\alpha+v}} d\lambda$$

$$= \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{\lambda^{(\alpha-1)-1}\, e^{-\frac{x}{\lambda}}}{(\lambda+\delta)^{\alpha+v}}\, d\lambda$$

Let,

$$\lambda = \frac{1}{t} \quad \therefore \quad d\lambda = -\frac{dt}{t^2} \tag{3.147}$$

Therefore,

$$f(x) = \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{e^{-xt}}{t^{(\alpha-1)-1}\, (\frac{1}{t}+\delta)^{\alpha+v}}\, \frac{dt}{t^2}$$

$$= \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{e^{-xt}}{t^{(\alpha-1)-1}}\, \frac{t^{\alpha+v}}{(1+\delta t)^{\alpha+v}}\, \frac{dt}{t^2}$$

$$= \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{e^{-xt}\, t^v}{(1+\delta t)^{\alpha+v}}\, dt$$

Let,

$$\delta t = y \quad \therefore \quad t = \frac{y}{\delta} \quad and \quad dt = \frac{dy}{\delta}$$

Therefore,

$$f(x) = \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{e^{-\frac{x}{\delta}y}\, \left(\frac{y}{\delta}\right)^v}{(1+y)^{\alpha+v}}\, \frac{dy}{\delta}$$

$$= \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{y^v\, (1+y)^{-\delta-v}\, e^{-\frac{x}{\delta}y}}{\delta^{v+1}}\, dy$$

$$= \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty \frac{y^v\, (1+y)^{2-\delta-v-1-1}\, e^{-\frac{x}{\delta}y}}{\delta^{v+1}}\, dy$$

$$= \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v}\, \Gamma(v+1)\, \Psi(v+1, 2-\alpha; \frac{x}{\delta})$$

$$= \frac{v}{\delta}\, \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)}\, \Psi(v+1, 2-\alpha; \frac{x}{\delta}) \tag{3.148}$$

(3.) The survival function of the mixture is

$$S(x) = \frac{\delta^v\, \Gamma(\alpha+v)}{\Gamma(\alpha)\,\Gamma v} \int_0^\infty e^{-\frac{x}{\lambda}}\, \frac{\lambda^{\alpha-1}}{(\lambda+\delta)^{\alpha+v}}\, d\lambda$$

$$= \frac{\delta^v \, \Gamma(\alpha+v)}{\Gamma(\alpha)\, \Gamma v} \int_0^\infty \frac{\lambda^{\alpha-1}\, e^{-\frac{x}{\lambda}}}{(\lambda+\delta)^{\alpha+v}}\, d\lambda$$

Using the substitution (3.147),

$$\begin{aligned} S(x) &= \frac{\delta^v \, \Gamma(\alpha+v)}{\Gamma(\alpha)\, \Gamma v} \int_0^\infty \frac{e^{-xt}}{t^{\alpha-1}\,(\frac{1}{t}+\delta)^{\alpha+v}}\, \frac{dt}{t^2} \\ &= \frac{\delta^v \, \Gamma(\alpha+v)}{\Gamma(\alpha)\, \Gamma v} \int_0^\infty \frac{e^{-xt}}{t^{\alpha-1}}\, \frac{t^{\alpha+v}}{(1+\delta t)^{\alpha+v}}\, \frac{dt}{t^2} \\ &= \frac{\delta^v \, \Gamma(\alpha+v)}{\Gamma(\alpha)\, \Gamma v} \int_0^\infty \frac{e^{-xt}\, t^{v-1}}{(1+\delta t)^{\alpha+v}}\, dt \end{aligned}$$

Put,

$$\delta t = y \quad \therefore \quad t = \frac{y}{\delta} \quad and \quad dt = \frac{dy}{\delta}$$

Therefore,

$$\begin{aligned} S(x) &= \frac{\delta^v \, \Gamma(\alpha+v)}{\Gamma(\alpha)\, \Gamma v} \int_0^\infty \frac{e^{-\frac{x}{\delta}y}\, \left(\frac{y}{\delta}\right)^{v-1}}{(1+y)^{\alpha+v}}\, \frac{dy}{\delta} \\ &= \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)\, \Gamma v} \int_0^\infty \frac{y^{v-1}\, e^{-\frac{x}{\delta}y}}{(1+y)^{\delta+v}}\, dy \\ &= \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)} \int_0^\infty \frac{y^{v-1}\, (1+y)^{1-\delta-v-1}\, e^{-\frac{x}{\delta}y}}{\Gamma v}\, dy \\ &= \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)}\, \Psi(v, 1-\alpha; \frac{x}{\delta}) \end{aligned} \tag{3.149}$$

The hazard function is,

$$\begin{aligned} h(x) &= \frac{\frac{v}{\delta}\, \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)}\, \Psi(v+1, 2-\alpha; \frac{x}{\delta})}{\frac{\Gamma(\alpha+v)}{\Gamma(\alpha)}\, \Psi(v, 1-\alpha; \frac{x}{\delta})} \\ &= \frac{v}{\delta}\, \frac{\Psi(v+1, 2-\alpha; \frac{x}{\delta})}{\Psi(v, 1-\alpha; \frac{x}{\delta})} \end{aligned} \tag{3.150}$$

The $r^{th}$ moment about zero is,

$$\begin{aligned} E(X^r) &= \int_{x=0}^\infty x^r\, f(x)\, dx \\ &= \int_{x=0}^\infty x^r\, \frac{v}{\delta}\, \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)}\, \Psi(v+1, 2-\alpha; \frac{x}{\delta})\, dx \end{aligned}$$

$$= \frac{v}{\delta} \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)} \int_{x=0}^{\infty} x^r \; \Psi(v+1, 2-\alpha; \frac{x}{\delta}) \; dx$$

Let

$$\frac{x}{\delta} = y \quad \therefore \quad x = \delta \; y \quad and \quad dx = \delta dy$$

$$E(X^r) = \frac{v}{\delta} \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)} \int_{x=0}^{\infty} \delta^r \; y^r \; \Psi(v+1, 2-\alpha; y) \; \delta dy$$

$$= v \; \delta^r \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)} \int_{x=0}^{\infty} y^{(r+1)-1} \; \Psi(v+1, 2-\alpha; y) \; dy$$

Applying equation (1.19)

$$E(X^r) = v \; \delta^r \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)} \frac{\Gamma(r+1) \; \Gamma(v+1-r-1) \; \Gamma(r+1-2+\alpha+1)}{\Gamma(v+1) \; \Gamma(v+1-2+\alpha+1)}$$

$$= v \; \delta^r \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)} \frac{\Gamma(r+1) \; \Gamma(v-r) \; \Gamma(\alpha+r)}{\Gamma(v+1) \; \Gamma(v+\alpha)}$$

$$= v \; \delta^r \frac{1}{\Gamma(\alpha)} \frac{\Gamma(r+1) \; \Gamma(v-r) \; \Gamma(\alpha+r)}{\Gamma(v+1)}$$

$$= \delta^r \; r! \frac{\Gamma(v-r)}{\Gamma v} \frac{\Gamma(\alpha+r)}{\alpha} \tag{3.151}$$

6. Using conditional expectation approach, we have

$$E(X^r) = r! \; E(\Lambda^r)$$

i.e.

$$E(X^r) = r! \int_0^{\infty} \lambda^r \frac{\Gamma(\alpha+v)}{\Gamma(\alpha) \; \Gamma v} \frac{\delta^v \; \lambda^{\alpha-1}}{(\lambda+\delta)^{\alpha+v}} \; d\lambda$$

$$= r! \frac{\delta^v \; \Gamma(\alpha+v)}{\Gamma(\alpha) \; \Gamma v} \int_0^{\infty} \frac{\lambda^{\alpha+r-1}}{(\lambda+\delta)^{\alpha+v}} \; d\lambda$$

Let

$$\lambda = \delta z \quad \therefore \quad d\lambda = \delta dz$$

$$E(X^r) = r! \frac{\delta^v \; \Gamma(\alpha+v)}{\Gamma(\alpha) \; \Gamma v} \int_0^{\infty} \frac{\delta^{\alpha+r-1} \; z^{\alpha+r-1}}{\delta^{\alpha+v} \; (1+z)^{\alpha+v}} \; \delta \; dz$$

$$= r! \frac{\delta^r \; \Gamma(\alpha+v)}{\Gamma(\alpha) \; \Gamma v} \int_0^{\infty} \frac{z^{\alpha+r-1}}{(1+z)^{\alpha+v}} \; dz$$

$$= r! \frac{\delta^r \; \Gamma(\alpha+v)}{\Gamma(\alpha) \; \Gamma v} \int_0^{\infty} \frac{z^{\alpha+r-1}}{(1+z)^{\alpha+r+v-r}} \; dz$$

$$= r! \frac{\delta^r \, \Gamma(\alpha+v)}{\Gamma(\alpha) \, \Gamma v} \, B(\alpha+r, v-r)$$
$$= r! \frac{\delta^r \, \Gamma(\alpha+v)}{\Gamma(\alpha) \, \Gamma v} \frac{\Gamma(\alpha+r) \, \Gamma(v-r)}{\Gamma(\alpha+v)}$$
$$= r! \, \delta^r \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{\Gamma(v-r)}{\Gamma v} \tag{3.152}$$

**Special cases of the generalized Pareto**

a.) When $\alpha = 1$,

Equation (3.146) becomes

$$g(\lambda) = \frac{\Gamma(1+v)}{\Gamma v} \frac{\delta^v}{(\lambda+\delta)^{1+v}}$$
$$= \frac{v \, \delta^v}{(\lambda+\delta)^{1+v}}$$

which is Pareto II (Lomax) distribution

It is for this reason that it is known as generalized Pareto distribution

b.) When $\delta = 1$,

Equation (3.146) becomes

$$g(\lambda) = \frac{\Gamma(\alpha+v)}{\Gamma(\alpha) \, \Gamma v} \frac{\lambda^{\alpha-1}}{(\lambda+1)^{\alpha+v}}$$
$$= \frac{\lambda^{\alpha-1}}{B(\alpha, v) \, (\lambda+1)^{1+v}}$$
$$\lambda > 0 \; \alpha, v > 0 \tag{3.153}$$

which is beta II distribution

It is for this reason that it is known as generalized beta distribution

## 3.6 Concluding remarks

In this chapter, eighteen (18) mixing distributions have been considered, and it is only the inverse gamma mixing distribution whose mixture is in explicit form.

It has been shown that the generalized inverse Gaussian mixing distribution nests inverse Gaussian, reciprocal inverse Gaussian, gamma and inverse gamma distributions. Pareto II (Lomax) and Beta II distributions are special cases of the Generalized Pareto distribution.

The direct method of obtaining moments using the Mellin transform technique fails when the mixing distribution is the generalized inverse Gaussian, inverse Gaussian and reciprocal inverse Gaussian distributions, hence the need for an alternative approach. The conditional expectation technique has also been used to verify results obtained using Mellin transform technique.

# Chapter 4

# CHARACTERIZING POISSON MIXTURES BY HAZARD FUNCTIONS OF EXPONENTIAL MIXTURES

## 4.1 Introduction

Motivated by the work of Walhin and Paris (1999), this chapter has defined mixed Poisson distribution in terms of hazard function of type I exponential mixture. Examples are given of Hofmann distributions and their associated hazard functions of exponential mixtures.

The chapter has the following sections: In section 4.2 mixed Poisson distribution is expressed in terms of Laplace transform. In section 4.3 the mixed Poisson distribution is expressed in terms of hazard function of exponential mixture.

Section 4.4 discusses infinite divisibility in relation to complete monotocity and Laplace transform leading to infinite divisible mixed Poisson distribution. Section 4.5 derives compound Poisson distribution in terms of pgfs and recursive form. Section 4.6 deals with applications of the models derived to various Parameterizations of Hoffmann distributions. Hoffman hazard function has been re-parameterized in section 4.7 and concluding remarks are in section 4.8.

## 4.2 Mixed Poisson Distribution in Terms of Laplace Transform

Let

$$p_n(t) = prob(N_t = n) \tag{4.1}$$

where $\mathbf{N_t}$ is a discrete random variable, defined as the number of changes in the time interval $[0, t]$ and T is the time until the occurrence of the first change.

A mixed Poisson distribution is given as

$$\begin{aligned} p_n(t) &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} g(\lambda) d\lambda \qquad n = 0, 1, 2, \ldots \\ &= E(e^{-\lambda t} \frac{(\lambda t)^n}{n!}) \\ &= \frac{(-1)^n t^n}{n!} E((-1)^n \lambda^n e^{-\lambda t}) \end{aligned}$$

When $\mathbf{n = 0}$, we have

$$p_0(t) = E(e^{-\lambda t})$$

$$= \boldsymbol{L_{\Lambda}(t)} \tag{4.2}$$

the Laplace transform of the mixing distribution $\mathbf{g(\lambda)}$

Differentiating $\mathbf{p_0(t)}$, n times we get

$$\begin{aligned} p_0^{(n)}(t) &= E[(-1)^n(\lambda)^n e^{-\lambda t}] \\ p_n(t) &= (-1)^n \frac{t^n}{n!}\, p_0^{(n)}(t) \\ &= (-1)^n \frac{t^n}{n!}\, L_{\Lambda}^{(n)}(t) \qquad n = 0, 1, 2, 3, \ldots \end{aligned} \tag{4.3}$$

which is the mixed Poisson distribution in terms of the Laplace transform of the mixing distribution.

## 4.3 Mixed Poisson Distribution in Terms of the Hazard Function of the Type I Exponential Mixture

The pdf of a type I exponential mixture is defined by

$$f(t) = \int_0^{\infty} \lambda\, e^{-\lambda\, t}\, g(\lambda)\, d\lambda$$

where $\mathbf{g(\lambda)}$ is the mixing distribution.

The corresponding survival function is

$$\begin{aligned} S(t) &= \int_0^{\infty} S(t|\lambda)\, g(\lambda)\, d\lambda \\ &= \int_0^{\infty} e^{-\lambda t}\ g(\lambda) d\lambda \\ &= L_{\Lambda}(t) \end{aligned} \tag{4.4}$$

**Remark 4.1.** *The survival function of a type I exponential mixture is the Laplace transform of the mixing distribution.*

The hazard function of the exponential mixture is given by

$$\begin{aligned} h(t) &= \frac{f(t)}{S(t)} \\ &= -\frac{1}{S(t)}\ \frac{dS}{dt} \\ &= -\frac{L'(t)}{L(t)} \end{aligned} \tag{4.5}$$

Let,

$$\theta(t) = In(\frac{1}{L_\Lambda(t)})$$

Therefore,

$$\begin{aligned}\theta^{'}(t) &= -\frac{L'(t)}{L(t)} \\ &= h(t) \end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned} p_0(t) &= L_\Lambda(t)) \\ &= e^{In\ L_\Lambda(t)} \\ &= e^{-In(\frac{1}{L_\Lambda(t)})} \\ &= e^{-\theta(t)} \end{aligned} \tag{4.7}$$

where,

$$p_0(t) = e^{\int\limits_0^t h(x)\, dx}$$

and,

$$\theta(t) = -\int\limits_0^t h(x)\, dx \tag{4.8}$$

is the cumulative or integrated hazard function.

Using 4.7 and 4.8, $\mathbf{p_0(t)}$ can be obtained and then use formula 4.3 to derive $\mathbf{p_n(t)}$.

The pgf of the mixed Poisson distribution is given by

$$\begin{aligned} H(s,t) &= \sum_{n=0}^{\infty} p_n(t)\ s^n \\ &= \sum_{n=0}^{\infty} [\int\limits_0^{\infty} e^{-\lambda t}\ \frac{(\lambda t)^n}{n!}\ g(\lambda)\ d\lambda]\ s^n \\ &= \int\limits_0^{\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t s)^n}{n!}\ g(\lambda)\ d\lambda \\ &= \int\limits_0^{\infty} e^{-\lambda t}\ e^{\lambda t s}\ g(\lambda)\ d\lambda \end{aligned}$$

$$= \int_{0}^{\infty} e^{-\lambda t(1-s)}\, g(\lambda)\, d\lambda$$
$$= L_{\Lambda}(t-ts)$$
$$= e^{-In \frac{1}{L_{\Lambda}(t-ts)}}$$
$$H(s,t) = e^{-\theta(t-ts)} \tag{4.9}$$

which is the survival function of the exponential mixture at time $\mathbf{t-ts}$.

**Remark 4.2.** *Given a hazard function of a type I exponential mixture, the pgf of the mixed Poisson distribution is the survival function of the exponential mixture at time* $\mathbf{t-ts}$.

The mean and variance of the mixed Poisson distribution can hence be obtained as follows:

$$H'(s,t) = \frac{d}{ds}\, H(s,t)$$
$$= t\theta'(t-ts)\, H(s,t)$$
$$H''(s,t) = t^2[(\theta'(t-ts))^2 - \theta''(t-ts)]\, H(s,t)] \tag{4.10}$$

Then,

$$H(1,t) = 1, \qquad H'(1,t) = t\,\theta'(0) \quad and \quad H''(1,t) = t^2\,[\theta'(0)]^2 - \theta''(0)]$$
$$E[z(t)] = H'(1,t)$$
$$= t\,\theta'(0) \tag{4.11}$$
$$Var[z(t)] = H''(1,t) + H'(1,t) - (H'(1,t))^2$$
$$= t\,\theta'(0) - t^2\,\theta''(0) \tag{4.12}$$

## 4.4 Infinite Divisibility

**Definition 4.1** (Infinite Divisibility (Karlis and Xekalaki, 2005))**.** *A random variable X is said to have an infinitely divisible distribution, if its characteristic function* $\Psi(t)$ *can be written in the form:*

$$\Psi(\mathrm{t}) = [\Psi_{\mathrm{n}}(\mathrm{t})]^{\mathrm{n}}$$

*where* $\Psi_{\mathrm{n}}(\mathrm{t})$ *is a characteristic function for any* $\mathbf{n = 1, 2, 3, \ldots}$.

In other words a distribution is infinitely divisible if it can be written as the distribution of the sum of an arbitrary number $\mathbf{n}$ of independently and identically distributed random variables.

Feller (1968) has defined infinite divisibility in terms of probability generating function as follows:

"A probability generating function $\mathbf{h}$ is called infinitely divisible if for each positive integer $\mathbf{n}$ the $\mathbf{n^{th}}$ root $\sqrt[n]{h}$ is again a probability generating function"

**Definition 4.2** (Completely Monotonicity). *A function* $\Psi$ *on* $[0, \infty]$ *is completely monotone if it possesses derivatives* $\Psi^{(n)}$ *of all orders and*

$$(-1)^n \Psi^{(n)}(t) \geq 0, \qquad t > 0$$

The link between infinite divisibility and complete monotonicity is given in the following propositions.

**Proposition 4.1** (Feller, 1971, Vol II, Chapter XIII). *The function* $\omega$ *is the Laplace transform of an infinitely divisible probability distribution iff*

$$\omega = e^{-\Psi}$$

*where* $\Psi$ *has a completely monotone derivative and* $\Psi(0) = 0$

**Remark 4.3.** *: In our situation*

$$P_0(t) = E(e^{-\lambda t}) = L_\Lambda(t) \tag{4.13}$$

*and*

$$P_0(t) = e^{-\theta(t)}$$

$\therefore$

$$L_\Lambda(t) = e^{-\theta(t)}$$

$$L_\Lambda(0) = e^{-\theta(0)}$$

*i.e*

$$1 = e^{-\theta(0)}$$

*and*

$$\theta(0) = 0$$

The derivative of $\theta(t)$ is $\theta'(t) = h(t)$ which is the hazard function of an exponential mixture.

Hesselager et. al. (1998) have stated the following theorem: "A distribution with a completely monotone hazard function is a mixed exponential distribution".

**Remark 4.4.** *The theorem implies that a hazard function of an exponential mixture is completely monotone.*

According to proposition 4.1, $\theta'(t) = h(t)$ is completely monotone and $\theta(0) = 0$ Therefore

$$P_0(t) = L_\Lambda(t)$$

is the Laplace transform of an infinitely divisible distribution.

Thus the mixing distribution, $g(\lambda)$ is infinitely divisible.

**Proposition 4.2** (Maceda, 1948). *If in a Poisson mixture the mixing distribution is infinitely divisible, the resulting mixture is also infinitely divisible.*

**Proposition 4.3** (Feller, 1968; Ospina and Gerbes, 1987). *Any discrete infinitely divisible distribution can arise as a compound Poisson distribution.*

From the above discussion we have the following:

**Theorem 4.1.** *Mixed Poisson distributions expressed in terms of Laplace transforms can also be expressed in terms of hazard functions of exponential mixtures. Such Poisson mixtures are infinitely divisible and hence are compound Poisson distributions.*

## 4.5 Compound Poisson Distribution

Let

$$\begin{aligned} Z(t) &= Z_{N(t)} \\ &= X_1 + X_2 + \cdots + X_{N(t)} \end{aligned} \tag{4.14}$$

where $\mathbf{X_i's}$ are iid random variables and $\mathbf{N(t)}$, is also a random variable independent of $\mathbf{X_is}$

Then $\mathbf{Z_{N(t)}}$ is said to have a compound Poisson distribution.

### 4.5.1 Compound Poisson Distribution in terms of pgf

Let

$$\begin{aligned} H(s,t) &= E[s^{Z_{N(t)}}] \\ &= \sum_{n=0}^{\infty} P_n(t)\ s^n \\ &= \text{the pgf of} \quad Z_{N(t))} \\ F(s,t) &= E(s^{N(t)}) \\ &= \sum_{j=0}^{\infty} f_j\ s^j \\ &= \text{the pgf of} \quad N(t) \end{aligned}$$

and,

$$\begin{aligned} G(s,t) &= E(s^{X_i}) \\ &= \sum_{x=0}^{\infty} g_x(t)\ s^x \\ &= \text{the pgf of} \quad X_i \end{aligned}$$

It can be proved that

$$\boldsymbol{H(s,t) = F[G(s,t)]} \tag{4.15}$$

If $\mathbf{N(t)}$ is Poisson with parameter $\boldsymbol{\theta}(\mathbf{t})$, then

$$\boldsymbol{H(s,t) = e^{-\theta(t)[1-G(s,t)]}} \tag{4.16}$$

### 4.5.2 The Distribution of iid Random Variables of the Compound Poisson Distribution

Since an infinitely divisible mixed Poisson distribution is also a compound Poisson distribution, we equate their pgfs given in 4.9 and 4.16 i.e

$$\boldsymbol{e^{-\theta(t-t\,s)} = e^{-\theta(t)[1-G(s,t)]}}$$

$$\boldsymbol{G(s,t) = 1 - \frac{\theta(t-t\,s)}{\theta(t)}} \tag{4.17}$$

which is the probability generating function of iid random variables expressed in terms of the cumulative hazard function of the exponential mixture.

The corresponding probability mass functions, which are the coefficients of $\boldsymbol{G(s,t)}$ obtained by differentiation method, are:

$$\boldsymbol{g_0(t) = 0} \tag{4.18}$$

$$\boldsymbol{g_x(t) = \frac{1}{x!}\,\frac{d^x}{ds^x}\,G(s,t)|_{s=0} \qquad for\;\; x = 1, 2, 3, \ldots} \tag{4.19}$$

### 4.5.3 Compound Poisson Distribution in Recursive Form

By differentiating equation 4.16,

$$\boldsymbol{H^{'}(s,t) = \theta(t)\; G^{'}(s,t)\; H(s,t)} \tag{4.20}$$

$$\boldsymbol{G^{'}(s,t) = \frac{d}{ds}\,G(s,t)}$$

By definition

$$\boldsymbol{H(s,t) = \sum_{n=0}^{\infty} p_n(t)\; s^n \quad \therefore \quad H'(s,t) = \sum_{n=1}^{\infty} n\; p_n(t)\; s^{n-1}}$$
$$\boldsymbol{G(s,t) = \sum_{x=0}^{\infty} g_x(t)\; s^x \quad \therefore \quad G'(s,t) = \sum_{x=1}^{\infty} x\; g_x(t)\; s^{x-1}} \tag{4.21}$$

By comparing 4.20 to 4.21 we have

$$\boldsymbol{\sum_{n=1}^{\infty} n\; p_n(t)\; s^{n-1} = \theta(t)\; G'(s,t)\; H(s,t)}$$

$$= \theta(t)\,[\sum_{x=1}^{\infty} x\, g_x(t)\, s^{x-1}] \sum_{n=0}^{\infty} p_n(t)\, s^n$$

$$= \theta(t) \sum_{x=1}^{\infty} x\, g_x(t)\, s^{x-1} \sum_{n=x}^{\infty} p_{n-x}(t)\, s^{n-x}$$

$$= \theta(t) \sum_{x=1}^{\infty} \sum_{n=x}^{\infty} x\, g_x(t)\; p_{n-x}(t)\, s^{n-1}$$

$$= \theta(t) \sum_{x=1}^{\infty} \sum_{n=1}^{\infty} x\, g_x(t)\; p_{n-x}(t)\, s^{n-1}$$

$$\sum_{n=1}^{\infty} n\, p_n(t)\, s^{n-1} = \theta(t) \sum_{n=1}^{\infty} \sum_{x=1}^{\infty} x\, g_x(t)\; p_{n-x}(t)\, s^{n-1}$$

$$\sum_{n=1}^{\infty} n\, p_n(t)\, s^{n} = \theta(t) \sum_{n=1}^{\infty} \sum_{x=1}^{\infty} x\, g_x(t)\; p_{n-x}(t)\, s^{n}$$

$$\sum_{n=1}^{\infty} n\, p_n(t)\, s^{n} = \sum_{n=1}^{\infty} [\theta(t) \sum_{x=1}^{\infty} x\, g_x(t)\; p_{n-x}(t)]\, s^{n}$$

$$n\, p_n(t) = \theta(t) \sum_{x=1}^{\infty} x\, g_x(t)\; p_{n-x}(t)$$

Since $x$ cannot exceed $n$,

$$n\, p_n(t) = \theta(t) \sum_{x=1}^{n} x\, g_x(t)\; p_{n-x}(t) \qquad n = 1, 2, 3, \ldots \tag{4.22}$$

Replace $n$ with $n+1$

$$(n+1)\, p_{(n+1)}(t) = \theta(t) \sum_{x=1}^{(n+1)} x\, g_x(t)\; p_{(n+1)-x}(t) \qquad n = 0, 1, 2, 3, \ldots \tag{4.23}$$

Put $x = i + 1$

$$(n+1)\, p_{n+1}(t) = \theta(t) \sum_{i=0}^{n} (i+1)\, g_{i+1}(t)\; p_{n-i}(t) n \qquad = 0, 1, 2, \ldots \tag{4.24}$$

This is the compound Poisson distribution expressed recursively in terms of the pmf of the iid random variables and the hazard function of the exponential mixture.

Equation 4.22 can be used to obtain $p_n(t)$ iteratively and also obtain the mean and the variance of the mixed Poisson distribution.

### 4.5.4 Panjer's Recursive Model

Let

$$p_n = (a + \frac{b}{n}\, p_{n-1}); \qquad n = 1, 2, 3, \ldots$$

where $a$ and $b$ are real numbers.

This recursive model is known as Panjer's recursive model of class zero denoted by $(a, b, 0)$ class of distributions.

We can extend it to

$$p_n = (a + \frac{b}{n}\, p_{n-1}); \qquad n = 2, 3, \ldots$$

which is Panjer's $(a, b, 1)$ class.

In general, we have

$$p_n = (a + \frac{b}{n}\, p_{n-1}); \qquad n = k+1, k+2, \ldots \tag{4.25}$$

which is Panjer's $(a, b, k)$ class for $k = 0, 1, 2, \ldots$
Some probability mass functions in this paper take Panjer's recursive form.

**Remark 4.5.** *In actuarial literature, this recursive relation is due to the work of Panjer (1981). In statistical literature, however, this relation had been published by Katz (1965) based on his PhD dissertation of 1945.*

## 4.6 Hofmann Hazard Function

A class of mixed Poisson distributions known as Hofmann distribution has been described by Walhin and Paris (1999) as:

$$p_0(t) = e^{-\theta(t)}$$

and,

$$p_n(t) = (-1)^n \frac{t^n}{n!} p_0^n(t) \qquad n = 1, 2, \ldots$$

where,

$$\theta'(t) = \frac{p}{(1+ct)^a} \qquad for \;\; p > 0, \; c > 0 \; and \; a \geq 0 \tag{4.26}$$

and,

$$\theta(0) = 0 \tag{4.27}$$

**Remark 4.6.** *From 4.6,* $\theta'(\mathrm{t})$ *is a hazard function of an exponential mixture. We shall refer to it as Hofmann hazard function of an exponential mixture; i.e*

$$\theta'(\mathrm{t}) = \mathrm{h(t)} = \frac{\mathrm{p}}{(1+\mathrm{ct})^{\mathrm{a}}} \qquad \text{for } \; \mathrm{p} > 0, \; \mathrm{c} > 0 \text{ and } \mathrm{a} \geq 0 \tag{4.28}$$

We will now consider various Parameterizations of $a$

### 4.6.1 When the hazard function in (4.26)is such that $a = 0$, $p > 0$ and $c > 0$,

$$\begin{aligned} \theta'(t) &= h(t) \\ &= p, \qquad \text{a constant} \end{aligned} \tag{4.29}$$

which is the hazard function of the exponential distribution with parameter **p**

Therefore,

$$h^{(n)}(t) = 0$$

$$(-1)^n \frac{d^n}{dt^n} h(t) = (-1)^n h^n(t) \geq 0$$

Therefore $\mathbf{h(t)}$ is completely monotone.

The cumulative hazard function is therefore

$$\begin{aligned} \theta(t) &= p \int_0^t dx \\ &= pt \end{aligned} \tag{4.30}$$

implying that

$$\begin{aligned} \theta\ (t - ts) &= p\, t\ (1 - s) \\ \theta(0) &= 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = 0 \\ \therefore \qquad p_0(t) &= e^{-\theta(t)} \\ &= e^{-pt} \\ p_0'(t) &= (-1)\ p\ e^{-pt} \\ p_0''(t) &= (-1)^2\ p^2\ e^{-pt} \\ p_0^{(n)}(t) &= (-1)^n\ p^n\ e^{-pt} \\ p_0^{(n)}(t) &= (-1)^n\ p^n\ e^{-pt} \end{aligned} \tag{4.31}$$

Therefore,

$$\begin{aligned} p_n(t) &= (-1)^n \frac{t^n}{n!} p_0^{(n)}(t) \\ p_n(t) &= \frac{e^{-pt}\ (pt)^n}{n!} \end{aligned} \tag{4.32}$$

which is a Poisson distribution with parameter **pt**.

$$H(s,t) = e^{-\theta\ (t - ts)}$$

$$= e^{-p\,t\,(1-s)} \tag{4.33}$$

which is the pgf of Poisson distribution with parameter **pt**

Therefore,

$$E[z(t)] = pt \qquad and \qquad Var(z_{N(t)}) = pt \tag{4.34}$$

Since $\theta(0) = 0$ and $\mathbf{h(t)} = \theta'(\mathbf{t})$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$\begin{aligned} G(s,t) &= 1 - \frac{pt\,(1-s)}{pt} \\ &= s \end{aligned}$$

The coefficients of $G(s,t)$ are

$$\begin{aligned} g_x(t) &= 1 \qquad x = 1 \\ g_x(t) &= 0 \qquad x \neq 1 \end{aligned}$$

From 4.22, the compound Poisson distribution in recursive form is

$$n\,p_n(t) = \theta(t)\,\sum_{x=1}^{n}\,x\,g_x(t)\;\;p_{n-x}(t)$$

$$\therefore \qquad n\,p_n(t) = \theta(t)\,p_{n-1}(t) \qquad for\;\; n = 1, 2, 3 \ldots \tag{4.35}$$

By iteration technique

For n=1

$$p_1(t) = \theta(t)\,p_0(t)$$

For n=2

$$\begin{aligned} 2\,p_2(t) &= \theta(t)\,p_1(t) \\ &= (\theta(t))^2\,p_0(t) \\ p_2(t) &= \frac{(\theta(t))^2}{2!}\,p_0(t) \end{aligned}$$

For n=3

$$\begin{aligned} 3\,p_3(t) &= \theta(t)\,p_2(t) \\ &= \frac{(\theta(t))^3}{2!}\,p_0(t) \end{aligned}$$

$$p_3(t) = \frac{(\theta(t))^3}{3!}\, p_0(t)$$

By induction, assume it is true for $n-1$

$$p_{n-1}(t) = \frac{(\theta(t))^{n-1}}{(n-1)!}\, p_0(t)$$

From (4.35)

$$n\, p_n(t) = \theta(t)\; \frac{(\theta(t))^{n-1}}{(n-1)!}\, p_0(t)$$

$$\therefore \qquad p_n(t) = \frac{(\theta(t))^{n}}{(n)!}\, p_0(t)$$

$$= \frac{(\theta(t))^{n}}{(n)!}\, e^{-\theta(t)}$$

and,

$$p_n(t) = \frac{e^{-pt}\,(pt)^n}{n!} n \qquad = 0, 1, 2 \ldots \tag{4.36}$$

which is a Poisson with parameter $pt$

**Moments**

$$E(Z(t))) = \sum_{n=1}^{\infty} n\, p_n(t)$$

Sum equation (4.35) over $n$; that is

$$\sum_{n=1}^{\infty} n\, p_n(t) = \sum_{n=1}^{\infty} \theta(t)\, p_{n-1}(t)$$

$$= pt \sum_{n=1}^{\infty} p_{n-1}(t)$$

$$E(Z(t))) = \theta(t) = pt$$

Multiply (4.35) by $n$ and then sum the result over $n$; that is

$$E(Z(t))^2 = \sum_{n=1}^{\infty} n^2\, p_n(t)$$

$$= \sum_{n=1}^{\infty} n\, n\, p_n(t)$$

$$= \sum_{n=1}^{\infty} n\, [pt\, p_{n-1}(t)] = pt \sum_{n=1}^{\infty} [n-1+1]\, [p_{n-1}(t)]$$

$$= pt \sum_{n=1}^{\infty} [n-1]\, p_{n-1}(t) + pt \sum_{n=1}^{\infty} p_{n-1}(t)$$

$$= pt\, E(Z(t))) + pt$$

$$E(Z(t))^2 = (pt)^2 + pt$$

$$Var(Z(t)) = E(Z(t))^2 - (E(Z(t))^2 = pt \tag{4.37}$$

### 4.6.2 When the hazard function in (4.26) is such that $a = 1,\ p = c > 0$

$$h(t) = \theta'(t) = \frac{p}{1+pt} \tag{4.38}$$

which is a hazard function of Pareto (exponential-exponential) distribution, with parameters $p$ and $p$.

Therefore,

$$\begin{aligned}
h'(t) &= -p^2\ (1+pt)^{-2} \\
h''(t) &= (-1)^2\ p^3\ (1+pt)^{-3} \\
h^{(n)}(t) &= (-1)^n\ p^{n+1}\ (1+pt)^{-(n+1)} \\
(-1)^n\ \frac{d^n}{dt^n}\ h(t) &= (-1)^n\ h^n(t) \geq 0
\end{aligned}$$

Therefore $\mathbf{h(t)}$ is completely monotone.

The cumulative hazard function is

$$\begin{aligned}
\theta(t) &= \int_0^t \frac{p}{1+px}\ dx \\
&= In(1+pt) \qquad (4.39)
\end{aligned}$$

Implying that,

$$\theta(t-ts) = In(1+pt-pts) \tag{4.40}$$

Therefore,

$$\begin{aligned}
&\theta(0) = 0, \qquad \theta'(0) = p \qquad \text{and} \qquad \theta''(0) = -p^2 \\
&p_0(t) = e^{-In(1+pt)} = \frac{1}{1+pt} \\
&p_0'(t) = (-1)^1\ (p)^1\ (1+pt)^{-2} \\
&p_0''(t) = (-1)^2\ (p)^2\ (1+pt)^{-3} \\
&p_0'''(t) = (-1)^3\ (p)^3\ (1+pt)^{-4} \\
&p_0^{(n)}(t) = (-1)^n\ n!\ p^n\ (1+pt)^{-n-1}
\end{aligned}$$

Therefore,

$$p_n(t) = (\frac{pt}{1+pt})^n\ \frac{1}{1+pt} \qquad n = 0,1,2,3,\ldots \tag{4.41}$$

which is a geometric(Poisson-exponential) distribution with parameter $\frac{pt}{1+pt}$

The pgf is given by

$$\boldsymbol{H(s,t) = (\frac{\frac{1}{1+pt}}{1-\frac{pt}{1+pt}\ s})} \tag{4.42}$$

which is the pgf of a geometric distribution with parameter $\boldsymbol{\frac{1}{1+pt}}$

Therefore,

$$\boldsymbol{E[z(t) = pt} \qquad \text{and} \qquad \boldsymbol{Var(Z(t)) = pt + (pt)^2}$$

Since $\boldsymbol{\theta(0) = 0}$ and $\mathbf{h(t) = \boldsymbol{\theta}'(t)}$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$\boldsymbol{G(s,t) = 1 - \frac{In(1+pt-pts))}{In\ (1+pt)}} \tag{4.43}$$

By power series expansion,

$$\boldsymbol{G(s,t) = \sum_{x=1}^{\infty} \frac{(\frac{pt}{1+pt})^x}{-x\ In\ (1-\frac{pt}{1+pt})}\ s^x}$$

$$\boldsymbol{g_x(t) = \frac{(\frac{pt}{1+pt})^x}{-x\ In\ (1-\frac{pt}{1+pt})}} \tag{4.44}$$

which is logarithmic series distribution with parameter $\boldsymbol{\frac{1}{1+pt}}$.

By the differentiation method, we have

$$\boldsymbol{G^x(s,t) = \frac{(x-1)!}{In(1+pt)}\ (\frac{pt}{1+pt})^x\ (1-\frac{pt}{1+pt}\ s)^{-x}}$$

$$\boldsymbol{g_x(t) = \frac{(\frac{pt}{1+pt})^x}{-x\ In(1-\frac{pt}{1+pt})}} \qquad \boldsymbol{x = 1, 2 \ldots}$$

$$\boldsymbol{\frac{g_x(t)}{g_{x-1}(t)} = \frac{x-1}{x}\ \frac{pt}{1+pt} = \frac{pt}{1+pt}\ (1-\frac{1}{x})}$$

$$\boldsymbol{g_x(t) = (a+\frac{b}{x})\ g_{x-1}(t)} \qquad \text{for} \quad \boldsymbol{x = 2, 3, \ldots}$$

which is Panjer's recursive model with

$$\boldsymbol{a = \frac{pt}{1+pt}} \qquad \text{and} \qquad \boldsymbol{b = -\frac{pt}{1+pt}} \tag{4.45}$$

The compound Poisson distribution in recursive form is:

$$\boldsymbol{n\, p_n(t) = In(1+pt)\sum_{x=1}^{n} \frac{x\,(\frac{pt}{1+pt})^x}{x\, In(1+pt)}\, p_{n-x}(t)}$$

$$\boldsymbol{n\, p_n(t) = \sum_{x=1}^{n} (\frac{pt}{1+pt})^x\, p_{n-x}(t)}\text{for} \quad \boldsymbol{x = 2, 3, \ldots} \tag{4.46}$$

For n=1,

$$\boldsymbol{p_1(t) = p\,\frac{pt}{1+pt}\, p_0(t)}$$

For n=2,

$$\boldsymbol{p_2(t) = (\frac{t}{1+t})^2\, p_0(t)}$$

By induction, assume it is true for $\boldsymbol{n-1}$

$$\boldsymbol{p_{n-1}(t) = (\frac{t}{1+t})^{n-1}\, p_0(t)}$$

$$\boldsymbol{where \qquad p_0(t) = e^{-In(1+pt)} = \frac{1}{1+pt}}$$

$$\boldsymbol{p_{n-1}(t) = (\frac{t}{1+t})^{n-1}\,\frac{1}{1+pt}}$$

$$\boldsymbol{p_n(t) = (\frac{pt}{1+pt})^n\,\frac{1}{1+pt} n} \qquad \boldsymbol{= 0, 1, 2, \ldots} \tag{4.47}$$

which is the geometric distribution with parameter $\mathbf{\frac{1}{1+pt}}$

**Moments**

Sum the recursive relation (4.46) over **n**; thus

$$\boldsymbol{\sum_{n=1}^{\infty} n\, p_n(t) = \frac{\frac{pt}{1+pt}}{1-\frac{pt}{1+pt}}}$$

$$\boldsymbol{E(Z(t)) = pt} \tag{4.48}$$

Next, multiply the recursive relation (4.46) by **n** and then sum the result over **n**; thus

$$\boldsymbol{E(Z(t))^2 = (pt)^2 + pt + (pt)^2} \qquad \text{and} \qquad \boldsymbol{Var(Z(t)) = pt + (pt)^2} \tag{4.49}$$

### 4.6.3 When the hazard function in (4.26) is such that $a = 1$, $c = 1$ and $p > 0$

$$\begin{aligned}\theta'(t) &= h(t)\\ &= \frac{p}{1+t} \qquad (4.50)\end{aligned}$$

which is a hazard function of Pareto (exponential-gamma) distribution, with parameters $p$ and $\mathbf{1}$.

Therefore,

$$\begin{aligned}h'(t) &= -p^1\,(1+pt)^{-2}\\ h''(t) &= (-1)^2\,p^2\,(1+pt)^{-3}\\ \textit{and} \qquad h^{(n)}(t) &= (-1)^n\,p^n\,(1+pt)^{-(n+1)}\\ (-1)^n\,\frac{d^n}{dt^n}\,h(t) &= (-1)^n\,h^n(t) \geq 0\end{aligned}$$

Therefore $\mathbf{h(t)}$ is completely monotone.

The cumulative hazard function is

$$\begin{aligned}\theta(t) &= p\int_0^t \frac{1}{1+x}\,dx\\ &= p\,In(1+t) \qquad (4.51)\end{aligned}$$

implying that,

$$\theta(t-ts) = p\,In(1+t-ts) \qquad (4.52)$$

Therefore,

$$\begin{aligned}\theta(0) &= 0, \qquad \theta'(0) = p \qquad \textit{and} \qquad \theta''(0) = -p\\ p_0(t) &= e^{-\theta(t)}\\ &= e^{-p\,In(1+t)}\\ &= (1+t)^{-p}\\ p_0'(t) &= (-1)\;\;p\,(1+t)^{-p-1}\\ &= (-1)\;\;\frac{p\,\Gamma(p)}{\Gamma(p)}\,(1+t)^{-p-1}\\ &= (-1)\;\;\frac{\Gamma(p+1)}{\Gamma(p)}\,(1+t)^{-p-1}\\ p_0''(t) &= (-1)^2\,\frac{\Gamma(p+2)}{\Gamma(p)}\,(1+t)^{-p-2}\end{aligned}$$

$$\textit{and hence} \qquad p_0^{(n)}(t) = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)} (1+t)^{-p-n}$$
$$= (-1)^n \; n! \frac{(p+n-1)!)}{n!\,(p-1)!} (1+t)^{-p-n}$$
$$p_n(t) = (-1)^n \frac{t^n}{n!} (-1)^n \; n! \frac{(p+n-1)!)}{n!\,(p-1)!} (1+t)^{-p-n}$$
$$p_n(t) = \binom{p+n-1}{n} (\frac{t}{1+t})^n (\frac{1}{1+t})^p \qquad for \quad n = 0,1,2,3,\ldots \tag{4.53}$$

which is a negative binomial (Poisson-gamma) distribution with parameters $p$ and $\frac{1}{1+t}$

The pgf is given by

$$H(s,t) = e^{-\theta(t-ts)}$$
$$= e^{-p\, In\,(1+t-ts)}$$
$$= (\frac{1}{1+t-ts})^p$$
$$= (\frac{\frac{1}{1+t}}{1-\frac{t}{1+t}\, s})^p \tag{4.54}$$

which is the pgf of a negative binomial distribution with parameters $p$ and $\frac{1}{1+t}$

$$E[z(t)] = t\,\theta'(0)$$
$$= pt$$
$$Var(Z(t)) = t\,\theta'(0) - t^2\,\theta''(0)$$
$$= tp + t^2\,p$$
$$= pt + pt^2 \tag{4.55}$$

Since $\theta(0) = 0$ and $\mathbf{h(t)} = \theta'(\mathbf{t})$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{In(1+t-ts))}{In\,(1+t)} \tag{4.56}$$

By power series expansion

$$G(s,t) = 1 - \frac{In[(1+t)(1-\frac{t}{1+t}\, s)]}{In(1+t)}$$

$$= 1 - \frac{In(1+t) + In(1 - \frac{t}{1+t}\ s)}{In(1+t)} = \frac{-In(1 - \frac{t}{1+t}\ s)}{In(1+t)}$$

$$= \sum_{x=1}^{\infty} \frac{1}{In\ (\frac{1+t}{1})} \frac{(\frac{t}{1+t})^x}{x}\ s^x = \sum_{x=1}^{\infty} \frac{(\frac{t}{1+t})^x}{-x\ In\ (1 - \frac{t}{1+t})}\ s^x$$

$$g_x(t) = \frac{(\frac{t}{1+t})^x}{-x\ In\ (1 - \frac{t}{1+t})} \qquad for \quad x = 1, 2, 3, \ldots \tag{4.57}$$

which is logarithmic series distribution with parameter $\frac{1}{1+t}$.

By the differentiation method, we have

$$G'(s,t) = \frac{\partial}{\partial\ s}\ G(s,t) = -\frac{1}{In(1+t)} \frac{-\frac{t}{1+t}}{1 - \frac{t}{1+t}\ s}$$

$$\frac{\partial\ G(s,t)}{\partial\ s} = \frac{\frac{t}{1+t}}{In(1+t)}\ (1 - \frac{t}{1+t}\ s)^{-1}$$

$$\frac{\partial^2\ G(s,t)}{\partial\ s^2} = \frac{\frac{t}{1+t}}{In(1+t)}\ (\frac{t}{1+t})\ (1 - \frac{t}{1+t}\ s)^{-2}$$

$$= \frac{1}{In(1+t)}\ (\frac{t}{1+t})^2\ (1 - \frac{t}{1+t}\ s)^{-2}$$

$$\frac{\partial^3\ G(s,t)}{\partial\ s^3} = \frac{2}{In(1+t)}\ (\frac{t}{1+t})^3\ (1 - \frac{t}{1+t}\ s)^{-3}$$

$$\frac{\partial^4\ G(s,t)}{\partial\ s^4} = \frac{3}{In(1+t)}\ (\frac{t}{1+t})^4\ (1 - \frac{t}{1+t}\ s)^{-4}$$

$$\frac{\partial^x\ G(s,t)}{\partial\ s^x} = \frac{(x-1)!}{In(1+t)}\ (\frac{t}{1+t})^x\ (1 - \frac{t}{1+t}\ s)^{-x}$$

$$g_x(t) = \frac{1}{x!}\ \frac{\partial^x\ G(s,t)}{\partial\ s^x}|_{s=0} = \frac{1}{x!}\ \frac{(x-1)!}{In(1+t)}\ (\frac{t}{1+t})^x$$

$$= \frac{1}{x}\ \frac{1}{In(1+t)}\ (\frac{t}{1+t})^x$$

$$= \frac{(\frac{t}{1+t})^x}{x\ In(1+t)} = \frac{(\frac{t}{1+t})^x}{-x\ In \frac{1}{1+t}} \qquad x = 1, 2 \ldots$$

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{x-1}{x}\ \frac{t}{1+t} = \frac{t}{1+t}\ (1 - \frac{1}{x})$$

$$g_x(t) = (a + \frac{b}{x})\ g_{x-1}(t) \qquad for \quad x = 2, 3, \ldots \tag{4.58}$$

which is Panjer's recursive model with

$$a = \frac{t}{1+t} \qquad and \qquad b = -\frac{t}{1+t}$$

The compound Poisson distribution in recursive form is:

$$
\begin{aligned}
n\, p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\ \ p_{n-x}(t) \\
&= p\, In(1+t) \sum_{x=1}^{n} \frac{x\, (\frac{t}{1+t})^x}{x\, In(1+t)}\, p_{n-x}(t) \\
n\, p_n(t) &= p \sum_{x=1}^{n} (\frac{t}{1+t})^x\, p_{n-x}(t) \qquad for \quad n = 1, 2, 3, \ldots
\end{aligned} \tag{4.59}
$$

By iteration

For n=1

$$p_1(t) = p\, \frac{t}{1+t}\, p_0(t)$$

For n=2

$$
\begin{aligned}
2\, p_2(t) &= \theta(t) \sum_{x=1}^{2} x\, g_x(t)\ \ p_{2-x}(t) \\
&= \theta(t)\ \ g_1(t)\ \ p_1(t) + 2\, \theta(t)\, g_2(t)\ \ p_0(t) \\
2\, p_2(t) &= p\, \frac{t}{1+t}\, p_1(t) + p\, (\frac{t}{1+t})^2\, p_0(t) = (\frac{t}{1+t})^2\ [p\ (p+1) p_0(t) \\
p_2(t) &= \frac{1}{2} (\frac{t}{1+t})^2\ [p\ (p+1) p_0(t) = \binom{p+1}{2}\ (\frac{t}{1+t})^2\, p_0(t)
\end{aligned}
$$

For n=3

$$
\begin{aligned}
3\, p_3(t) &= \theta(t) \sum_{x=1}^{3} x\, g_x(t)\ \ p_{3-x}(t) \\
&= \theta(t)\ \ g_1(t)\ \ p_2(t) + 2\, \theta(t)\, g_2(t)\ \ p_1(t) + 3\, \theta(t)\, g_3(t)\ \ p_0(t) \\
3\, p_3(t) &= p\, \frac{t}{1+t}\, p_2(t) + p\, (\frac{t}{1+t})^2\, p_1(t) + p\, (\frac{t}{1+t})^3\, p_0(t) \\
&= p\, \frac{t}{1+t}\ [\binom{p+1}{2}\ (\frac{t}{1+t})^2\, p_0(t)] + p\, (\frac{t}{1+t})^2\ [p\, \frac{t}{1+t}\, p_0(t)] + \\
&= p\, [\binom{p+1}{2} + \binom{p+1}{1}]\ (\frac{t}{1+t})^3\, p_0(t)
\end{aligned}
$$

But

$$\binom{r+1}{j} = \binom{r}{j} + \binom{r}{j-1}$$

Therefore,

$$3\, p_3(t) = p\, \binom{p+2}{2}\ (\frac{t}{1+t})^3\, p_0(t)$$

$$= \frac{p\,(p+2)(p+1)}{2!}\,(\frac{t}{1+t})^3\,p_0(t) = \frac{(p+2)(p+1)p}{2!}\,(\frac{t}{1+t})^3\,p_0$$

$$p_3(t) = \frac{(p+2)(p+1)p}{3!}\,(\frac{t}{1+t})^3\,p_0(t)$$

$$= \binom{p+2}{3}\,(\frac{t}{1+t})^3\,p_0(t)$$

By induction, assume that it is true for $n-1$

$$p_{n-1}(t) = \binom{p+n-2}{n-1}\,(\frac{t}{1+t})^{n-1}\,p_0(t)$$

$$\therefore \qquad n\,p_n(t) = p\,\sum_{x=1}^{n}\,(\frac{t}{1+t})^x\,p_{n-x}(t)$$

$$= p\,\frac{t}{1+t}\,p_{n-1}(t) + (\frac{t}{1+t})^2\,p_{n-2}(t) + \cdots +$$

$$(\frac{t}{1+t})^{n-1}\,p_1(t) + (\frac{t}{1+t})^n\,p_0(t)$$

$$= p\,[\binom{p+n-2}{n-1} + \binom{p+n-3}{n-2} + \cdots +$$

$$\binom{p+2}{3} + \binom{p+1}{2} + \binom{p+1}{1}](\frac{t}{1+t})^n\,p_0(t)$$

$$= p\,[\binom{p+n-2}{n-1} + \binom{p+n-3}{n-2} + \cdots +$$

$$\binom{p+2}{3} + \binom{p+2}{2}](\frac{t}{1+t})^n\,p_0(t)$$

$$= p\,[\binom{p+n-2}{n-1} + \binom{p+n-3}{n-2} + \binom{p+3}{3}](\frac{t}{1+t})^n\,p_0(t)$$

$$= p\,[\binom{p+n-2}{n-1} + \binom{p+n-3}{n-2} + \binom{p+n-3}{n-1}](\frac{t}{1+t})^n\,p_0(t)$$

$$= p\,[\binom{p+n-2}{n-1} + \binom{p+n-2}{n-2}](\frac{t}{1+t})^n\,p_0(t)$$

$$= \frac{1}{n}\,p\,\frac{(p+n-1)\,(p+n-2)\ldots(p+n-(n-1))(p)}{(n-1)!}\,(\frac{t}{1+t})^n\,p$$

$$= \binom{p+n-1}{n}\,(\frac{t}{1+t})^n\,p_0(t)$$

$$n\,p_n(t) = \binom{p+n-1}{n}\,(\frac{t}{1+t})^n\,(\frac{1}{1+t})^p \qquad (4.60)$$

which is the negative binomial distribution with parameters $\mathbf{p}$ and $\frac{1}{1+t}$

The negative binomial distribution is a special example of a mixed Poisson distribution. When an aggregate claims variable $z_N = Y_1 + Y_2 + \ldots.. + Y_N$ has a mixed Poisson distribution, the number of claims $N$ follows a Poisson distribution, but the Poisson parameter $\Lambda$ is uncertain. The uncertainty could be due to an heterogeneity of risks across the insureds in the insurance portfolio (or across various rating classes). If the information of the risk parameter $\Lambda$ can be captured in a gamma distribution, then the unconditional number of claims in a given fixed period has a negative binomial distribution.

**Moments**

Let,

$$
\begin{aligned}
& M_1(t) = \sum_{n=1}^{\infty} n\, p_n(t) \\
and \qquad & M_2(t) = \sum_{n=1}^{\infty} n^2\, p_n(t)
\end{aligned}
$$

Sum equation (4.59) over n

$$
\begin{aligned}
\sum_{n=1}^{\infty} n\, p_n(t) &= p \sum_{n=1}^{\infty} \sum_{x=1}^{n} (\frac{t}{1+t})^x\, p_{n-x}(t) \\
&= p \sum_{x=1}^{n} (\frac{t}{1+t})^x \sum_{n=x}^{\infty} p_{n-x}(t) \\
&= p \sum_{x=1}^{n} (\frac{t}{1+t})^x \\
&= p\, \frac{\frac{t}{1+t}}{1-\frac{t}{1+t}} \\
E(Z(t)) &= pt \\
E(Z(t))^2 &= \sum_{n=1}^{\infty} n^2\, p_n(t) \\
\sum_{n=0}^{\infty} n^2\, p_n(t) &= \sum_{n=0}^{\infty} n\, n\, p_n(t) = p \sum_{n=1}^{\infty} n\, [\sum_{x=1}^{n} (\frac{t}{1+t})^x\, p_{n-x}(t)] \\
&= p \sum_{x=1}^{n} (\frac{t}{1+t})^x \sum_{n=x}^{\infty} [n-x+x]\, p_{n-x}(t) \\
&= p \sum_{x=1}^{n} (\frac{t}{1+t})^x\, [\sum_{n=x}^{\infty} [n-x]\, p_{n-x}(t) + \sum_{n=x}^{\infty} x\, p_{n-x}(t)] \\
&= p \sum_{x=1}^{n} (\frac{t}{1+t})^x\, [M_1(t) + x] = p\, M_1(t) \sum_{x=1}^{n} (\frac{t}{1+t})^x + p \sum_{x=1}^{n} x \\
&= p\, M_1(t)\, \frac{t}{1+t}\, \frac{1}{1-\frac{t}{1+t}} + p\, \frac{t}{1+t}\, (\frac{1}{1-\frac{t}{1+t}})^2
\end{aligned}
$$

$$= p\,pt\,\frac{t}{1+t}\,(1+t) + p\,\frac{t}{1+t}\,(1+t)^2$$
$$= p^2\,t^2 + pt(1+t) = (pt)^2 + pt + pt^2$$
$$But \qquad E(Z(t)) = M_1(t) = pt$$
$$Var(Z(t)) = M_2(t) - (M_1(t))^2 = (pt)^2 + pt + pt^2 - (pt)^2$$
$$= pt + pt^2 \tag{4.61}$$

### 4.6.4 When the hazard function in (4.26) is such that $a = 1$, $p > 0$ and $c > 0$

$$\theta'(t) = h(t) = \frac{p}{1+ct} \tag{4.62}$$

which is a hazard function of Pareto (exponential-gamma) distribution, with parameters $p$ and $c$.

Therefore

$$h'(t) = (-1)^1\,p\,(c)^1\,(1+ct)^{-2})$$
$$h''(t) = (-1)^2\,p\,(c)^2\,(1+ct)^{-3})$$
$$h'''(t) = (-1)^3\,p\,(c)^3\,(1+ct)^{-4})$$
$$\therefore \qquad h^{(n)}(t) = (-1)^n\,p\,(c)^n\,(1+ct)^{-(n+1)})$$
$$(-1)^n\,\frac{d^n}{dt^n}\,h(t) = (-1)^n\,h^n(t) \geq 0$$

Therefore $\mathbf{h(t)}$ is completely monotone.

The cumulative hazard function is

$$\theta(t) = p\int_0^t \frac{1}{1+cx}\,dx = \frac{p}{c}\,In(1+ct) \tag{4.63}$$

implying that,

$$\theta(t-ts) = \frac{p}{c}\,In(1+ct-cts) \tag{4.64}$$

Therefore,

$$\theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -pc$$
$$p_0(t) = e^{-\theta(t)} = e^{-\frac{p}{c}\,In(1+ct)}$$
$$p_0(t) = (1+ct)^{-\frac{p}{c}}$$
$$p_0'(t) = (-1)\,c\,\frac{p}{c}\,(1+ct)^{-\frac{p}{c}-1}$$

$$= (-1)\, c\, \frac{\frac{p}{c}\,\Gamma(\frac{p}{c})}{\Gamma(\frac{p}{c})}\, (1+ct)^{-\frac{p}{c}-1}$$

$$= (-1)\, c\, \frac{\Gamma(\frac{p}{c}+1)}{\Gamma(\frac{p}{c})}\, (1+ct)^{-\frac{p}{c}-1}$$

$$p_0''(t) = (-1)^2\, c^2\, \frac{\Gamma(\frac{p}{c}+2)}{\Gamma(\frac{p}{c})}\, (1+ct)^{-\frac{p}{c}-2}$$

$$\textit{Generally,} \qquad p_0^{(n)}(t) = (-1)^n\, c^n\, \frac{\Gamma(\frac{p}{c}+n)}{\Gamma(\frac{p}{c})}\, (1+ct)^{-\frac{p}{c}-n}$$

$$p_0^{(n)}(t) = (-1)^n\, c^n\, n!\, \frac{(\frac{p}{c}+n-1)!)}{n!\,(\frac{p}{c}-1)!}\, (1+ct)^{-\frac{p}{c}-n}$$

$$p_n(t) = (-1)^n \frac{t^n}{n!}\, (-1)^n\, c^n\, n!\, \frac{(\frac{p}{c}+n-1)!)}{n!\,(\frac{p}{c}-1)!}\, (1+ct)^{-\frac{p}{c}-n}$$

$$p_n(t) = \binom{\frac{p}{c}+n-1}{n}\, (\frac{ct}{1+ct})^n\, (\frac{1}{1+ct})^{\frac{p}{c}} \qquad for \quad n = 0,1,2,\ldots \tag{4.65}$$

which is a negative binomial (Poisson-gamma) distribution with parameter $\frac{p}{c}$ and $\frac{1}{1+ct}$

The pgf is given by

$$H(s,t) = e^{-\theta(t-ts)} = e^{-\frac{p}{c}\, In\,(1+ct-cts)}$$

$$= (\frac{1}{1+ct-cts})^{\frac{p}{c}}$$

$$= (\frac{\frac{1}{1+ct}}{1-\frac{ct}{1+ct}\, s})^{\frac{p}{c}} \tag{4.66}$$

which is the pgf of a negative binomial distribution with parameter $\frac{p}{c}$ and $\frac{1}{1+ct}$

$$E[z(t) = t\,\theta'(0) = t\,h(0) = tp = pt$$
$$Var(Z(t)) = t\,\theta'(0) - t^2\,\theta''(0) = tp + t^2\,pc = pt + pct^2 \tag{4.67}$$

Since $\theta(0) = 0$ and $\mathbf{h(t)} = \theta'(\mathbf{t})$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)}$$

$$= 1 - \frac{In(1+ct-cts))}{In\,(1+ct)} \tag{4.68}$$

By power series expansion

$$
\begin{aligned}
G(s,t) &= 1-\frac{In[(1+ct)(1-\frac{ct}{1+ct}\ s)]}{In(1+ct)} \\
&= 1-\frac{In(1+ct)+In(1-\frac{ct}{1+ct}\ s)}{In(1+ct)} = \frac{-In(1-\frac{ct}{1+ct}\ s)}{In(1+ct)} \\
&= \sum_{x=1}^{\infty}\frac{1}{In\ (\frac{1+ct}{1})}\ \frac{(\frac{ct}{1+ct})^x}{x}\ s^x = \sum_{x=1}^{\infty}\frac{(\frac{ct}{1+ct})^x}{-x\ In\ (1-\frac{ct}{1+ct})}\ s^x \\
g_x(t) &= \frac{(\frac{ct}{1+ct})^x}{-x\ In\ (1-\frac{ct}{1+ct})} \qquad for \quad x1,2,3,\ldots \qquad (4.69)
\end{aligned}
$$

which is logarithmic series distribution with parameter $\frac{1}{1+ct}$.

By the differentiation method, we have

$$
\begin{aligned}
G'(s,t) &= \frac{\partial}{\partial s}\ G(s,t) = -\frac{1}{In(1+ct)}\ \frac{-\frac{ct}{1+ct}}{1-\frac{ct}{1+ct}\ s} \\
\frac{\partial\ G(s,t)}{\partial s} &= \frac{\frac{ct}{1+ct}}{In(1+ct)}\ (1-\frac{ct}{1+ct}\ s)^{-1} \\
\frac{\partial^2\ G(s,t)}{\partial s^2} &= \frac{\frac{ct}{1+ct}}{In(1+ct)}\ (\frac{ct}{1+ct})\ (1-\frac{ct}{1+ct}\ s)^{-2} \\
&= \frac{1}{In(1+ct)}\ (\frac{ct}{1+ct})^2\ (1-\frac{ct}{1+ct}\ s)^{-2} \\
\frac{\partial^3\ G(s,t)}{\partial s^3} &= \frac{2}{In(1+ct)}\ (\frac{ct}{1+ct})^3\ (1-\frac{ct}{1+ct}\ s)^{-3} \\
\frac{\partial^4\ G(s,t)}{\partial s^4} &= \frac{3}{In(1+ct)}\ (\frac{ct}{1+ct})^4\ (1-\frac{ct}{1+ct}\ s)^{-4} \\
\frac{\partial^x\ G(s,t)}{\partial s^x} &= \frac{(x-1)!}{In(1+ct)}\ (\frac{ct}{1+ct})^x\ (1-\frac{ct}{1+ct}\ s)^{-x} \\
g_x(t) &= \frac{1}{x!}\ \frac{\partial^x\ G(s,t)}{\partial s^x}|_{s=0} = \frac{1}{x!}\ \frac{(x-1)!}{In(1+ct)}\ (\frac{ct}{1+ct})^x \\
&= \frac{1}{x}\ \frac{1}{In(1+ct)}\ (\frac{ct}{1+ct})^x \\
&= \frac{(\frac{ct}{1+ct})^x}{x\ In(1+ct)} = \frac{(\frac{ct}{1+ct})^x}{-x\ In\frac{1}{1+ct}} \qquad x=1,2\ldots \\
\frac{g_x(t)}{g_{x-1}(t)} &= \frac{x-1}{x}\ \frac{ct}{1+ct} = \frac{ct}{1+ct}\ (1-\frac{1}{x}) \\
g_x(t) &= (a+\frac{b}{x})\ g_{x-1}(t) \qquad for \quad x=2,3,\ldots \qquad (4.70)
\end{aligned}
$$

which is Panjer's recursive model with

$$a = \frac{ct}{1+ct} \qquad and \qquad b = -\frac{ct}{1+ct}$$

The compound Poisson distribution in recursive form is:

$$\begin{aligned} n\, p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\ \ p_{n-x}(t) \qquad for\ n = 1,2,3,\ldots \\ &= \frac{p}{c}\, In(1+ct) \sum_{x=1}^{n} \frac{x\, (\frac{ct}{1+ct})^x}{x\, In(1+ct)}\, p_{n-x}(t) \qquad for \quad x = 2,3,4\ldots \\ &= \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x\, p_{n-x}(t) \qquad for \quad x = 2,3,4\ldots \end{aligned} \tag{4.71}$$

For n=1

$$p_1(t) = \frac{p}{c}\, \frac{ct}{1+ct}\, p_0(t) \tag{4.72}$$

For n=2

$$\begin{aligned} 2\, p_2(t) &= \theta(t) \sum_{x=1}^{2} x\, g_x(t)\ \ p_{2-x}(t) \\ &= \theta(t)\ \ g_1(t)\ \ p_1(t) + 2\, \theta(t)\, g_2(t)\ \ p_0(t) \\ 2\, p_2(t) &= \frac{p}{c}\, \frac{ct}{1+ct}\, p_1(t) + \frac{p}{c}\, (\frac{ct}{1+ct})^2\, p_0(t) = (\frac{ct}{1+ct})^2\, [\frac{p}{c}\, (\frac{p}{c}+1) p_0(t) \\ p_2(t) &= \frac{1}{2} (\frac{ct}{1+ct})^2\, [\frac{p}{c}\, (\frac{p}{c}+1) p_0(t) = \binom{\frac{p}{c}+1}{2}\, (\frac{ct}{1+ct})^2\, p_0(t) \end{aligned} \tag{4.73}$$

For n=3

$$\begin{aligned} 3\, p_3(t) &= \theta(t) \sum_{x=1}^{3} x\, g_x(t)\ \ p_{3-x}(t) \\ &= \theta(t)\ \ g_1(t)\ \ p_2(t) + 2\, \theta(t)\, g_2(t)\ \ p_1(t) + 3\, \theta(t)\, g_3(t)\ \ p_0(t) \\ 3\, p_3(t) &= \frac{p}{c}\, \frac{ct}{1+ct}\, p_2(t) + \frac{p}{c}\, (\frac{ct}{1+ct})^2\, p_1(t) + \frac{p}{c}\, (\frac{ct}{1+ct})^3\, p_0(t) \\ &= \frac{p}{c}\, \frac{ct}{1+ct}\, [\binom{\frac{p}{c}+1}{2}\, (\frac{ct}{1+ct})^2\, p_0(t)] + \\ &\quad \frac{p}{c}\, (\frac{ct}{1+ct})^2\, [\frac{p}{c}\, \frac{ct}{1+ct}\, p_0(t)] + \frac{p}{c}\, (\frac{ct}{1+ct})^3\, p_0(t) \\ &= \frac{p}{c}\, [\binom{\frac{p}{c}+1}{2} + \binom{\frac{p}{c}+1}{1}]\, (\frac{ct}{1+ct})^3\, p_0(t) \end{aligned}$$

But $$\binom{r+1}{j} = \binom{r}{j} + \binom{r}{j-1}$$

$$\therefore \quad 3\, p_3(t) = \frac{p}{c} \binom{\frac{p}{c}+2}{2} \left(\frac{ct}{1+ct}\right)^3 p_0(t)$$

$$= \frac{\frac{p}{c}\,(\frac{p}{c}+2)(\frac{p}{c}+1)}{2!} \left(\frac{ct}{1+ct}\right)^3 p_0(t) = \frac{(\frac{p}{c}+2)(\frac{p}{c}+1)\frac{p}{c}}{2!} \left(\frac{ct}{1+ct}\right)^3 p_0(t)$$

$$p_3(t) = \frac{(\frac{p}{c}+2)(\frac{p}{c}+1)\frac{p}{c}}{3!} \left(\frac{ct}{1+ct}\right)^3 p_0(t)$$

$$= \binom{\frac{p}{c}+2}{3} \left(\frac{ct}{1+ct}\right)^3 p_0(t)$$

By induction, assume that it is true for $n-1$

$$p_{n-1}(t) = \binom{\frac{p}{c}+n-2}{n-1} \left(\frac{ct}{1+ct}\right)^{n-1} p_0(t)$$

Then,

$$n\, p_n(t) = \frac{p}{c} \sum_{x=1}^{n} \left(\frac{ct}{1+ct}\right)^x p_{n-x}(t)$$

$$= \frac{p}{c} \frac{ct}{1+ct}\, p_{n-1}(t) + \left(\frac{ct}{1+ct}\right)^2 p_{n-2}(t) + \cdots + \left(\frac{ct}{1+ct}\right)^{n-1} p_1(t) + \left(\frac{ct}{1+ct}\right.$$

$$= \frac{p}{c} \left[\binom{\frac{p}{c}+n-2}{n-1} + \binom{\frac{p}{c}+n-3}{n-2} + \cdots \right.$$

$$\left. + \binom{\frac{p}{c}+2}{3} + \binom{\frac{p}{c}+1}{2} + \binom{\frac{p}{c}+1}{1}\right] \left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \frac{p}{c} \left[\binom{\frac{p}{c}+n-2}{n-1} + \binom{\frac{p}{c}+n-3}{n-2} + \cdots + \binom{\frac{p}{c}+2}{3} + \binom{\frac{p}{c}+2}{2}\right] \left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \frac{p}{c} \left[\binom{\frac{p}{c}+n-2}{n-1} + \binom{\frac{p}{c}+n-3}{n-2} + \binom{\frac{p}{c}+3}{3}\right] \left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \frac{p}{c} \left[\binom{\frac{p}{c}+n-2}{n-1} + \binom{\frac{p}{c}+n-3}{n-2} + \binom{\frac{p}{c}+n-3}{n-1}\right] \left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \frac{p}{c} \left[\binom{\frac{p}{c}+n-2}{n-1} + \binom{\frac{p}{c}+n-2}{n-2}\right] \left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \frac{1}{n} \frac{p}{c} \frac{(\frac{p}{c}+n-1)\,(\frac{p}{c}+n-2)\ldots(\frac{p}{c}+n-(n-1))(\frac{p}{c})}{(n-1)!}$$

$$\left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \binom{\frac{p}{c}+n-1}{n} \left(\frac{ct}{1+ct}\right)^n p_0(t)$$

$$= \binom{\frac{p}{c}+n-1}{n} \left(\frac{ct}{1+ct}\right)^n \left(\frac{1}{1+ct}\right)^{\frac{p}{c}}, \qquad n = 0, 1, 2, \ldots \tag{4.74}$$

which is the negative binomial distribution with parameters $\frac{\mathbf{p}}{\mathbf{c}}$ and $\frac{\mathbf{1}}{\mathbf{1+ct}}$

**Moments**

Sum equation (4.71) over n

$$\begin{aligned}
\sum_{n=1}^{\infty} n\, p_n(t) &= \frac{p}{c} \sum_{n=1}^{\infty} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x\, p_{n-x}(t) \\
&= \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x \sum_{n=x}^{\infty} p_{n-x}(t) \\
&= \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x \\
&= \frac{p}{c} \frac{\frac{ct}{1+ct}}{1-\frac{ct}{1+ct}} \\
E(Z(t)) &= pt \\
E(Z(t))^2 &= \sum_{n=1}^{\infty} n^2\, p_n(t)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2\, p_n(t) &= \sum_{n=0}^{\infty} n\, n\, p_n(t) = \frac{p}{c} \sum_{n=1}^{\infty} n\, [\sum_{x=1}^{n} (\frac{ct}{1+ct})^x \;\; p_{n-x}(t)] \\
&= \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x \sum_{n=x}^{\infty} [n-x+x] \;\; p_{n-x}(t) \\
&= \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x [\sum_{n=x}^{\infty} [n-x] \;\; p_{n-x}(t) + \sum_{n=x}^{\infty} x \;\; p_{n-x}(t)] \\
&= \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x [M_1(t)+x] = \frac{p}{c} M_1(t) \sum_{x=1}^{n} (\frac{ct}{1+ct})^x + \frac{p}{c} \sum_{x=1}^{n} x\, (\frac{t}{1+ct} \\
&= \frac{p}{c} M_1(t) \frac{ct}{1+ct} \frac{1}{1-\frac{ct}{1+ct}} + \frac{p}{c} \frac{ct}{1+ct} (\frac{1}{1-\frac{ct}{1+ct}})^2 \\
&= \frac{p}{c} pt \frac{ct}{1+ct} (1+ct) + \frac{p}{c} \frac{ct}{1+ct} (1+ct)^2 \\
&= p^2\, t^2 + pt(1+ct) = (pt)^2 + pt + pct^2
\end{aligned}$$

But

$$E(Z(t)) = M_1(t) = pt \tag{4.75}$$

$$Var(Z(t)) = M_2(t) - (M_1(t))^2 = (pt)^2 + pt + pct^2 - (pt)^2 = pt + pct^2 \tag{4.76}$$

### 4.6.5 When the hazard function in (4.26) is such that $a = \frac{1}{2}$ , $p > 0$ and $c > 0$

$$h(t) = \theta'(t) = \frac{p}{(1+ct)^{\frac{1}{2}}} \quad for \quad p > 0, \quad and \quad c > 0 \tag{4.77}$$

which is a hazard function of the exponential-inverse Gaussian distribution.

Therefore,

$$h'(t) = \theta''(t) = -\frac{1}{2}\, p\, c\, (1+ct)^{-\frac{3}{2}}$$

$$h''(t) = \theta''(t) = (-1)^2\, \frac{3}{2}\, \frac{1}{2}\, p\, c^2\, (1+ct)^{-\frac{5}{2}}$$

$$h'''(t) = \theta'''(t) = (-1)^3\, \frac{5}{2}\, \frac{3}{2}\, \frac{1}{2}\, p\, c^3\, (1+ct)^{-\frac{7}{2}}$$

$$and \qquad h^{(n)}(t) = (-1)^n\, \frac{(2n-1)}{2}\, \frac{(2n-3)}{2}\, \frac{(2n-5)}{2} \,\ldots\ldots \frac{5}{2}\, \frac{3}{2}\, \frac{1}{2}\, (1+ct)^{-\frac{(2n+1)}{2}}$$

$$(-1)^n\, \frac{d^n}{dt^n}\, h(t) = (-1)^n\, h^n(t) \geq 0$$

Therefore $\mathbf{h(t)}$ is completely monotone.
The cumulative hazard function is

$$\theta(t) = p \int_0^t (1+cx)^{-\frac{1}{2}}\, dx = \frac{2p}{c}\, [(1+ct)^{\frac{1}{2}} - 1] \tag{4.78}$$

Therefore,

$$\theta(t-ts) = \frac{2p}{c}\, [(1+ct-cts)^{\frac{1}{2}} - 1] \tag{4.79}$$

$$\therefore \qquad \theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -\frac{1}{2}pc$$

$$p_0(t) = e^{-\frac{2p}{c}\, [(1+ct)^{\frac{1}{2}} - 1]}$$

$$p_0'(t) = -p\, (1+ct)^{-\frac{1}{2}}\, p_0(t)$$

$$\begin{aligned}
p_0''(t) &= -pc\, (1+ct)^{-\frac{1}{2}}\, p_0'(t) + \frac{1}{2}\, pc\, (1+ct)^{-\frac{3}{2}}\, p_0(t) \\
&= [p^2\, (1+ct)^{-1} + \frac{1}{2}\, pc\, (1+ct)^{-\frac{3}{2}}]\, p_0(t) \\
&= [p^2\, (1+ct)^{-1} + 2\, p^2 \frac{c}{4p}\, (1+ct)^{-\frac{3}{2}}]\, p_0(t) \\
&= p^2\, [(1+ct)^{-\frac{2}{2}} + \frac{2!}{1!}\, \frac{c}{4p}\, (1+ct)^{-\frac{3}{2}}]\, p_0(t) \\
&= p^2\, [\frac{(1+0)!}{(1-0)!0!}\, (\frac{c}{4p})^0\, (1+ct)^{-\frac{2}{2}} + \frac{(1+1)!}{1!}\, \frac{c}{4p}\, (1+ct)^{-\frac{3}{2}}]\, p_0(t)
\end{aligned}$$

$$p_0''(t) = p^2\, [\frac{(1+0)!}{(1-0)!0!}\, (\frac{c}{4p})^0\, (1+ct)^{-\frac{(2+0)}{2}} + \frac{(1+1)!}{(1-1)!1!}\, (\frac{c}{4p})\, (1+ct)^{-\frac{(2+1)}{2}}]\, p_0$$

$$\begin{aligned}
p_0'''(t) &= p^2\, [(1+ct)^{-\frac{(2)}{2}} + \frac{(2)!}{1!}\, (\frac{c}{4p})\, (1+ct)^{-\frac{3}{2}}]\, p_0(t) \\
&+ p^2\, [-\frac{2}{2}\, c(1+ct)^{-\frac{(4)}{2}} - \frac{3}{2}\, c\, \frac{(2)!}{1!}\, (\frac{c}{4p})\, (1+ct)^{-\frac{5}{2}}]\, p_0(t)
\end{aligned}$$

$$= p^2\ [-p[(1+ct)^{-\frac{3}{2}} + \frac{(2)!}{1!}\ (\frac{c}{4p})\ (1+ct)^{-\frac{(4)}{2}}]$$

$$-p[4\frac{c}{4p}\ (1+ct)^{-\frac{(4)}{2}} + \frac{3!}{2}\ (\frac{c}{4p})^2\ (1+ct)^{-\frac{5}{2}}]\ p_0(t)$$

$$= -p^3\ [(1+ct)^{-\frac{3}{2}} + 6\ (\frac{c}{4p})\ (1+ct)^{-\frac{(4)}{2}}]$$

$$+ \frac{4!}{2}\ (\frac{c}{4p})^2\ (1+ct)^{-\frac{(5)}{2}}]\ p_0(t)$$

$$= -p^3\ [(1+ct)^{-\frac{3}{2}} + \frac{3!}{1!}\ (\frac{c}{4p})\ (1+ct)^{-\frac{(4)}{2}}]$$

$$+ \frac{4!}{2}\ (\frac{c}{4p})^2\ (1+ct)^{-\frac{(5)}{2}}]\ p_0(t)$$

$$p_0'''(t) = (-1)^3\ p^3\ [\frac{(2+0)!}{(2-0)!0!}(\frac{c}{4p})^0\ (1+ct)^{-\frac{(3+0)}{2}} + \frac{(2+1)!}{(2-1)!1!}\ (\frac{c}{4p})\ (1+ct)^{-\frac{(3+}{}}$$

$$+ \frac{(2+2)!}{(2-2)!2!}\ (\frac{c}{4p})^2\ (1+ct)^{-\frac{(3+2)}{2}}]\ p_0(t)$$

$$p_0^{iv}(t) = (-p)^3\ [[(1+ct)^{-\frac{3}{2}} + 3!\ (\frac{c}{4p})\ (1+ct)^{-\frac{(4)}{2}}]$$

$$+ \frac{4!}{2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{(5)}{2}}]p_0(t) + [-\frac{3!}{2}\ c(1+ct)^{-\frac{5}{2}} - \frac{4!}{2}c(\frac{c}{4p})\ (1+ct)^{-\frac{(6)}{2}}]$$

$$= (-p)^3\ [-p[(1+ct)^{-\frac{4}{2}} + 3!\ (\frac{c}{4p})\ (1+ct)^{-\frac{5}{2}}$$

$$+ \frac{4!}{2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{(6)}{2}}] - p[3!\frac{c}{4p}\ (1+ct)^{-\frac{5}{2}} + 4!2\ (\frac{c}{4p})^2\ (1+ct)^{-\frac{(6)}{2}} +$$

$$\frac{6!}{3!}\ (\frac{c}{4p})^3\ (1+ct)^{-\frac{(7)}{2}}]\ p_0(t)]$$

$$= p^4\ [(1+ct)^{-\frac{4}{2}} + 3!\ 2!(\frac{c}{4p})\ (1+ct)^{-\frac{5}{2}}$$

$$+ 4!(\frac{1}{2} + 2)(\frac{c}{4p})^2\ (1+ct)^{-\frac{(6)}{2}} + \frac{6!}{3!}(\frac{c}{4p})^3\ (1+ct)^{-\frac{7}{2}}]\ p_0(t)$$

$$= p^4\ [(1+ct)^{-\frac{4}{2}} + \frac{4!}{2!}(\frac{c}{4p})\ (1+ct)^{-\frac{5}{2}}$$

$$+ \frac{5!}{2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{(6)}{2}} + \frac{6!}{3!}(\frac{c}{4p})^3\ (1+ct)^{-\frac{7}{2}}]\ p_0(t)$$

$$p_0^{iv}(t) = (-1)^4\ p^4\ [\frac{(3+0)!}{(3-0)!0!}(\frac{c}{4p})^0\ (1+ct)^{-\frac{(4+0)}{2}} + \frac{(3+1)!}{(3-1)!1!}(\frac{c}{4p})\ (1+ct)^{-\frac{(4+}{2}}$$

$$+ \frac{(3+2)!}{(3-2)!2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{(4+2)}{2}} + \frac{(3+3)!}{(3-3)!3!}(\frac{c}{4p})^3\ (1+ct)^{-\frac{(4+3)}{2}}]\ p_0(t)$$

$$p_0^{v}(t) = (-1)^4\ p^4\ [(1+ct)^{-\frac{4}{2}} + \frac{4!}{2!}(\frac{c}{4p})\ (1+ct)^{-\frac{5}{2}}$$

$$+\frac{5!}{2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{(6)}{2}}+\frac{6!}{3!}(\frac{c}{4p})^3\ (1+ct)^{-\frac{7}{2}}]\ p_0(t)+[-\frac{4}{2}c(1+ct)^{-\frac{6}{2}}-$$

$$\frac{5!}{4}c(\frac{c}{4p})\ (1+ct)^{-\frac{7}{2}}-\frac{6!}{4}c(\frac{c}{4p})^2\ (1+ct)^{-\frac{(8)}{2}}-\frac{7!}{3!2}c(\frac{c}{4p})^3\ (1+ct)^{-\frac{9}{2}}]\ p_0$$

$$=(-1)^5\ p^5\ [(1+ct)^{-\frac{5}{2}}+\frac{5!}{3!})(\frac{c}{4p})\ (1+ct)^{-\frac{6}{2}}$$

$$+\frac{6!}{2!2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{(7)}{2}}+\frac{7!}{3!}(\frac{c}{4p})^3\ (1+ct)^{-\frac{8}{2}}+\frac{8!}{4!}(\frac{c}{4p})^4\ (1+ct)^{-\frac{9}{2}}]\ p_0($$

$$=(-1)^5\ p^5\ [\frac{(4+0)!}{(4+0)!0!}(\frac{c}{4p})^0\ (1+ct)^{-\frac{(5+0)}{2}}+\frac{(4+1)!}{(4-1)!1!}(\frac{c}{4p})^1\ (1+ct)^{-\frac{(5}{}}$$

$$+\frac{(4+2)!}{(4-2)!2!}(\frac{c}{4p})^2\ (1+ct)^{-\frac{((5+2))}{2}}+\frac{(4+3)!}{(4-3)!3!}(\frac{c}{4p})^3\ (1+ct)^{-\frac{(5+3)}{2}}+$$

$$\frac{(4+4)!}{(4-4)!4!}(\frac{c}{4p})^4\ (1+ct)^{-\frac{(5+4)}{2}}]\ p_0(t)$$

$$=(-1)^5\ p^5\ \sum_{k=0}^{4}\frac{(4+k)!}{(4-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(5+k)}{2}}]\ p_0(t)$$

$$=(-1)^5\ p^5\ \sum_{k=0}^{5-1}\frac{(5-1+k)!}{(5-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(5+k)}{2}}]\ p_0(t)$$

Generally,

$$p_0^n(t)=(-1)^n\ p^n\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}]\ p_0(t)$$

$$\therefore\ p_n(t)=(-1)^n\frac{t^n}{n!}\ p_0^{(n)}(t)$$

$$p_n(t)=(-1)^n\frac{t^n}{n!}\ (-1)^n\ p^n\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}]\ p_0(t)$$

$$n=0,1,2,3,\ldots$$

$$p_n(t)=\frac{(pt)^n}{n!}\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}]\ p_0(t)\qquad n=0,1,2,3,\ldots \tag{4.80}$$

where,

$$p_0(t)=e^{-\frac{2p}{c}\ [(1+ct)^{\frac{1}{2}}-1]}$$

The pgf is

$$H(s,t)=e^{-\theta(t-ts)}$$
$$=e^{-\frac{2p}{c}\ [(1+ct-cts)^{\frac{1}{2}}-1]} \tag{4.81}$$

Therefore,

$$E(Z(t))=t\,\theta'(0)=pt\qquad and\qquad Var(Z(t))=t\,\theta'(0)-t^2\,\theta''(0)=pt+\frac{1}{2}\,pc\,t^2 \tag{4.82}$$

Since $\boldsymbol{\theta(0) = 0}$ and $\mathbf{h(t)} = \boldsymbol{\theta}'\mathbf{(t)}$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$\begin{aligned} G(s,t) &= 1 - \frac{\theta(t-ts)}{\theta(t)} \\ &= 1 - \frac{(1+ct-cts)^{\frac{1}{2}} - 1}{(1+ct)^{\frac{1}{2}} - 1} \\ &= \frac{(1+ct)^{\frac{1}{2}} - (1+ct-cts)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}} - 1} \end{aligned} \tag{4.83}$$

By power series expansion

$$\begin{aligned} G(s,t) &= \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}-1}\,[1-(1-\frac{ct}{1+ct}\,s)^{\frac{1}{2}}]} \\ &= \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}-1}\,[-\sum_{x=1}^{\infty}\binom{\frac{1}{2}}{x}\,(-\frac{ct}{1+ct})^x\,s^x]} \end{aligned}$$

But

$$\begin{aligned} \binom{\frac{1}{2}}{x} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots[\frac{1}{2}-(x-2)][\frac{1}{2}-(x-1)]}{x!} \\ &= \frac{1}{2}\,\frac{(-1)^{x-1}}{x!}\,(1-\frac{1}{2})(2-\frac{1}{2})\ldots[(x-2)-\frac{1}{2}][(x-1)-\frac{1}{2}] \\ &= \frac{1}{2}\,\frac{(-1)^{x-1}}{x!}\,\frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\ldots(x-\frac{5}{2})\,(x-\frac{3}{2}) \\ &= \frac{1}{2}\,\frac{(-1)^{x-1}}{x!}\,\frac{\frac{1}{2}\,\Gamma(\frac{1}{2})\,\frac{3}{2}\,\frac{5}{2}\ldots(x-\frac{5}{2})\,(x-\frac{3}{2})}{\Gamma(\frac{1}{2})} \\ &= \frac{\frac{1}{2}\,(-1)^{x-1}}{x!\,\Gamma(\frac{1}{2})}\,\frac{3}{2}\,\Gamma(\frac{3}{2})\,\frac{5}{2}\ldots(x-\frac{5}{2})\,(x-\frac{3}{2}) \\ &= \frac{\frac{1}{2}\,(-1)^{x-1}}{x!\,\Gamma(\frac{1}{2})}\,\frac{5}{2}\Gamma(\frac{5}{2})\,\ldots(x-\frac{5}{2})\,(x-\frac{3}{2}) \\ &= \frac{\frac{1}{2}\,(-1)^{x-1}}{x!\,\Gamma(\frac{1}{2})}\,(x-\frac{5}{2})\,\Gamma(x-\frac{5}{2})\,(x-\frac{3}{2}) \\ &= \frac{\frac{1}{2}\,(-1)^{x-1}}{x!\,\Gamma(\frac{1}{2})}\,(x-\frac{3}{2})\,\Gamma(x-\frac{3}{2}) \\ &= \frac{\frac{1}{2}\,(-1)^{x-1}}{x!\,\Gamma(\frac{1}{2})}\,\Gamma(x-\frac{1}{2}) \end{aligned}$$

$$
\begin{aligned}
G(s,t) &= \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\,[-\sum_{x=1}^{\infty} \frac{\frac{1}{2}\,(-1)^{x-1}}{x!\,\Gamma(\frac{1}{2})}\,\Gamma(x-\frac{1}{2})\,(\frac{-ct}{1+ct})^{x}\,s^{x}] \\
&= \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\,[\sum_{x=1}^{\infty} \frac{\frac{1}{2}\,(-1)^{x}}{x!\,\Gamma(\frac{1}{2})}\,\Gamma(x-\frac{1}{2})\,(\frac{-ct}{1+ct})^{x}\,s^{x}] \\
&= \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\,\sum_{x=1}^{\infty} \frac{\frac{1}{2}}{x!\,\Gamma(\frac{1}{2})}\,\Gamma(x-\frac{1}{2})\,(\frac{ct}{1+ct})^{x}\,s^{x} \\
g_x(t) &= \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\,\frac{\frac{1}{2}\,\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x} \qquad x=1,2,3,\ldots \\
&= \frac{(1+ct)^{\frac{1}{2}}\,\frac{ct}{1+ct}}{(1+ct)^{\frac{1}{2}}-1}\,\frac{\frac{1}{2}\,\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \\
&= \frac{pct}{2p}\,\frac{1}{(1+ct)^{\frac{1}{2}}}\,\frac{1}{(1+ct)^{\frac{1}{2}}-1}\,\frac{\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \\
&= \frac{pt\,(1+ct)^{-\frac{1}{2}}}{\frac{2p}{c}\,[(1+ct)^{\frac{1}{2}}-1]}\,\frac{\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \\
&= \frac{pt\,(1+ct)^{-\frac{1}{2}}}{\theta(t)}\,\frac{\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \qquad x=1,2,3,\ldots
\end{aligned}
\tag{4.84}
$$

By differentiation method

$$
\begin{aligned}
\frac{\partial G(s,t)}{\partial s} &= \frac{\frac{1}{2}\,ct\,(1+ct-cts)^{-\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1} \\
\frac{\partial^2 G(s,t)}{\partial s^2} &= \frac{\frac{1}{2}\,\frac{1}{2}\,(ct)^2\,(1+ct-cts)^{-\frac{3}{2}}}{(1+ct)^{\frac{1}{2}}-1} \\
\frac{\partial^3 G(s,t)}{\partial s^3} &= \frac{\frac{3}{2}\,\frac{1}{2}\,\frac{1}{2}\,(ct)^3\,(1+ct-cts)^{-\frac{5}{2}}}{(1+ct)^{\frac{1}{2}}-1} \\
\frac{\partial^4 G(s,t)}{\partial s^4} &= \frac{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\frac{1}{2}\,(ct)^4\,(1+ct-cts)^{-\frac{7}{2}}}{(1+ct)^{\frac{1}{2}}-1} = \frac{\frac{2.3-1}{2}\,\frac{2.2-1}{2}\,\frac{2.1-1}{2}\,\frac{1}{2}\,(ct)^4\,(1+c\ldots}{(1+ct)^{\frac{1}{2}}-1} \\
\therefore \qquad \frac{\partial^x G(s,t)}{\partial s^x} &= \frac{\frac{[2.(x-1)-1]}{2}\,\frac{[2.(x-2)-1]}{2}\,\frac{[2.(x-3)-1]}{2}\ldots\frac{2.3-1}{2}\,\frac{2.2-1}{2}\,\frac{2.1-1}{2}\,\frac{1}{2}\,(ct)^x\,(1-\ldots}{(1+ct)^{\frac{1}{2}}-1} \\
&= \frac{\frac{[2.(x-1)-1]}{2}\,\frac{[2.(x-2)-1]}{2}\ldots.\frac{[2.2-1]}{2}\,\frac{\frac{1}{2}\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})}\,\frac{1}{2}\,(ct)^x\,(1+ct-cts)^{-\frac{(2x\ldots}{}}}{(1+ct)^{\frac{1}{2}}-1} \\
&= \frac{[(x-1)-\frac{1}{2}]\,\ldots.\frac{3}{2}\,\Gamma(\frac{3}{2})\,\frac{1}{2}\,(ct)^x\,(1+ct-cts)^{-\frac{(2x-1)}{2}}}{\Gamma(\frac{1}{2})\,(1+ct)^{\frac{1}{2}}-1}
\end{aligned}
$$

$$\frac{\partial^x G(s,t)}{\partial s^x} = \frac{[(x-1)-\frac{1}{2}]\,\Gamma[(x-1)-\frac{1}{2}]\,(\frac{1}{2})\,(ct)^x\,(1+ct-cts)^{-\frac{(2x-1)}{2}}}{\Gamma(\frac{1}{2})\,[(1+ct)^{\frac{1}{2}}-1]}$$

$$= \frac{\Gamma[x-\frac{1}{2}]\,(\frac{1}{2})\,(ct)^x\,(1+ct-cts)^{-\frac{(2x-1)}{2}}}{\Gamma(\frac{1}{2})\,[(1+ct)^{\frac{1}{2}}-1]}$$

$$\frac{\partial^x G(s,t)}{\partial s^x}|_{s=0} = \frac{\Gamma[x-\frac{1}{2}]\,(\frac{1}{2})\,(ct)^x\,(1+ct)^{-\frac{(2x-1)}{2}}}{\Gamma(\frac{1}{2})\,[(1+ct)^{\frac{1}{2}}-1]}$$

$$g_x(t) = \frac{(\frac{1}{2})\,\Gamma[x-\frac{1}{2}]}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^x\,\frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1} g_x(t)$$

$$= \frac{pct}{2p(1+ct)}\,\frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\,\frac{\Gamma[x-\frac{1}{2}]}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}$$

$$= \frac{pt\,(1+ct)^{-\frac{1}{2}}}{\frac{2p}{c}\,(1+ct)^{\frac{1}{2}}-1}\,\frac{\Gamma[x-\frac{1}{2}]}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}$$

$$= \frac{pt\,(1+ct)^{-\frac{1}{2}}}{\theta(t)}\,\frac{\Gamma[x-\frac{1}{2}]}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \qquad for \quad x = 1,2,3\ldots$$

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{\Gamma(x-\frac{1}{2})}{x!}\,(\frac{ct}{1+ct})^{x-1}\,\frac{(x-1)!}{\Gamma[x-1-\frac{1}{2}]\,(\frac{ct}{1+ct})^{x-2}}$$

$$= \frac{1}{x}\,\frac{\Gamma(x-\frac{1}{2})}{\Gamma(x-\frac{3}{2})}\,\frac{ct}{1+ct} = \frac{1}{x}\,(x-\frac{3}{2}\,\frac{ct}{1+ct}$$

$$= (1-\frac{3}{2x})\,\frac{ct}{1+ct} \qquad x = 2,3,\ldots \tag{4.85}$$

which is in Panjer's recursive form, where $\mathbf{a} = \frac{\mathbf{ct}}{\mathbf{1+ct}}$ and $\mathbf{b} = -\frac{\mathbf{3}}{\mathbf{2}}\,\frac{\mathbf{ct}}{\mathbf{1+ct}}$

$$n\,p_n(t) = \theta(t)\,\sum_{x=1}^{n}\,x\,g_x(t)\;p_{n-x}(t)$$

$$= \theta(t)\,\sum_{x=1}^{n}\,x\,\frac{pt\,(1+ct)^{-\frac{1}{2}}}{\theta(t)}\,\frac{\Gamma[x-\frac{1}{2}]}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}\;p_{n-x}(t)$$

$$= pt\,(1+ct)^{-\frac{1}{2}}\,\sum_{x=1}^{n}\,\frac{\Gamma[x-\frac{1}{2}]}{(x-1)!!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}\;p_{n-x}(t) \qquad for\ n \tag{4.86}$$

Replace $\mathbf{n}$ by $\mathbf{n+1}$ to get

$$(n+1)p_{n+1}(t) = pt\,(1+ct)^{-\frac{1}{2}}\,\sum_{x=1}^{n+1}\,\frac{\Gamma[x-\frac{1}{2}]}{(x-1)!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}\;p_{n+1-x}(t)$$

Let x=i+1

$$(n+1)p_{n+1}(t) = pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{n} \frac{\Gamma[i+\frac{1}{2}]}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^i\ p_{n-i}(t) \qquad n = 0,1,2\ldots \tag{4.87}$$

By iteration,

$$n=0 \Rightarrow\ p_1(t) = pt\ (1+ct)^{-\frac{1}{2}}\ p_0(t)$$

$$n=1 \Rightarrow\ 2p_2(t) = pt\ (1+ct)^{-\frac{1}{2}} \sum_{x=1}^{2} \frac{\Gamma[i+\frac{1}{2}]}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^i\ p_{1-i}(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ [p_1(t) + \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}\ \frac{ct}{1+ct}\ p_0(t)]$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ [pt\ (1+ct)^{-\frac{1}{2}} + \frac{1}{2}\ \frac{ct}{1+ct}]\ p_0(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ [pt\ (1+ct)^{-\frac{1}{2}} + pt\ \frac{c}{2p}\ \frac{1}{1+ct}]\ p_0(t)$$

$$= (pt)^2\ [(1+ct)^{-1} + 2\ \frac{c}{4p}\ (1+ct)^{-\frac{3}{2}}]\ p_0(t)$$

$$p_2(t) = \frac{(pt)^2}{2!}\ [(1+ct)^{-\frac{2}{2}} + 2!\ \frac{c}{4p}\ (1+ct)^{-\frac{3}{2}}]\ p_0(t)$$

$$n=2 \Rightarrow\ 3p_3(t) = pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{2} \frac{\Gamma[i+\frac{1}{2}]}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^i\ p_{2-i}(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ [p_2(t) + \frac{ct}{2\ (1+ct)}\ p_1(t) + \frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ (\frac{ct}{1+ct})^2\ p_0(t)]$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ [\frac{(pt)^2}{2!}\ [(1+ct)^{-\frac{2}{2}} + 2!\ \frac{c}{4p}\ (1+ct)^{-\frac{3}{2}}] +$$

$$\frac{ct}{2\ (1+ct)}.pt\ (1+ct)^{-\frac{1}{2}} + \frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ (\frac{ct}{1+ct})^2]\ p_0(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ [\frac{(pt)^2}{2!}\ [(1+ct)^{-\frac{2}{2}} + 2!\ \frac{c}{4p}\ (1+ct)^{-\frac{3}{2}}] +$$

$$\frac{c}{p\ (1+ct)}.\frac{(pt)^2}{2!}\ (1+ct)^{-\frac{1}{2}} + \frac{3}{4}\ \frac{(pt)^2}{2!}\ \frac{1}{(pt)^2}\ (\frac{ct}{1+ct})^2]\ p_0(t)$$

$$= \frac{(pt)^3}{2!}\ [(1+ct)^{-\frac{3}{2}} + 2!\ \frac{c}{4p}\ (1+ct)^{-\frac{4}{2}} +$$

$$\frac{c}{p}\ (1+ct)^2 + \frac{3}{4}\ (\frac{ct}{pt})^{-2}\ (\frac{ct}{1+ct})^{-\frac{5}{2}}]\ p_0(t)$$

$$= \frac{(pt)^3}{2!}\ [(1+ct)^{-\frac{3}{2}} + 3!\ \frac{c}{4p}\ (1+ct)^{-\frac{4}{2}} + \frac{2.3.4}{2!}\ (\frac{c}{4p})^2\ (\frac{ct}{1+ct})^{-\frac{5}{2}}]\ p$$

$$= \frac{(pt)^3}{2!}\ [(1+ct)^{-\frac{3}{2}} + 3!\ \frac{c}{4p}\ (1+ct)^{-\frac{4}{2}} + \frac{4!}{2!}\ (\frac{c}{4p})^2\ (\frac{ct}{1+ct})^{-\frac{5}{2}}]\ p_0(t$$

$$= \frac{(pt)^3}{3!} [\frac{(2+0)!}{(2-0)!0!} (\frac{c}{4p})^0 (1+ct)^{-\frac{(3+0)}{2}} + \frac{(2+1)!}{(2-1)!1!} \frac{c}{4p} (1+ct)^{-\frac{(3}{}}$$

$$\frac{(2+2)!}{(2-2)!2!} (\frac{c}{4p})^2 (\frac{ct}{1+ct})^{-\frac{(3+2)}{2}}] \, p_0(t)$$

$$= \frac{(pt)^3}{3!} \sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(3+k)}{2}} \, p_0(t)$$

$$n = 3 \Rightarrow 4p_4(t) = pt\,(1+ct)^{-\frac{1}{2}} \sum_{i=0}^{3} \frac{\Gamma[i+\frac{1}{2}]}{i!\,\Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^i \; p_{3-i}(t)$$

$$= pt\,(1+ct)^{-\frac{1}{2}} [p_3(t) + \frac{ct}{2\,(1+ct)} p_2(t) + \frac{3}{2} \frac{1}{2} \frac{1}{2!} (\frac{ct}{1+ct})^2 p_1(t) +$$

$$\frac{5}{2} \frac{3}{2} \frac{1}{2} \frac{1}{3!} (\frac{ct}{1+ct})^3 p_0(t)]$$

$$4p_4(t) = pt\,(1+ct)^{-\frac{1}{2}} \sum_{i=0}^{3} \frac{\Gamma[i+\frac{1}{2}]}{i!\,\Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^i \; p_{3-i}(t)$$

$$= pt\,(1+ct)^{-\frac{1}{2}} [p_3(t) + \frac{ct}{2\,(1+ct)} p_2(t) + \frac{\Gamma(\frac{5}{2})}{2!\,\Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^2 p_1(t) -$$

$$\frac{\Gamma(\frac{7}{2})}{3!\,\Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^3 p_0(t)]$$

$$= pt\,(1+ct)^{-\frac{1}{2}} [\frac{(pt)^3}{3!} \sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(3+k)}{2}} +$$

$$\frac{ct}{2\,(1+ct)} \frac{(pt)^2}{2!} \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(2+k)}{2}} +$$

$$\frac{3}{2} \frac{1}{2} \frac{1}{2!} (\frac{ct}{1+ct})^2 pt\,(1+ct)^{-\frac{1}{2}} + \frac{5}{2} \frac{3}{2} \frac{1}{2} \frac{1}{3!} (\frac{ct}{1+ct})^3 p_0(t)]$$

$$= pt\,(1+ct)^{-\frac{1}{2}} [\frac{(pt)^3}{3!} \sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(3+k)}{2}} +$$

$$\frac{(pt)^3}{3!} \frac{3}{pt} \frac{ct}{2\,(1+ct)} \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(2+k)}{2}} +$$

$$\frac{(pt)^3}{3!} \frac{3}{4} \frac{3}{(pt)^2} (\frac{ct}{1+ct})^2 (1+ct)^{-\frac{1}{2}} + \frac{(pt)^3}{3!} \frac{15}{8} \frac{1}{(pt)^3} (\frac{ct}{1+ct})^3]$$

$$4\,p_4(t) = \frac{(pt)^4}{3!} [\sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(4+k)}{2}} +$$

$$\frac{3c}{2p} (1+ct)^{-\frac{3}{2}} \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(2+k)}{2}} +$$

$$\frac{3}{4}\, 3\, (\frac{c}{p})^2 (1+ct)^{-\frac{6}{2}} + \frac{15}{8} (\frac{c}{p})^2 (1+ct)^{-\frac{7}{2}}] \, p_0(t)$$

$$p_4(t) = \frac{(pt)^4}{4!} [\sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(4+k)}{2}} +$$

$$3! \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!} (\frac{c}{4p})^{k+1} (1+ct)^{-\frac{(5+k)}{2}} +$$

$$3.3.4 (\frac{c}{p})^2 (1+ct)^{-\frac{6}{2}} + \frac{15.4.2}{2.2.4.4} (\frac{c}{p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [\sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(4+k)}{2}} +$$

$$3! \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!} (\frac{c}{4p})^{k+1} (1+ct)^{-\frac{(5+k)}{2}} +$$

$$3.3.4 (\frac{c}{p})^2 (1+ct)^{-\frac{6}{2}} + \frac{15.4.2}{2.2.4.4} (\frac{c}{p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [\sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!} (\frac{c}{4p})^k (1+ct)^{-\frac{(4+k)}{2}} +$$

$$3! \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!} (\frac{c}{4p})^{k+1} (1+ct)^{-\frac{(5+k)}{2}} +$$

$$3.3 (\frac{c}{4p})^2 (1+ct)^{-\frac{6}{2}} + \frac{2.3.4.5.6}{6} (\frac{c}{p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [(1+ct)^{-\frac{4}{2}} + 3! \frac{c}{4p} (1+ct)^{-\frac{5}{2}} + \frac{4!}{2!} \frac{c}{4p} (1+ct)^{-\frac{5}{2}} +$$

$$3!\ 2! (\frac{c}{4p})^2 (1+ct)^{-\frac{6}{2}} + 3!\ 3! (\frac{c}{p})^2 (1+ct)^{-\frac{6}{2}} +$$

$$\frac{6!}{3!} (\frac{c}{4p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [(1+ct)^{-\frac{4}{2}} + 3!\ 2 \frac{c}{4p} (1+ct)^{-\frac{5}{2}} + (\frac{4!}{2!} + 3!2 + 3!3!) (\frac{c}{4p})^2$$

$$\frac{6!}{3!} (\frac{c}{p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [(1+ct)^{-\frac{4}{2}} + \frac{4!}{2!} \frac{c}{4p} (1+ct)^{-\frac{5}{2}} + 3! (2+2+6) (\frac{c}{4p})^2 (1+$$

$$\frac{6!}{3!} (\frac{c}{p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [(1+ct)^{-\frac{4}{2}} + \frac{4!}{2!} \frac{c}{4p} (1+ct)^{-\frac{5}{2}} + 3!\ 10 (\frac{c}{4p})^2 (1+ct)^{-\frac{6}{2}} +$$

$$\frac{6!}{3!} (\frac{c}{p})^3 (1+ct)^{-\frac{7}{2}}] p_0(t)$$

$$= \frac{(pt)^4}{4!} [(1+ct)^{-\frac{4}{2}} + \frac{4!}{2!} \frac{c}{4p} (1+ct)^{-\frac{5}{2}} + 2.3.2.5 (\frac{c}{4p})^2 (1+ct)^{-\frac{6}{2}}$$

$$\frac{6!}{3!}\ (\frac{c}{p})^3\ (1+ct)^{-\frac{7}{2}}]\ p_0(t)$$

$$= \frac{(pt)^4}{4!}\ [(1+ct)^{-\frac{4}{2}} + \frac{4!}{2!}\ \frac{c}{4p}\ (1+ct)^{-\frac{5}{2}} + \frac{5!}{2!}\ (\frac{c}{4p})^2\ (1+ct)^{-\frac{6}{2}} +$$

$$\frac{6!}{3!}\ (\frac{c}{p})^3\ (1+ct)^{-\frac{7}{2}}]\ p_0(t)$$

$$= \frac{(pt)^4}{4!}\ [\frac{(3+0)!}{(3-0)!0!}\ (\frac{c}{4p})^0\ (1+ct)^{-\frac{(4+0)}{2}} + \frac{(3+1)!}{(3-1)!1!}\ (\frac{c}{4p})^1\ (1+ct$$

$$\frac{(3+2)!}{(3-2)!2!}\ (\frac{c}{p})^2\ (1+ct)^{-\frac{(4+2)}{2}} + \frac{(3+3)!}{(3-3)!3!}\ (\frac{c}{p})^3\ (1+ct)^{-\frac{(4+3)}{2}}]$$

$$= \frac{(pt)^4}{4!}\ \sum_{k=0}^{3}\ \frac{(3+k)!}{(3-k)!k!}\ (\frac{c}{4p})^k\ (1+ct)^{-\frac{(4+k)}{2}}\ p_0(t)$$

$$\therefore \quad p_n(t) = \frac{(pt)^n}{n!}\ \sum_{k=0}^{n-1}\ \frac{(n-1+k)!}{(n-1-k)!k!}\ (\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}\ p_0(t)$$

$$for \quad n = 1,2,3\ldots \tag{4.88}$$

where,

$$p_0(t) = e^{-\frac{2p}{c}\ [(1+ct)^{\frac{1}{2}}-1]} \tag{4.89}$$

**Moments**

Sum the recursive relation (4.86) over **n**; thus

$$\sum_{n=1}^{\infty}\ (n+1)\ p_{n+1}(t) = pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{n=1}^{\infty}\ \sum_{i=0}^{n}\ \frac{\Gamma(i+\frac{1}{2})}{i!\ \Gamma(\frac{1}{2})}\ (\frac{ct}{1+ct})^i\ p_{n-i}(t)$$

$$\sum_{n=1}^{\infty}\ (n+1)\ p_{n+1}(t) = pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{n=1}^{\infty}\ \sum_{i=0}^{\infty}\ \frac{\Gamma(i+\frac{1}{2})}{i!\ \Gamma(\frac{1}{2})}\ (\frac{ct}{1+ct})^i\ p_{n-i}(t)$$

$$M_1(t) = pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{n=1}^{\infty}\ \sum_{i=0}^{\infty}\ \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})}\ (\frac{ct}{1+ct})^i\ p_{n-i}(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{i=0}^{\infty}\ \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})}\ (\frac{ct}{1+ct})^i\ \sum_{n=0}^{\infty}\ p_{n-i}(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{i=0}^{\infty}\ \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})}\ (\frac{ct}{1+ct})^i\ \sum_{n=i}^{\infty}\ p_{n-i}(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{i=0}^{\infty}\ \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})}\ (\frac{ct}{1+ct})^i$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{i=0}^{\infty}\ \binom{\frac{1}{2}+i-1}{i}\ (\frac{ct}{1+ct})^i$$

$$= pt\ (1+ct)^{-\frac{1}{2}}\ \sum_{i=0}^{\infty}\ \binom{(-1)^i\ -\frac{1}{2}}{i}\ (\frac{ct}{1+ct})^i$$

$$= pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} (-\frac{ct}{1+ct})^{i}$$

$$= pt\ (1+ct)^{-\frac{1}{2}} (1-\frac{ct}{1+ct})^{-\frac{1}{2}} = pt\ (1+ct)^{-\frac{1}{2}} (\frac{1}{1+ct})^{-\frac{1}{2}}$$

$$= pt\ (1+ct)^{-\frac{1}{2}} (1+ct)^{\frac{1}{2}}$$

$$E(Z(t))) = pt$$

Next, multiply the recursive relation (4.86) by **n** and then sum the result over **n**; thus

$$\sum_{n=1}^{\infty} (n+1)^2\ p_{n+1}(t) = pt\ (1+ct)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}\ (n+1)\ p_{n-i}(t)$$

$$M_2(t) = pt\ (1+ct)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}\ (n-i+i+1)\ p_{n-i}(t)$$

$$= pt\ (1+ct)^{-\frac{1}{2}} [\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}\ (n-i)\ p_{n-i}(t)+$$

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}\ (i+1)\ p_{n-i}(t)]$$

$$= pt\ (1+ct)^{-\frac{1}{2}} [\sum_{i=0}^{\infty} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i} \sum_{n=0}^{\infty} (n-i)\ p_{n-i}(t)+$$

$$\sum_{i=0}^{\infty} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}\ (i+1) \sum_{n=0}^{\infty} p_{n-i}(t)]$$

$$= pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}\ M_1(t)+$$

$$pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} [\frac{\Gamma(\frac{1}{2}+i)}{(i-1)!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i} + \frac{\Gamma(\frac{1}{2}+i)}{i!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}]$$

$$= pt\ (1+ct)^{-\frac{1}{2}} (M_1(t)+1) \sum_{i=0}^{\infty} \binom{\frac{1}{2}+i-1}{i} (\frac{ct}{1+ct})^{i}+$$

$$pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(\frac{1}{2}+i-1)\ \Gamma(\frac{1}{2}+i-1)}{(i-1)!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i} \binom{-\frac{1}{2}}{i} (\frac{ct}{1+$$

$$pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{\Gamma(i+\frac{1}{2})}{(i-1)!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}$$

$$= pt\ (1+ct)^{-\frac{1}{2}} (M_1(t)+1) (1-\frac{ct}{1+ct})^{-\frac{1}{2}}+$$

$$pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{\Gamma(i+\frac{1}{2})}{(i-1)!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}$$

$$= pt\ (1+ct)^{-\frac{1}{2}} (M_1(t)+1) (1+ct)^{\frac{1}{2}}+$$

$$pt\ (1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{\Gamma(i+\frac{1}{2})}{(i-1)!\ \Gamma(\frac{1}{2})} (\frac{ct}{1+ct})^{i}$$

$$
\begin{aligned}
M_2(t) &= pt\,(M_1(t)+1)+ \\
&\quad pt\,(1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(i+\frac{1}{2}-1)(i+\frac{1}{2}-2)\ldots.(i+\frac{1}{2}-(i-1))\Gamma(i+\frac{1}{2}-}{(i-1)!\,\Gamma(\frac{1}{2})} \\
&= pt\,(M_1(t)+1)+ \\
&\quad pt\,(1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(i+\frac{1}{2}-1)(i+\frac{1}{2}-2)\ldots.\frac{3}{2}\Gamma(\frac{3}{2})}{(i-1)!\,\Gamma(\frac{1}{2})} \left(\frac{ct}{1+ct}\right)^i \\
&= pt\,(M_1(t)+1)+ \\
&\quad pt\,(1+ct)^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(i+\frac{1}{2}-1)(i+\frac{1}{2}-2)\ldots.\frac{3}{2}\,\frac{1}{2}}{(i-1)!} \left(\frac{ct}{1+ct}\right)^i \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct} \sum_{i=0}^{\infty} \binom{i+\frac{1}{2}-1}{i-1} \left(\frac{ct}{1+ct}\right)^{i-} \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct} \sum_{i=0}^{\infty} \binom{(\frac{1}{2}+1)+(i-1)-1}{i-1} \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct} \sum_{i=0}^{\infty} (-1)^{i-1} \binom{-\frac{3}{2}}{i-1} \left(\frac{ct}{1+\right. \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct} \sum_{i=1}^{\infty} \binom{-\frac{3}{2}}{i-1} \left(-\frac{ct}{1+ct}\right)^{i-1} \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct} \left(1-\frac{ct}{1+ct}\right)^{-\frac{3}{2}} \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct} \left(\frac{1}{1+ct}\right)^{-\frac{3}{2}} \\
&= pt\,(M_1(t)+1) + \frac{pt}{2}\,(1+ct)^{-\frac{1}{2}}\,\frac{ct}{1+ct}\,(1+ct)^{\frac{3}{2}} = pt\,(M_1(t)+1 \\
M_2(t) &= (pt)^2 + pt + \frac{pc\,t^2}{2} \\
Var(Z(t)) &= M_2(t) - (M_1(t))^2 \\
&= (pt)^2 + pt + \frac{pc\,t^2}{2} - (pt)^2 \\
&= pt + \frac{pc\,t^2}{2}
\end{aligned}
$$

Using notations by Willmot (1986), let

$$t = 1, \qquad c = 2\beta \qquad p = \mu \qquad and \qquad s = z$$

Then we get

$$p_n(t) = \frac{p_0\,(\mu)^n}{n!} \sum_{k=0}^{n-1} \frac{(n-1+k)!}{(n-1-k)!k!} \left(\frac{\beta}{2\mu}\right)^k (1+2\beta)^{-\frac{(n+k)}{2}}\, p_0(t) \qquad for \tag{4.90}$$

where,

$$p_0(t) = e^{-\frac{\mu}{\beta}\,[(1+2\beta)^{\frac{1}{2}}-1]}$$

The pgf $H(s,t)$ is given by

$$P(z) = e^{-\frac{\mu}{\beta}\,[(1-2\beta(z-1))^{\frac{1}{2}}-1]}$$

The mean and variance are

$$E(N) = E(Z(t)) = \mu \qquad Var(N) = Var(z(z(t)) = \mu(1+\beta)$$

The pgf, $\mathbf{G(s,t)}$ of the iid random variables is given by

$$Q(z) = \frac{[(1-2\beta(z-1))]^{\frac{1}{2}} - (1+2\beta)^{\frac{1}{2}}}{1-(1+2\beta)^{\frac{1}{2}}}$$

The pmf $g_x(t)$ of the iid random variables is given by

$$q_n = \frac{\frac{1}{2}\Gamma(n-\frac{1}{2})\ (1+2\beta)^{\frac{1}{2}}\ (\frac{2\beta}{1+2\beta})^n}{n!\Gamma(\frac{1}{2})\ [(1+2\beta)^{\frac{1}{2}-1]}} \qquad for \quad n = 1,2,3\ldots$$

which satisfies Panjer's recursive model with

$$a = \frac{2\beta}{1+2\beta} \qquad and \qquad b = \frac{-3\beta}{1+2\beta}$$

Using notations of Sankara (1968), let

$$t = 1, \qquad n = r \qquad \frac{p}{(1+c)^{\frac{1}{2}}} \qquad b = -\frac{c}{1+c} \qquad and \qquad i = k$$

The recursive formula for the Poisson-inverse Gaussian distribution becomes

$$\begin{aligned}(r+1)p_{r+1}(t) &= a\sum_{k=0}^{r}\frac{\Gamma(k+\frac{1}{2})}{k!\Gamma(\frac{1}{2})}\,(-b)^k\,p_{r-k} = a\sum_{k=0}^{r}\binom{\frac{1}{2}+k-1}{k}\,(-b)^k\,p_{r-k} \\ &= a\sum_{k=0}^{r}\frac{(k+\frac{1}{2}-1)(k+\frac{1}{2}-2)\ldots.(k+\frac{1}{2}-k)}{k!}\,(-b)^k\,p_{r-k} \\ &= a\sum_{k=0}^{r}\frac{(2k-1)(2k-3)\ldots.(5.3.1)\ (\frac{-b}{k})^k\ \ p_{r-k}}{k!} \qquad for \qquad r = 0\end{aligned} \tag{4.91}$$

### 4.6.6 When the hazard function in (4.26) is such that a = 2, $p > 0$ and $c > 0$

$$h(t) = \theta'(t) = \frac{p}{(1+ct)^2} \quad for \quad p > 0, \quad and \quad c > 0 \tag{4.92}$$

which we shall refer to as Polya-Aeppli hazard function.

Therefore,

$$\begin{aligned}
h'(t) &= \theta'(t) = -2pc\,(1+ct)^{-3} \\
h''(t) &= \theta''(t) = (-1)^2\, 3!pc^2(1+ct)^{-4} \\
h'''(t) &= \theta'''(t) = (-1)^3\, 4!pc^3(1+ct)^{-5} \\
\therefore \quad h^{(n)}(t) &= (-1)^n\,(n+1)!pc^n(1+ct)^{-(n+2)}
\end{aligned}$$

$$(-1)^n \frac{d^n}{dt^n}\, h(t) = (-1)^n\, h^n(t) = (-1)^n\,(n+1)!pc^n(1+ct)^{-(n+2)} \geq 0$$

Therefore $\mathbf{h(t)}$ is completely monotone.

$$\begin{aligned}
\theta(t) &= p\int_0^t (1+cx)^{-2}\,dx = \frac{p}{-c}\,[(1+cx)^{-1}]_0^t \\
&= -\frac{p}{c}\,[\frac{1}{1+ct} - 1] = \frac{pt}{1+ct} \qquad (4.93)
\end{aligned}$$

$$\Rightarrow \quad \theta(t-ts) = \frac{p(t-ts)}{1+ct-cts} = \frac{pt-pts}{1-ct-cts}$$

$$\therefore \quad \theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -2pc$$

$$p_0(t) = e^{-\frac{pt}{1+ct}}$$

$$p_n(t) = (-1)^n \frac{t^n}{n!}\, p_0^{(n)}(t)$$

Therefore,

$$p_0(t) = e^{-\frac{2p}{c}\,[(1+ct)^2-1]}$$

The pgf is

$$H(s,t) = e^{-\theta(t-ts)} = e^{-\frac{pt-pts}{1-ct-cts}} \qquad (4.94)$$

Therefore,

$$E(Z(t)) = t\,\theta'(0) = pt \qquad and \qquad Var(Z(t)) = t\,\theta'(0) - t^2\,\theta''(0) = pt + 2\,pc\,t^2$$

Since $\boldsymbol{\theta(0) = 0}$ and $\mathbf{h(t)} = \boldsymbol{\theta'}\mathbf{(t)}$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \frac{pt-pts}{1-ct-cts}\,\frac{1+ct}{pt}$$

$$= 1 - \frac{1-s}{1-ct-cts}\,(1+ct) = \frac{s}{1+ct-cts}$$

$$= \frac{s}{(1+ct)(1-\frac{ct}{1+ct}\,s)} = \frac{\frac{1}{1+ct}\,s}{1-\frac{ct}{1+ct}\,s} \tag{4.95}$$

which is the pgf of zero-truncated (shifted) geometric distribution.

By power series expansion

$$
\begin{aligned}
G(s,t) &= \frac{1}{1+ct}\,s\sum_{x=0}^{\infty}(\frac{ct}{1+ct})^{x}\,s^{x} \\
&= \sum_{x=0}^{\infty}\frac{1}{1+ct}\,(\frac{ct}{1+ct})^{x}\,s^{x+1} \\
&= \frac{1}{1+ct}\,[s+\frac{ct}{1+ct}\,s^2+(\frac{ct}{1+ct})^2\,s^3\ldots] \\
&= \sum_{x=1}^{\infty}\frac{1}{1+ct}\,(\frac{ct}{1+ct})^{x-1}\,s^{x} \\
g_x(t) &= \frac{1}{1+ct}\,(\frac{ct}{1+ct})^{x-1} \qquad x=1,2,3.. \qquad and \qquad g_0(t)=0 \\
\frac{g_x(t)}{g_{x-1}(t)} &= (\frac{ct}{1+ct})^{x-1}\,(\frac{(1+ct)}{(ct)})^{x-2} = \frac{ct}{1+ct} \qquad x=2,3,\ldots \\
&= (\frac{ct}{1+ct}+\frac{0}{x}) \qquad for \qquad x=2,3,\ldots
\end{aligned} \tag{4.96}
$$

which is Panjer's form with $\mathbf{a} = \frac{\mathbf{ct}}{\mathbf{1+ct}}$ and $\mathbf{b = 0}$

The compound Poisson distribution in the recursive form is given by

$$
\begin{aligned}
np_n(t) &= \theta(t)\,\sum_{i=0}^{n}(i+1)\,g_{i+1}(t)\,p_{n-i}(t) \\
&= \frac{pt}{1+ct}\,\sum_{i=0}^{n}(i+1)\,\frac{1}{1+ct}\,(\frac{ct}{1+ct})^{i}\,p_{n-i}(t) \\
&= \frac{pt}{(1+ct)^2}\,\sum_{i=0}^{n}(i+1)\,\frac{1}{1+ct}\,(\frac{ct}{1+ct})^{i}\,p_{n-i}(t); \qquad n=0,1,2\ldots
\end{aligned} \tag{4.97}
$$

By iteration,

n=0

$$p_1(t) = \frac{pt}{(1+ct)^2}\,p_0(t)$$

n=1

$$
\begin{aligned}
2p_2(t) &= \frac{pt}{(1+ct)^2}\ \sum_{i=0}^{1} (i+1)\ (\frac{ct}{1+ct})^i\ p_{1-i}(t) \\
&= \frac{pt}{(1+ct)^2}\ [p_1(t) + \frac{ct}{1+ct}\ p_0(t)] \\
&= \frac{pt}{(1+ct)^2}\ [\frac{pt}{(1+ct)^2}\ p_0(t) + 2\ \frac{ct}{(1+ct)}\ p_0(t)] \\
2\ p_2(t) &= [\frac{(pt)^2}{(1+ct)^4} + 2\ \frac{pt\ ct}{(1+ct)^3}]\ p_0(t) \\
p_2(t) &= [\frac{1}{2}\ \frac{(pt)^2}{(1+ct)^4} + \frac{pt\ ct}{(1+ct)^3}]\ p_0(t)
\end{aligned}
$$

n=2

$$
\begin{aligned}
3\ p_3(t) &= \frac{pt}{(1+ct)^2}\ \sum_{i=0}^{2} (i+1)\ (\frac{ct}{1+ct})^i\ p_{2-i}(t) \\
&= \frac{pt}{(1+ct)^2}\ [p_2(t) + 2\ \frac{ct}{1+ct}\ p_1(t) + 3\ \frac{(ct)^2}{(1+ct)^2}\ p_0(t)] \\
&= \frac{pt}{(1+ct)^2}\ [[\frac{1}{2}\ \frac{(pt)^2}{(1+ct)^4} + \frac{pt\ ct}{(1+ct)^3}]\ p_0(t) + \\
&\quad 2\ \frac{ct}{1+ct}\ \frac{pt}{(1+ct)^2}\ p_0(t) + 3\ \frac{(ct)^2}{(1+ct)^2}\ p_0(t)] \\
&= [\frac{1}{2}\ \frac{(pt)^3}{(1+ct)^6} + \frac{(pt)^2\ ct}{(1+ct)^5} + 2\ \frac{(pt)^2\ ct}{(1+ct)^5} + 3\ \frac{pt\ (ct)^2}{(1+ct)^4}]\ p_0(t) \\
p_3(t) &= [\frac{1}{3!}\ \frac{(pt)^3}{(1+ct)^6} + \frac{(pt)^2\ ct}{(1+ct)^5} + \frac{pt\ (ct)^2}{(1+ct)^4}]\ p_0(t)
\end{aligned}
$$

n=3

$$
\begin{aligned}
4\ p_4(t) &= \frac{pt}{(1+ct)^2}\ \sum_{i=0}^{3} (i+1)\ (\frac{ct}{1+ct})^i\ p_{3-i}(t) \\
&= \frac{pt}{(1+ct)^2}\ [p_3(t) + 2\ \frac{ct}{1+ct}\ p_2(t) + 3\ \frac{(ct)^2}{(1+ct)^2}\ p_1(t) + 4\ \frac{(ct)^3}{(1+ct)^3}\ p_0(t \\
&= \frac{pt}{(1+ct)^2}\ [[\frac{1}{3!}\ \frac{(pt)^3}{(1+ct)^6} + \frac{(pt)^2\ ct}{(1+ct)^5} + \frac{pt\ (ct)^2}{(1+ct)^4}]\ p_0(t) \\
&\quad + 2\ \frac{ct}{1+ct}\ [\frac{1}{2}\ \frac{(pt)^2}{(1+ct)^4} + \frac{pt\ ct}{(1+ct)^3}]\ p_0(t) + 3\ \frac{(ct)^2}{(1+ct)^2}\ \frac{pt}{(1+ct)^2}\ p_0(t) \\
&\quad 4\ \frac{(ct)^3}{(1+ct)^3}\ p_0(t)] \\
&= [\frac{1}{3!}\ \frac{(pt)^4}{(1+ct)^8} + \frac{(pt)^3\ ct}{(1+ct)^7} + \frac{(pt)^2\ (ct)^2}{(1+ct)^6}
\end{aligned}
$$

$$+2\,\frac{1}{2}\,\frac{(pt)^3\,ct}{(1+ct)^7}+2\,\frac{(pt)^2\,(ct)^2}{(1+ct)^6}+3\,\frac{(pt)^2\,(ct)^2}{(1+ct)^6}+4\,\frac{pt\,(ct)^3}{(1+ct)^5}]\,p_0(t)$$

$$=[\frac{1}{3!}\,\frac{(pt)^4}{(1+ct)^8}+2\,\frac{(pt)^3\,ct}{(1+ct)^7}+6\,\frac{(pt)^2\,(ct)^2}{(1+ct)^6}+4\,\frac{pt\,(ct)^3}{(1+ct)^5}]\,p_0(t)$$

$$p_4(t)=[\frac{pt\,(ct)^3}{(1+ct)^5}+\frac{3}{2}\,\frac{(pt)^2\,(ct)^2}{(1+ct)^6}+\frac{1}{2}\,\frac{(pt)^3\,ct}{(1+ct)^7}+\frac{1}{4!}\,\frac{(pt)^4}{(1+ct)^8}]\,p_0(t)$$

$$=\frac{pt}{(1+ct)^5}\,[\frac{(ct)^3}{1}+\frac{3}{2}\,\frac{pt\,(ct)^2}{1+ct}+\frac{1}{2}\,\frac{(pt)^2\,ct}{(1+ct)^2}+\frac{1}{4!}\,\frac{(pt)^3}{(1+ct)^3}]\,p_0(t)$$

$$=\frac{pt}{(1+ct)^5}\,[\frac{1}{1!}\,\frac{(pt)^0}{(1+ct)^0}\,(ct)^3+3\,\frac{1}{2!}\,\frac{(pt)^1}{(1+ct)^1}\,(ct)^2+$$
$$3\,\frac{1}{3!}\,\frac{(pt)^2}{(1+ct)^2}\,(ct)^1+\frac{1}{4!}\,\frac{(pt)^3}{(1+ct)^3}\,(ct)^0]\,p_0(t)$$

$$=\frac{pt}{(1+ct)^5}\,[\binom{3}{0}\,\frac{1}{1!}\,\frac{(pt)^0}{(1+ct)^0}\,(ct)^3+\binom{3}{1}\,\frac{1}{2!}\,\frac{(pt)^1}{(1+ct)^1}\,(ct)^2+$$
$$\binom{3}{2}\,\frac{1}{3!}\,\frac{(pt)^2}{(1+ct)^2}\,(ct)^1+\binom{3}{3}\,\frac{1}{4!}\,\frac{(pt)^3}{(1+ct)^3}\,(ct)^0]\,p_0(t)$$

$$p_4(t)=\frac{pt}{(1+ct)^5}\,\sum_{j=1}^{4}\binom{4-1}{j-1}\,[\frac{1}{j!}\,(\frac{pt}{1+ct})^{j-1}\,(ct)^{4-j}]\,p_0(t)$$

$$p_4(t)=(\frac{ct}{1+ct})^4\,p_0(t)\,\sum_{j=1}^{4}\binom{4-1}{j-1}\,(\frac{pt}{ct\,(1+ct)})^j\,\frac{1}{j!}$$

$$\therefore\quad p_n(t)=(\frac{ct}{1+ct})^n\,p_0(t)\,\sum_{j=1}^{n}\binom{n-1}{j-1}\,(\frac{pt}{ct\,(1+ct)})^j\,\frac{1}{j!}\qquad(4.98)$$

which is the Polya-Aeppli distribution and $p_0(t)=e^{-\frac{pt}{1+ct}}$

### 4.6.7 When the hazard function in (4.26) is such that $p>0$ and $b=ac$ as a $\to\infty$

$$h(t)=\theta'(t)$$
$$=\lim_{a\to\infty}\,p\,(1+c_2t)^{-a}\quad for\quad p>0,\quad and\quad c>0$$
$$=\lim_{a\to\infty}\,p\,\sum_{k=0}^{\infty}\frac{-a\,(-a-1)\,(-a-2)\ldots[-a-(k-1)]\,(c_2t)^k}{k!}$$
$$=\lim_{a\to\infty}\,p\,\sum_{k=0}^{\infty}\frac{(-1)^k\,a^k\,(1+\frac{1}{a})\,(1+\frac{2}{a})\ldots[(1+\frac{(k-1)}{a})]\,(c_2t)^k}{k!}$$
$$=p\,\sum_{k=0}^{\infty}\,\lim_{a\to\infty}\,\frac{(-ac)^k\,(t)^k}{k!}$$

Let,

$$\begin{aligned} b &= ac \quad as \quad a \rightarrow \infty \\ h(t) &= \theta'(t) \\ &= p \sum_{k=0}^{\infty} \frac{(-bt)^k}{k!} \\ &= p\, e^{-bt} \end{aligned} \tag{4.99}$$

which is the hazard function for Gompertz distribution.

Therefore,

$$\begin{aligned} h^{(n)}(t) &= (-1)^n\, b^n\, p\, e^{-bt} \\ (-1)^n \frac{d^n}{dt^n}\, h(t) &= (-1)^n\, h^n(t) \geq 0 \end{aligned}$$

Therefore $\mathbf{h(t)}$ is completely monotone.

The cumulative hazard function is

$$\begin{aligned} \theta(t) &= p \int_0^t e^{-bx}\, dx \\ &= \frac{p}{b}\, [1 - e^{-bt}] \end{aligned} \tag{4.100}$$

implying that

$$\theta(t - ts) = \frac{p}{b}\, [1 - e^{-bt(1-s)}]$$

Therefore,

$$\theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -b\, p$$

Since Gompertz distribution is an exponential mixture, the survival function is the Laplace transform of the mixing distribution and hence,

$$\begin{aligned} p_0(t) &= e^{\frac{p}{b}\, [e^{-bt} - 1]} \\ &= e^{-\frac{p}{b}} \sum_{j=0}^{\infty} [\frac{p}{b}\, e^{-bt}]^j\, \frac{1}{j!} \\ &= \sum_{j=0}^{\infty} e^{-btj}\, [e^{-\frac{p}{b}}\, \frac{[\frac{p}{b}]^j}{j!}] \\ p_0^{(n)}(t) &= \sum_{j=0}^{\infty} (-bj)^n\, e^{-btj}\, p_j \end{aligned}$$

$$p_n(t) = (-1)^n \frac{t^n}{n!} \sum_{j=0}^{\infty} (-bj)^n \, e^{-btj} \, p_j$$

$$p_n(t) = \sum_{j=0}^{\infty} (-1)^n \frac{t^n}{n!} (-bj)^n \, e^{-btj} \, p_j$$

$$= \sum_{j=0}^{\infty} \frac{(btj)^n}{n!} \, e^{btj} \, e^{-\frac{p}{b}} \, \frac{[\frac{p}{b}]^j}{j!}; \qquad j = 0, 1, 2 \ldots. \tag{4.101}$$

which is Poisson mixture of Poisson distribution also known as Neyman Type A distribution.

The pgf is

$$H(s,t) = e^{-\theta(t - ts)}$$

$$= e^{-\frac{p}{b}\,[1 - e^{-bt(1-s)}]} \tag{4.102}$$

Therefore,

$$E(Z(t)) = t\,\theta'(0) = pt \qquad and \qquad Var(Z(t)) = t\,\theta'(0) - t^2\,\theta''(0) = pt + pb\,t^2$$

Since $\theta(0) = 0$ and $\mathbf{h(t)} = \theta'(\mathbf{t})$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{1 - e^{-bt\,(1-s)}}{1 - e^{-bt}} = \frac{e^{-bt\,(1-s)} - e^{-bt}}{1 - e^{-bt}}$$

$$= \frac{e^{-bt}}{1 - e^{-bt}} \, [e^{bts} - 1]$$

$$= \frac{e^{-bt}}{1 - e^{-bt}} \, [\sum_{x=0}^{\infty} \frac{(bt)^x \, s^x}{x!} - 1]$$

$$g_x(t) = \frac{1}{e^{bt} - 1} \, \frac{(bt)^x}{x!} \qquad x = 1, 2 \ldots \tag{4.103}$$

which is zero-truncated Poisson distribution with parameter $bt$

$$\frac{g_x(t)}{g_{x-1}(t)} = 0 + \frac{bt}{x} \qquad x = 1, 2, \ldots \tag{4.104}$$

and it is in Panjer's recursive form, where $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{bt}$

**Remark 4.7.** . *Whereas Klugman et. al. (2008) have stated that the distribution of the iid random variables is Poisson, in this study, the distribution is zero-truncated Poisson distribution. The difference is due to the assumption that $N$ is Poisson with a constant parameter $\lambda$ instead of $\theta(t)$, which is a cumulative hazard function.*

The recursive formula for the compound Poisson distribution is given by:

$$(n+1)\, p_{n+1}(t) = \frac{p}{b}\, [1 - e^{-bt}] \sum_{i=0}^{n} (i+1) \, \frac{1}{e^{bt} - 1} \, \frac{(bt)^{i+1}}{(i+1)!} \, p_{n-i}(t)$$

$$= pt\, e^{-bt} \sum_{i=0}^{n} \frac{(bt)^i}{i!}\, p_{n-i}(t) \qquad n = 0, 1, 2, \ldots \tag{4.105}$$

### 4.6.8 When the hazard function in (4.26) is such that $p > 0$ $c > 0$, $a \neq 0$ *and* $a \neq 1$

$$h(t) = \theta'(t) = \frac{p}{(1+ct)^a} \quad for \quad p > 0, \quad c > 0 \quad \text{and} \quad a > 0 \quad \text{but} \tag{4.106}$$

Therefore

$$h'(t) = \theta''(t) = -acp\,(1+ct)^{-a-1}$$
$$h''(t) = \theta''(t) = (-1)^2\, a(a+1)c^2 p\,(1+ct)^{-a-2}$$
$$h'''(t) = \theta'''(t) = (-1)^3\, a(a+1)(a+2)c^3 p\,(1+ct)^{-a-3}$$
$$h^{(n)}(t) = \theta^{(n)}(t) = (-1)^n\, a(a+1)a+3)\ldots(a+n-1)c^n p\,(1+ct)^{-}$$

Therefore $h(t)$ is completely monotone.

$$\theta(t) = p \int_0^t (1+cx)^{-a}\, dx = \frac{p}{c(1-a)}\left[(1+ct)^{1-a} - 1\right] \tag{4.107}$$

implying that

$$\theta(t-ts) = \frac{p}{c(1-a)}\left[(1+ct-cts)^{1-a} - 1\right]$$

and

$$\theta(0) = 0 \qquad \text{and} \qquad \theta''(0) = -acp$$

also

$$p_0(t) = e^{-\frac{p}{c(1-a)}\left[(1+ct)^{1-a}-1\right]} \tag{4.108}$$

and,

$$H(s,t) = e^{-\theta(t-ts)} = e^{-\frac{p}{c(1-a)}\left[(1+ct-cts)^{1-a}-1\right]} \tag{4.109}$$

Therefore,

$$E(Z(t)) = t\,\theta'(0) = pt \qquad and \qquad Var(Z(t)) = t\,\theta'(0) - t^2\,\theta''(0) =$$

Since $\theta(0) = 0$ and $\mathbf{h(t)} = \theta'(\mathbf{t})$ is completely monotone, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$
\begin{aligned}
G(s,t) &= 1 - \frac{\theta(t-ts)}{\theta(t)} \\
&= 1 - \frac{1+ct-cts)]^{1-a} - 1}{(1+ct)^{1-a}-1} = \frac{(1+ct)^{1-a} - (1+ct-cts)^{1-a}}{(1+ct)^{1-a}-1} \\
&= \frac{(1+ct)^{1-a}}{(1+ct)^{1-a}-1} \; (1-(1-\frac{ct}{1+ct}\; s)^{1-a}) \\
&= \frac{(1+ct)^{1-a}}{(1+ct)^{1-a}-1} \; (1-\sum_{x=0}^{\infty} \binom{1-a}{x} \; (\frac{-cts}{1+ct})^{x}) \\
&= \frac{(1+ct)^{1-a}}{(1+ct)^{1-a}-1} \; (-1) \; \sum_{x=1}^{\infty} \binom{1-a}{x} \; (\frac{-ct}{1+ct})^{x} \; s^{x} \\
g_x(t) &= \frac{(1+ct)^{1-a}}{(1+ct)^{1-a}-1} \; (-1) \; \binom{1-a}{x} \; (\frac{-ct}{1+ct})^{x}
\end{aligned}
$$

But

$$
\begin{aligned}
(-1) \; \binom{1-a}{x} &= \frac{-(1-a)[(1-a)-1][(1-a)-2]\ldots\ldots[(1-a)-(x-1)]}{x!} \\
&= \frac{-(1-a)}{x!} \; [-a][-a-1][-a-2]\ldots\ldots[(-a-x+2)] \\
&= \frac{-(1-a)}{x!} \; (-1)^{x-1} \; [a][a+1][a+2]\ldots\ldots[(a+x-2)] \\
&= \frac{(-1)^{x}}{x!} \; \frac{[1-a]}{1} \; [(a+x-2)][(a+x-3)]\ldots\ldots[a+2][a+ \\
&= \frac{(-1)^{x}}{x!} \; \frac{[1-a]}{\Gamma(a)} \; [(a+x-2)][(a+x-3)]\ldots\ldots[a+2][a+ \\
&= \frac{(-1)^{x}}{x!} \; \frac{[1-a]}{\Gamma(a)} \; [(a+x-2)][(a+x-3)]\ldots\ldots[a+2][a+ \\
&= \frac{(-1)^{x}}{x!} \; \frac{[1-a]}{\Gamma(a)} \; [(a+x-2)][(a+x-3)]\ldots\ldots[a+2] \; \Gamma(a \\
&= \frac{(-1)^{x}}{x!} \; \frac{[1-a]}{\Gamma(a)} \; [(a+x-2)] \; \Gamma(a+x-2) = \frac{(-1)^{x}}{x!} \; \frac{[1-a}{\Gamma(a)} \\
g_x(t) &= \frac{1-a}{x!} \; \frac{\Gamma(a+x-1)}{\Gamma(a)} \; (\frac{ct}{1+ct})^{x} \; \frac{(1+ct)^{1-a}}{(1+ct)^{1-a}-1} \qquad for\; x =
\end{aligned}
\tag{4.110}
$$

and,

$$
g_0 = 0 \tag{4.111}
$$

The recursive formula of the compound Poisson distribution is therefore

$$
n \; p_n(t) = \theta(t) \; \sum_{x=1}^{n} x \; g_x(t) \; p_{n-x}(t)
$$

$$= \frac{p}{c(1-a)} \, [(1+ct)^{1-a} - 1] \sum_{x=1}^{n} x \, \frac{1-a}{x!} \, \frac{\Gamma(a+x-1)}{\Gamma(a)} \, (\frac{ct}{1+c$$

$$= \frac{pt}{1+ct} \sum_{x=1}^{n} \frac{(1+ct)^{1-a)}}{(x-1)!} \, \frac{\Gamma(a+x-1)}{\Gamma(a)} \, (\frac{ct}{1+ct})^{x-1} \, p_{n-x}(t)$$

$$= \frac{pt}{1+ct} \, (1+ct)^{1-a)} \sum_{x=1}^{n} \frac{\Gamma(a+x-1)}{(x-1)! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{x-1} \, p_{n-x}(t)$$

$$= \frac{pt}{(1+ct)^{a}} \sum_{x=1}^{n} \frac{\Gamma(a+x-1)}{(x-1)! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{x-1} \, p_{n-x}(t) \quad for \quad x \tag{4.112}$$

Therefore by replacing $n$ by $n+1$, we have

$$(n+1) \, p_{(n+1)}(t) = \frac{pt}{(1+ct)^{a}} \sum_{x=1}^{n+1} \frac{\Gamma(a+x-1)}{(x-1)! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{x-1} \, p_{n+1-x}(t)$$

Put $x = i+1$ we get

$$(n+1) \, p_{(n+1)}(t) = \frac{pt}{1+ct} \, (1+ct)^{1-a)} \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i} \, p_{n-i}(t) \quad for \quad n \tag{4.113}$$

as given by Walhin and Paris (2002)

**Moments**

To obtain the first moment, sum 4.113 over **n** to get

$$\sum_{n=0}^{\infty} (n+1) \, p_{n+1}(t) = \frac{pt}{(1+ct)^{a}} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i} \, p_{n-i}(t)$$

$$= \frac{pt}{(1+ct)^{a}} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i} \, p_{n-i}(t)$$

$$= \frac{pt}{(1+ct)^{a}} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i} \, p_{n-i}(t)$$

$$= \frac{pt}{(1+ct)^{a}} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i} \sum_{n=0}^{\infty} p_{n-i}(t)$$

$$= \frac{pt}{(1+ct)^{a}} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i} \sum_{n=i}^{\infty} p_{n-i}(t)$$

$$\sum_{n=0}^{\infty} (n+1) \, p_{n+1}(t) = \frac{pt}{(1+ct)^{a}} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i! \, \Gamma(a)} \, (\frac{ct}{1+ct})^{i}$$

Let,

$$M_1(t) = \sum_{n=1}^{\infty} n \, p_n(t) = \sum_{n=0}^{\infty} (n+1) \, p_{n+1}(t)$$

$$
\begin{aligned}
M_1(t) &= \frac{pt}{(1+ct)^a} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i \\
&= \frac{pt}{(1+ct)^a} \sum_{i=0}^{\infty} \binom{a+i-1}{i} \left(\frac{ct}{1+ct}\right)^i \\
&= \frac{pt}{(1+ct)^a} \sum_{i=0}^{\infty} (-1)^i \binom{-a}{i} \left(\frac{ct}{1+ct}\right)^i \\
&= \frac{pt}{(1+ct)^a} \sum_{i=0}^{\infty} \binom{-a}{i} \left(\frac{-ct}{1+ct}\right)^i \\
&= \frac{pt}{(1+ct)^a}) \left(1-\frac{ct}{1+ct}\right)^{-a} \\
M_1(t) &= \frac{pt}{(1+ct)^a} \left(\frac{1}{1+ct}\right)^{-a} = pt \qquad (4.114)
\end{aligned}
$$

Next, multiply 4.113 by $\mathbf{n+1}$ and then sum the result over $\mathbf{n}$, to get

$$
\begin{aligned}
M_2(t) &= \sum_{n=0}^{\infty} (n+1)^2\, p_{n+1}(t) \\
&= \sum_{n=0}^{\infty} [\frac{pt}{(1+ct)^a})\, (n+1) \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i p_{n-i}(t) \\
&= \frac{pt}{(1+ct)^a}) \sum_{n=0}^{\infty} \sum_{i=0}^{n} (n+1) \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i p_{n-i}(t) \\
&= \frac{pt}{(1+ct)^a}) \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i (n+1)\, p_{n-i}(t) \\
&= \frac{pt}{(1+ct)^a}) \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i \sum_{n=0}^{\infty} (n+1)\, p_{n-i}(t) \\
&= \frac{pt}{(1+ct)^a}) \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i \sum_{n=0}^{\infty} (n-i+i+1) p_{n-i} \\
&= \frac{pt}{(1+ct)^a}) \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i [M_1(t)+(i+1)] \\
&= \frac{pt}{(1+ct)^a})\, [M_1(t)+1] \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i \\
&\quad + \sum_{i=0}^{n} \frac{\Gamma(a+i)}{(i-1)!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i \\
&= \frac{pt}{(1+ct)^a})\, [(pt+1) \sum_{i=0}^{n} \frac{\Gamma(a+i)}{i!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^i \\
&\quad + \frac{ct}{1+ct} \sum_{i=0}^{n} \frac{\Gamma(a+i)}{(i-1)!\,\Gamma(a)} \left(\frac{ct}{1+ct}\right)^{i-1}]
\end{aligned}
$$

$$
\frac{\Gamma(a+i)}{(i-1)!\,\Gamma(a)} = \frac{(a+i-1)(a+i-2)(a+i-3)\ldots(a+i-(i-1))(a+i-}{(i-1)!\,\Gamma(a)}
$$

$$
\begin{aligned}
&= \frac{(a+i-1)(a+i-2)(a+i-3)\dots(a+i-(i-1))\ a}{(i-1)!} \\
&= \binom{a+i-1}{i-1}\ a \\
\sum_{i=0}^{n}\ \frac{\Gamma(a+i)}{(i-1)!\,\Gamma(a)}\ (\frac{ct}{1+ct})^{i} &= \frac{act}{1+ct}\ \sum_{i=0}^{n}\ \binom{a+i-1}{i-1}\ (\frac{ct}{1+ct})^{i-1} \\
&= \frac{act}{1+ct}\ \sum_{i=0}^{n}\ (-1)^{(i-1)}\ \binom{-(a+1)}{i-1}\ (\frac{ct}{1+ct})^{i-1} \\
&= \frac{act}{1+ct}\ \sum_{i=1}^{n}\ \binom{-(a+1)}{i-1}\ (\frac{-ct}{1+ct})^{i-1} \\
&= \frac{act}{1+ct}\ (1-\frac{ct}{1+ct})^{-(a+1)} = \frac{act}{1+ct}\ \frac{1}{(1+ct)^{-(a+1)}} = act\ (1 \\
M_2(t) &= \frac{pt}{(1+ct)^{a}}\ [(pt+1)(1+ct)^{a} + act(1+ct)^{a}] = pt\ [(pt+1) - \\
Var(Z(t)) &= M_2(t) - (M_1(t))^2 = pt[pt+1+act] - (pt)^2 \\
&= (pt)^2 + pt(1+act) - (pt)^2 \\
Var(Z(t)) &= M_2(t) - (M_1(t))^2 \\
&= pt[pt+1+act] - (pt)^2 = (pt)^2 + pt(1+act) - (pt)^2 \\
&= pt + acpt^2 \qquad (4.115)
\end{aligned}
$$

## 4.7 Parameterization of Hofmann Hazard Function

This section identifies the hazard function of an exponential mixture which accommodates the extended truncated negative binomial (ETNB) distribution as the distribution of the iid random variables for the compound Poisson distribution.

### 4.7.1 Extended Truncated Negative Binomial Distribution

$$
\begin{aligned}
[1-q(t)]^{-r} &= \sum_{k=0}^{\infty}\ \binom{-r}{k}\ (-q(t))^{k} = 1 + \sum_{k=1}^{\infty}\ \binom{-r}{k}\ (-q(t))^{k} \\
[1-q(t)]^{-r} - 1 &= \sum_{k=1}^{\infty}\ \binom{-r}{k}\ (-q(t))^{k} = \sum_{k=1}^{\infty}\ (-1)^{k}\ \binom{-r}{k}\ q(t)^{k} \\
[1-q(t)]^{-r}) - 1 &= \sum_{k=1}^{\infty}\ \binom{r+k-1}{k}\ q(t)^{k} \\
1 &= \sum_{k=1}^{\infty}\ \frac{\binom{r+k-1}{k}\ q(t)^{k}}{(1-q(t))^{-r}) - 1} \\
\therefore \qquad Prob(X=k) &= \frac{\binom{r+k-1}{k}\ q(t)^{k}}{(1-q(t))^{-r} - 1} \qquad k = 1, 2, 3, \dots \qquad (4.116)
\end{aligned}
$$

is a probability mass function called zero-truncated negative binomial distribution with parameter $\boldsymbol{q(t)}$. It is also called the extended truncated negative binomial (ETNB) distribution according to Klugman et. al. (2008) because the parameter $\boldsymbol{r}$ can extend below zero.

Let,

$$q(t) = \frac{ct}{1+ct}, \qquad c > 0, t > 0.$$

$$\textit{Then} \qquad Prob(X = k) = \frac{\binom{r+k-1}{k} \left(\frac{ct}{1+ct}\right)^k}{(1+ct)^r - 1} \qquad k = 1, 2, 3, \ldots$$

The probability generating function is given by:

$$G(s,t) = \sum_{k=1}^{\infty} p_k\, s^k = \sum_{k=1}^{\infty} \frac{\binom{r+k-1}{k} \left(\frac{ct}{1+ct}\, s\right)^k}{(1+ct)^r - 1} = \frac{\sum_{k=1}^{\infty} \binom{-r}{k} \left(-\frac{ct}{1+ct}\right.}{(1+ct)^r - 1}$$

$$G(s,t) = \frac{(1 - \frac{ct}{1+ct}\, s)^{-r} - 1}{(1+ct)^r - 1} \tag{4.117}$$

### 4.7.2 Identifying the Hazard Function

Let us parameterize Hofmann hazard function by putting $\mathbf{a = r + 1}$.
Thus

$$h(t) = \frac{p}{(1+ct)^{r+1}} \qquad p > 0, \quad c > 0 \quad \textit{and} \quad r \geqslant -1 \tag{4.118}$$

$$\begin{aligned} \theta(t) &= \int_0^t h(x) = \int_0^t p(1+cx)^{-r-1}\, dx \\ &= -\frac{p}{rc}\, [(1+ct)^{-r}]_0^t \\ &= -\frac{p}{rc}\, [(1+ct)^{-r} - 1] \end{aligned}$$

$$\theta(t) = \frac{p}{rc}\, [1 - (1+ct)^{-r}] \tag{4.119}$$

$$\Rightarrow \qquad \theta(t - t\, s) = \frac{p}{rc}\, [1 - (1 + ct - cts)^{-r}]$$

$$\textit{and} \qquad \theta(0) = 0 \tag{4.120}$$

For an infinitely mixed Poisson distribution

$$G(s,t) = 1 - \frac{\theta(t - ts)}{\theta(t)}$$

$$= 1 - \frac{[1-(1+ct-cts)^{-r}}{1-(1+ct)^{-r}}$$
$$= \frac{(1+ct-cts)^{-r}-(1+ct)^{-r}}{1-(1+ct)^{-r}}$$
$$= \frac{(1+ct)^{-r}\ (1-\frac{ct}{1+ct}\ s)^{-r}-(1+ct)^{-r}}{1-(1+ct)^{-r}}$$
$$= \frac{(1-\frac{ct}{1+ct}\ s)^{-r}-1}{(1+ct)^{r}-1} \tag{4.121}$$

which is the pgf of ETNB distribution.

$$G(s,t) = \frac{\sum_{x=0}^{\infty}\binom{-r}{x}\ (-\frac{ct}{1+ct}\ s)^{x}-1]}{(1+ct)^{r}-1}$$
$$= \frac{\sum_{x=1}^{\infty}\binom{-r}{x}\ (-\frac{ct}{1+ct}\ s)^{x}-1]}{(1+ct)^{r}-1}$$
$$g(x) = \frac{\binom{r+x-1}{x}\ (\frac{ct}{1+ct})^{x}}{[(1+ct)^{r}-1]} \qquad x = 1,2,3\ldots$$

which is the pmf of a zero-truncated negative binomial distribution or extended truncated negative binomial distribution with parameters $\frac{\mathbf{ct}}{\mathbf{1+ct}}$ **and r**.

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{\binom{r+x-1}{x}\ (\frac{ct}{1+ct})^{x}}{\binom{r+x-2}{x-1}\ (\frac{ct}{1+ct})^{x-1}} = \frac{ct}{1+ct}\ \frac{r+x-1}{x} \qquad for \quad x = 2,3,\ldots$$
$$= \frac{ct}{1+ct}\ [1+\frac{r-1}{x}] \equiv a + \frac{b}{x} \tag{4.122}$$

and this is Panjer's model

## 4.8 Concluding remarks

The hazard function of an exponential mixture characterizes an infinitely divisible mixed Poisson distribution which is also a compound Poisson distribution.

Hofmann hazard function is a good illustration of the theory. For further research other classes of hazard functions should be considered, in particular those based on frailty models.
Hazard functions expressed in terms of the modified Bessel function of the third kind and those expressed in terms of confluent hyper-geometric functions would be of interest since we have not explored the link between these hazard functions and mixed Poisson distribution in this study.

# Chapter 5

# SUMS OF HAZARD FUNCTIONS OF EXPONENTIAL MIXTURES AND CONVOLUTIONS OF POISSON MIXTURES

## 5.1 Introduction

In this chapter, it has been shown that a sum of hazard functions of exponential mixtures characterizes a convolution of infinitely divisible mixed Poisson distributions which is also a convolution of compound Poisson distributions.

For each sum of two special Parameterizations of Hofmann hazard function, the following have been obtained:

- the probability generating function (pgf) of the convolution of the mixed Poisson distributions.
- the pgf of the independent and identically distributed (iid) random variables for the convolution of the compound Poisson distributions.
- the recursive form of the convolution of the compound Poisson distribution.

It has been determined that Panjer's recursive model does not hold for all Parameterizations.

Pairs of Hofmann hazard functions have been considered to identify the convolutions and generally the chapter has been organised as follows: Section 5.2 briefly discusses the relationship between a hazard function of an exponential mixture and the corresponding infinitely divisible mixed Poisson distribution. Section 5.3 proves that a sum of two hazard functions of exponential mixtures gives rise to a convolution of two mixed Poisson distributions and a convolution of two corresponding compound Poisson distributions. Section 5.4 is an illustration of the results obtained using sums of various Parameterizations of Hofmann hazard function. Concluding remarks are given in section 5.5.

## 5.2 A Single Hazard Function of an Exponential Mixture

A mixed Poisson distribution can be expressed in terms of a Laplace transform as

$$p_n(t) = (-1)^n \frac{t^n}{n!} L_{\Lambda}^{(n)}(t) \qquad n = 0, 1, 2, \ldots \tag{5.1}$$

where $L_{\Lambda}(t)$ is Laplace transform of the mixing distribution

and,

$$L_{\Lambda}^{(n)}(t) = \frac{d^n}{d\,t^n}\,L_{\Lambda}(t) \tag{5.2}$$

When $\mathbf{n = 0}$, we have

$$\begin{aligned} p_0(t) &= L_{\Lambda}(t) \\ &= e^{In\, L_{\Lambda}(t)} \\ &= e^{-In\, \frac{1}{L_{\Lambda}(t)}} \\ &= e^{-\theta(t)} \end{aligned} \tag{5.3}$$

where,

$$\begin{aligned} \theta(t) &= In\frac{1}{L_{\Lambda}(t)} \\ \therefore \qquad \theta'(t) &= -\frac{L'_{\Lambda}(t)}{L_{\Lambda}(t)} \\ &= h(t) \end{aligned}$$

which is a hazard function of the exponential mixture.

Since $\mathbf{h(t) = \theta'(t)}$ is completely monotone and $\boldsymbol{\theta(0) = 0}$, then $\mathbf{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

Hence the mixed Poisson distribution $\mathbf{p_n(t)}$ is also infinitely divisible (Feller, Chapter XIII, Vol. 2, 1971).Furthermore, an infinitely divisible mixed Poisson distribution is a compound Poisson distribution (Feller, Chapter XII, Vol. I, 1968; Ospina and Gerbes, 1987) whose pgf is given by

$$H(s,t) = e^{-\theta(t)(1-G(s,t))}$$

where $\mathbf{G(s,t)}$ is the pgf of the iid random variables.

Since the pgf of the mixed Poisson distribution is

$$H(s,t) = e^{-\theta(t-ts)}$$

by equating the two formulae for pgf, $\mathbf{H(s,t)}$, we get

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)}$$

Therefore the probability mass functions (pmfs) of the iid random variables are

$$g_x(t) = \frac{1}{x!}\frac{d^x}{ds^x}\,G(s,t)|_{s=0} \tag{5.4}$$

$$= (-1)^{x-1} \frac{t^x}{x!} \frac{\theta^x(t)}{\theta(t)}, \qquad x = 1, 2, 3 \ldots \tag{5.5}$$

and,

$$g_0(t) = 0$$

Let $x = i + 1$, which implies that $x - 1 = i$, and hence

$$g_{i+1}(t) = (-1)^i \frac{t^{i+1}}{(i+1)!} \frac{\theta^{i+1}(t)}{\theta(t)}, \qquad i = 0, 1, 2, 3 \ldots$$

The recursive form for the compound Poisson distribution is

$$n\, p_n(t) = \theta(t) \sum_{x=0}^{n} x\, g_x(t) p_{n-x}(t) \qquad n = 1, 2, 3 \ldots$$

or

$$\begin{aligned}(n+1)p_{n+1}(t) &= \theta(t) \sum_{i=0}^{n} (i+1) g_{i+1}(t) p_{n-i}(t) \\ &= \sum_{i=0}^{n} (-1)^i \frac{t^{i+1}}{i!} \theta^{i+1}(t) p_{n-i}(t) \qquad n = 0, 1, 2, 3 \ldots\end{aligned}$$

Using the recursive relation, $p_n(t)$ can be obtained iteratively.

## 5.3 A sum of two hazard functions of exponential mixtures

### 5.3.1 Derivations of key results for convolutions

Let,

$$h_1(t) = \theta_1^{'}(t) \qquad \text{and} \qquad h_2(t) = \theta_2^{'}(t) \tag{5.6}$$

be two hazard functions of exponential mixtures.

Further, let

$$\theta(t) = \theta_1(t) + \theta_2(t) \tag{5.7}$$

Therefore, the pgf of the mixed Poisson distribution is

$$\begin{aligned}H(s,t) &= e^{-\theta(t-ts)} \\ &= e^{-\{\theta_1(t-ts)+\theta_2(t-ts)\}} \\ &= e^{-\theta_1(t-ts)} e^{-\theta_2(t-ts)}\end{aligned} \tag{5.8}$$

which is a product of two pgfs of mixed Poisson distributions.

Hence a sum of two hazard functions of exponential mixtures gives rise to a convolution of two random variables from mixed Poisson distributions.

Since infinitely divisible mixed Poisson distributions are also compound Poisson distributions, then the pgf can be expressed as

$$\begin{aligned} H(s,t) &= e^{-\{\theta_1(t)+\theta_2(t)\}\{1-G(s,t)\}} \\ &= e^{-\theta_1(t)\{1-G(s,t)\}} e^{-\theta_2(t)\{1-G(s,t)\}} \end{aligned} \tag{5.9}$$

implying a convolution of two compound Poisson random variables.

Equating the two formulae for $H(s,t)$, we get the pgf of the iid random variables for the convolution of the compound Poisson random variables

$$G(s,t) = 1 - \frac{\theta_1(t-ts)+\theta_2(t-ts)}{\theta_1(t)+\theta_2(t)} \tag{5.10}$$

with corresponding pmf of the iid random variables being

$$g_0(t) = 0 \tag{5.11}$$

and

$$g_x(t) = (-1)^{x-1} \frac{t^x}{x!} \frac{\theta_1^x(t)+\theta_2^x(t)}{\theta(t)}, \qquad x = 1,2,3\ldots \tag{5.12}$$

or

$$g_{i+1}(t) = (-1)^i \frac{t^{i+1}}{(i+1)!} \frac{\theta_1^{i+1}(t)+\theta_2^{i+1}(t)}{\theta(t)}, \qquad i = 0,1,2,\ldots \tag{5.13}$$

The recursive form for the compound Poisson distribution is either given by

$$n\, p_n(t) = \sum_{x=1}^{n} (-1)^{x-1} \frac{t^x}{(x-1)!} \left(\theta_1^x(t)+\theta_2^x(t)\right) p_{n-x}(t), \qquad n = 1,2,3,\ldots \tag{5.14a}$$

or

$$(n+1)p_{n+1}(t) = \sum_{i=0}^{n} (-1)^i \frac{t^{i+1}}{i!} \left(\theta_1^{i+1}(t)+\theta_2^{i+1}(t)\right) p_{n-i}(t), \qquad n = 0,1,2,3\ldots \tag{5.14b}$$

(5.14c)

Using this recursive relation, $p_n(t)$ can be obtained iteratively.

### 5.3.2 A Special Parameterization

When the first hazard function is a constant, we have:

$$h_1(t) = \theta_1'(t) = \delta$$

and hence

$$\theta_1(t) = \delta t \tag{5.15}$$

Therefore,

$$\theta_1'(t) + \theta_2'(t) = \delta + \theta_2'(t) \tag{5.16a}$$
$$\theta_1(t) + \theta_2(t) = \delta t + \theta_2(t) \tag{5.16b}$$

and,

$$H(s,t) = e^{-\delta t(1-s)} e^{-\theta_2(t-ts)} \tag{5.17}$$

which is a product of the pgf of a Poisson distribution with parameter $\delta t$ and a pgf of a mixed Poisson distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distributions

$$\begin{aligned} G(s,t) &= 1 - \frac{\theta_1(t-ts) + \theta_2(t-ts)}{\theta_1(t) + \theta_2(t)} \\ &= 1 - \frac{\delta * (t-ts) + \theta_2(t-ts)}{\delta t + \theta_2(t)} \end{aligned}$$

By differentiating $G(s,t)$

$$\begin{aligned} \frac{\partial\, G(s,t)}{\partial s} &= -\frac{-\delta t - t\theta_2'(t-ts)}{\delta t + \theta_2(t)} \\ &= \frac{\delta t + t\theta_2'(t-ts)}{\delta t + \theta_2(t)} \\ \frac{\partial^2\, G(s,t)}{\partial s^2} &= \frac{-t^2\theta_2''(t-ts)}{\delta t + \theta_2(t)} \end{aligned}$$

we obtain

$$G^x(s,t) = \frac{(-1)^{x-1} t^x \theta_2^x(t-ts)}{\delta t + \theta_2(t)} \qquad x = 2,3,\ldots \tag{5.18}$$

and the pmfs of the iid random variables

$$\therefore \quad g_0(t) = G(0,t) = 0 \tag{5.19a}$$

$$g_1(t) = \frac{\partial\, G(s,t)}{\partial s}\Big|_{s=0} = \frac{\delta t + t\theta_2'(t)}{\delta t + \theta_2(t)} \tag{5.19b}$$

$$g_x(t) = (-1)^{x-1}\frac{t^x}{x!}\frac{\partial^x\, G(s,t)}{\partial s^x}\Big|_{s=0} \tag{5.19c}$$

$$= (-1)^{x-1}\frac{t^x}{x!}\frac{\theta_2^x(t)}{\delta t + \theta_2(t)} \qquad x = 2,3,\ldots \tag{5.19d}$$

The recursive form of the compound Poisson distribution is

$$\begin{aligned} n\, p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\;\; p_{n-x}(t) \\ &= \theta(t) g_1(t) p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x\, g_x(t)\;\; p_{n-x}(t) \\ &= \left(\delta t + t\theta_2'(t)\right) p_{n-1}(t) + \sum_{x=2}^{n} x\, \frac{(-1)^{x-1}\, t^x\, \theta_2^x(t)}{x!}\, p_{n-x}(t) \\ &= \left(\delta t + t\theta_2'(t)\right) p_{n-1}(t) + \sum_{x=2}^{n} \frac{(-1)^{x-1}\, t^x\, \theta_2^x(t)}{(x-1)!}\, p_{n-x}(t) \end{aligned} \tag{5.20}$$

as given by Walhin and Paris (2002)

Replacing $n$ by $n+1$ in (5.20), we have

$$(n+1)\, p_{n+1}(t) = \left(\delta t + t\theta_2'(t)\right) p_n(t) + \sum_{x=2}^{n+1} \frac{(-1)^{x-1}\, t^x\, \theta_2^x(t)}{(x-1)!}\, p_{n-(x-1)}(t) \tag{5.21}$$

Let $x = i+1$, which implies that $x-1 = i$, and therefore

$$(n+1)\, p_{n+1}(t) = \left(\delta t + t\theta_2'(t)\right) p_n(t) + \sum_{i=1}^{n} \frac{(-1)^{i}\, t^{i+1}\, \theta_2^{i+1}(t)}{i!}\, p_{n-i}(t) \qquad n = 0,1,2\ldots \tag{5.22}$$

## 5.4 Sums of Hofmann hazard functions

Walhin and Paris (1999) defined Hofmann distribution as:

$$p_0(t) = e^{-\theta(t)}$$

and,

$$p_n(t) = (-1)^n \frac{t^n}{n!} p_0^n(t) \qquad n = 1, 2, 3, \ldots$$

where,

$$\theta'(t) = \frac{p}{(1+ct)^a} \qquad p > 0, \quad c > 0, \quad a \geq 0$$

and,

$$\theta(0) = 0$$

Wakoli and Ottieno (2015) determined that $\theta'(t)$ is in fact a hazard function of an exponential mixture and referred to it as Hofmann hazard function. Let the sum of two hazard functions of exponential mixture be in the form of Hofmann hazard functions; i.e,

$$h(t) = \frac{p_1}{(1+c_1t)^{a_1}} + \frac{p_2}{(1+c_2t)^{a_2}} \tag{5.23}$$

We wish to obtain the following:

- the pgf of mixed Poisson distribution.
- the pgf of the iid random variables.
- the recursive form of the compound Poisson distribution for Parameterizations of $a_i$, where $i = 1, 2$

We also wish to find out whether Panjer's recursive model still holds for all Parameterizations.

### 5.4.1 When the first hazard function is a constant

**In the equation (5.23) $a_1 = 0$ and $a_2 = \frac{1}{2}$**

$$h(t) = \theta'(t) = p_1 + \frac{p_2}{(1+c_2t)^{\frac{1}{2}}} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0 \tag{5.24}$$

where the second hazard function is that of an exponential-inverse Gaussian distribution.

Therefore,

(5.25)

$$\theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{2p_2}{c_2}\left((1+c_2 t)^{\frac{1}{2}} - 1\right) \tag{5.26a}$$

$$\theta_1(t-ts) = p_1 * (t-ts) \quad ; \quad \theta_2(t-ts) = \frac{2p_2}{c_2}\left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right) \tag{5.26b}$$

The pgf of the convolution is

$$\begin{aligned} H(s,t) &= e^{-\theta_1(t-ts)} e^{-\theta_2(t-ts)} \\ &= e^{-p_1 t(1-s)}\, e^{\frac{2p_2}{c_2}\left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right)} \end{aligned} \tag{5.27}$$

The the sum of hazard functions of exponential distribution and that of the exponential-inverse Gaussian distribution, therefore, gives rise to the convolution of the Poisson distribution and the Poisson-inverse Gaussian (Sichel) distribution. Using (5.10) the pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$\begin{aligned} G(s,t) &= 1 - \frac{1}{\theta(t)}\,\{p_1 t - p_1 ts + \frac{2p_2}{c_2}\,\{(1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\} \\ G'(s,t) &= \frac{1}{\theta(t)}\left\{p_1 t + \frac{p_2}{c_2}\,(c_2 t)^1 (1+c_2 t - c_2 ts)^{-\frac{1}{2}}\right\} \\ G''(s,t) &= \frac{1}{\theta(t)}\,(\frac{1}{2})\,\frac{p_2}{c_2}\,(c_2 t)^2 (1+c_2 t - c_2 ts)^{-\frac{3}{2}} \\ G'''(s,t) &= \frac{1}{\theta(t)}\,(\frac{1}{2})(\frac{3}{2})\,\frac{p_2}{c_2}\,(c_2 t)^3 (1+c_2 t - c_2 ts)^{-\frac{5}{2}} \\ G^{iv}(s,t) &= \frac{1}{\theta(t)}\,(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})\,\frac{p_2}{c_2}\,(c_2 t)^4 (1+c_2 t - c_2 ts)^{-\frac{7}{2}} \\ G^{v}(s,t) &= \frac{1}{\theta(t)}\,\frac{2.3-1}{2}\,\frac{2.2-1}{2}\,\frac{2.1-1}{2}\,\frac{p_2}{c_2}\,(c_2 t)^4 (1+c_2 t - c_2 ts)^{-\frac{(2.4-1)}{2}} \\ G^{x}(s,t) &= \frac{1}{\theta(t)}\left(\frac{2(x-1)-1}{2}\right)\left(\frac{2(x-2)-1}{2}\right)\cdots\left(\frac{2.2-1}{2}\right)\left(\frac{2.1-1}{2}\right) \\ &\quad \frac{p_2}{c_2}\,(c_2 t)^x \{(1+c_2 t - c_2 ts)^{-\frac{(2x-1)}{2}} \\ &= \frac{1}{\theta(t)}\left(x-1-\frac{1}{2}\right)\left(x-2-\frac{1}{2}\right)\cdots\left(2-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) \\ &\quad \frac{p_2}{c_2}\,(c_2 t)^x \{(1+c_2 t - c_2 ts)^{-x+\frac{1}{2}} \\ &= \frac{1}{\theta(t)}\left(-\frac{1}{2}+x-1\right)\left(-\frac{1}{2}+x-2\right)\cdots\left(-\frac{1}{2}+x-x+2\right)\left(-\frac{1}{2}+x-x+1\right) \\ &\quad \frac{p_2}{c_2}\,(c_2 t)^x \{(1+c_2 t - c_2 ts)^{-x+\frac{1}{2}} \end{aligned}$$

$$= \frac{(x-1)!}{\theta(t)} \binom{-\frac{1}{2}+x-1}{x-1} \frac{p_2}{c_2} (c_2 t)^x \{(1+c_2 t - c_2 t s)^{\frac{1}{2}-x}$$
$$= \frac{(x-1)!}{\theta(t)} p_2 t \binom{\frac{1}{2}+x-1-1}{x-1} (\frac{c_2 t}{(1+c_2 t - c_2 t s})^{x-1} (\frac{1}{(1+c_2 t - c_2 t s})^{\frac{1}{2}}$$
$$for \;\; x = 2, 3, \ldots$$

Therefore the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.28a}$$
$$g_1(t) = \frac{1}{\theta(t)} \left\{ p_1 t + \frac{p_2 t}{(1+c_2 t)^{\frac{1}{2}}} \right\} \tag{5.28b}$$
$$g_x(t) = \frac{1}{\theta(t)} \frac{1}{x} p_2 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1+c_2 t})^{x-1} (1+c_2 t)^{-\frac{1}{2}} \quad for \;\; x = 2, 3, \ldots \tag{5.28c}$$

where,

$$\theta(t) = \theta_1(t) + \theta_2(t)$$
$$= p_1 t + \frac{2 p_2}{c_2} \left( (1+c_2 t)^{\frac{1}{2}} - 1 \right) \tag{5.29}$$

And so,

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{\frac{1}{x} \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1+c_2 t})^{x-1}}{\frac{1}{(x-1)} \binom{-\frac{1}{2}}{x-2} (-\frac{c_2 t}{(1+c_2 t})^{x-2}}$$
$$= \frac{x-1}{x} \frac{\binom{-\frac{1}{2}}{x-1}}{\binom{-\frac{1}{2}}{x-2}} (\frac{-c_2 t}{(1+c_2 t})$$
$$= (\frac{x-1}{x}) (\frac{\frac{3}{2}-x}{x-1}) (\frac{-c_2 t}{1+c_2 t})$$
$$= \frac{c_2 t}{(1+c_2 t)} - \frac{3 \, c_2 t}{2 \, (1+c_2 t)} \frac{1}{x} \tag{5.30}$$

which is in Panjer's recursive form with

$$a = \frac{c_2 t}{(1+c_2 t)} \qquad \text{and} \qquad b = -\frac{3}{2} \frac{c_2 t}{(1+c_2 t)}$$

The recursive form for the convolution of compound Poisson distributions is

$$
\begin{aligned}
n\, p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\ \ p_{n-x}(t) \\
&= \theta(t)\, g_1(t)\, p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x\, g_x(t)\ \ p_{n-x}(t) \\
&= \left( p_1 t + \frac{p_2 t}{(1+c_2 t)^{\frac{1}{2}}} \right)\ p_{n-1}(t) + \\
&\quad (1+c_2 t)^{-\frac{1}{2}}\, p_2 t \sum_{x=2}^{n} \binom{-\frac{1}{2}}{x-1} \left(-\frac{c_2 t}{(1+c_2 t}\right)^{x-1}\ \ p_{n-x}(t) \qquad n = 1, 2, \ldots
\end{aligned}
\tag{5.31}
$$

**In the equation (5.23) $a_1 = 0$ and $a_2 = 1$**

$$
h(t) = p_1 + \frac{p_2}{(1+c_2 t)} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0 \tag{5.32}
$$

where the second hazard function is that of Pareto.

This sum of hazard functions can be obtained by considering an exponential mixture whose mixing distribution is the shifted-gamma distribution. The mixture is constructed below:

The pdf of the shifted-gamma distribution is

$$
g(\lambda) = \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1} \qquad \lambda > \mu, \quad \alpha > 0, \quad \beta > 0 \tag{5.33}
$$

The survival function of the exponential mixture is

$$
\begin{aligned}
S(t) &= \int_{\lambda=0}^{\infty} S(t|\lambda)\, g(\lambda)\, d\lambda \\
&= \int_{\lambda=\mu}^{\infty} e^{-\lambda\, t}\, \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1}\, d\lambda
\end{aligned}
$$

Let $\quad y = \lambda - \mu \quad$ hence $\quad \lambda = y + \mu \quad$ and $\quad d\lambda = dy$

$$
\begin{aligned}
\therefore \qquad S(t) &= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_{y=0}^{\infty} e^{-(y+\mu)\, t}\, e^{-\beta\, y}\, y^{\alpha-1}\, dy \\
&= \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\mu\, t} \int_{y=0}^{\infty} e^{-y\, (\beta+t)}\, y^{\alpha-1}\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\mu\, t}\, \frac{\Gamma\alpha}{(\beta+t)^{\alpha}} \\
&= \beta^{\alpha}\, e^{-\mu\, t}\, (\beta+t)^{-\alpha} \qquad (5.34)\\
-S'(t) &= \beta^{\alpha}\,(\mu)\, e^{-\mu\, t}\, (\beta+t)^{-\alpha} + \beta^{\alpha}\, e^{-\mu\, t}\,(\alpha)\,(\beta+t)^{-\alpha-1} \qquad (5.35)\\
h(t) &= \frac{-S'(t)}{S(t)} \\
&= \frac{\beta^{\alpha}\,(\mu)\, e^{-\mu\, t}\, (\beta+t)^{-\alpha} + \beta^{\alpha}\, e^{-\mu\, t}\,(\alpha)\,(\beta+t)^{-\alpha-1}}{\beta^{\alpha}\, e^{-\mu\, t}\,(\beta+t)^{-\alpha}} \\
&= \mu + \alpha\,(\beta+t)^{-1} \\
&= \mu + \frac{\alpha}{\beta+t} \\
&= \mu + \frac{\frac{\alpha}{\beta}}{1+\frac{1}{\beta}\, t} \qquad (5.36)
\end{aligned}
$$

which is the hazard function of the exponential-shifted gamma distribution.

The hazard function of the exponential-shifted gamma distribution is therefore the sum of Hofmann hazard functions given by

$$
h(t) = p_1 + \frac{p_2}{1+c_2 t} \qquad (5.37)
$$

where $p_1 = \mu, \quad p_2 = \frac{\alpha}{\beta} \quad and \quad c_2 = \frac{1}{\beta}$

Therefore,

$$
\theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{p_2}{c_2}\, In(1+c_2 t)
$$

$$
\theta_1(t-ts) = p_1 * (t-ts) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\, In(1+c_2 t - c_2 ts)
$$

The pgf of the convolution is

$$
\begin{aligned}
H(s,t) &= e^{p_1 t(s-1)}\, e^{\frac{p_2}{c_2}\, In\left(\frac{1}{1+c_2 t - c_2 ts}\right)} \\
&= e^{p_1 t(s-1)}\, \left(\frac{1}{1+c_2 t - c_2 ts}\right)^{\frac{p_2}{c_2}} \\
&= e^{p_1 t(s-1)}\, \left(\frac{\frac{1}{1+c_2 t}}{1-\frac{ct}{1+c_2 t} s}\right)^{\frac{p_2}{c_2}} \\
&= e^{p_1 t(s-1)}\, \left(\frac{\frac{1}{1+c_2 t}}{1-\frac{ct}{1+c_2 t} s}\right)^{\frac{p_2}{c_2}} \qquad (5.38)
\end{aligned}
$$

Therefore the sum of hazard functions of exponential distribution and that of the exponential-gamma (Pareto)distribution gives rise to the convolution of the Poisson distribution and the Poisson-gamma (negative binomial) distribution.

By differentiating the pgf of the iid random variables of the convolution of the compound Poisson distribution:

$$
\begin{aligned}
G(s,t) &= 1-\frac{1}{\theta(t)}\ \{p_1 t - p_1 ts + \frac{p_2}{c_2}\ In(1+c_2 t - c_2 ts)\} \\
G^{'}(s,t) &= \frac{\partial\, G(s,t)}{\partial s} = -\frac{1}{\theta(t)}\ \{-p_1 t + \frac{p_2}{c_2}\ (-c_2 t)(1+c_2 t - c_2 ts)^{-1}\} \\
G^{''}(s,t) &= -\frac{1}{\theta(t)}\ \{(-1)\frac{p_2}{c_2}\ (-c_2 t)^2 (1+c_2 t - c_2 ts)^{-2})\} \\
G^{'''}(s,t) &= -\frac{1}{\theta(t)}\ \{(-1)(-2)\frac{p_2}{c_2}\ (-c_2 t)^3 (1+c_2 t - c_2 ts)^{-3})\} \\
G^{x}(s,t) &= \frac{1}{\theta(t)}\ (x-1)!\ \frac{p_2}{c_2}\ (\frac{c_2 t}{1+c_2 t - c_2 ts})^x
\end{aligned} \tag{5.39}
$$

we obtain the pmfs of the iid random variables

$$
\begin{aligned}
g_0(t) &= G(0,t) = 0 \\
g_1(t) &= G'(0,t) \\
&= \frac{1}{\theta(t)}\ \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)
\end{aligned} \tag{5.40}
$$

$$
\begin{aligned}
g_x(t) &= \frac{1}{x!}\ G^{(x)}(s,t)|_{s=0} \\
&= \frac{1}{x!}\ (x-1)!\frac{p_2 t}{\theta(t)}\ (1+c_2 t)^{-x}\ (c_2 t)^{x-1} \\
&= \frac{1}{x}\ \frac{1}{\theta(t)}\ \frac{p_2}{c_2}\ \left(\frac{c_2 t}{1+c_2 t}\right)^x \\
&= \frac{1}{x}\ \left(\frac{1}{p_1 t + \frac{p_2}{c_2}\ In(1+c_2 t)}\right)\ \frac{p_2}{c_2}\ \left(\frac{c_2 t}{1+c_2 t}\right)^x \qquad x = 2,3,\ldots
\end{aligned} \tag{5.41}
$$

Therefore,

$$
\begin{aligned}
\frac{g_x(t)}{g_{x-1}(t)} &= \frac{x-1}{x}\ \frac{c_2 t}{1+c_2 t} \\
&= \frac{ct}{1+c_2 t} + \frac{-\frac{c_2 t}{1+c_2 t}}{x} \qquad x = 2,3,\ldots
\end{aligned} \tag{5.42}
$$

which is Panjer's recursive model with

$$
a = \frac{c_2 t}{1+c_2 t} \qquad and \qquad b = -\frac{c_2 t}{1+c_2 t}
$$

**Remark 5.1.** *This Panjer's model is the same as that of a logarithmic series distribution with parameter* $\frac{c_2 t}{1+c_2 t}$

The recursive form for the convolution of compound Poisson distributions is:

$$\begin{aligned}
n\, p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\;\; p_{n-x}(t) \\
&= \theta(t)\, g_1(t)\, p_{n-1}(t) + \;\theta(t) \sum_{x=2}^{n} x\, g_x(t)\, p_{n-x}(t) \\
&= \theta(t)\, \frac{1}{\theta(t)} \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x\, \frac{1}{x}\, \frac{1}{\theta(t)}\, \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^{x} \\
&= \theta(t)\, g_1(t)\, p_{n-1}(t) + \sum_{x=2}^{n} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^{x} p_{n-x}(t) \qquad for \qquad n = 1, 2,
\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}
p_0(t) &= e^{-\theta(t)} \\
&= e^{-p_1 t - \frac{p_2}{c_2}\, In(1+c_2 t)} \\
&= e^{-p_1 t} \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}
\end{aligned} \tag{5.44}$$

Replace $n$ by $n+1$ in (5.43)

$$(n+1) p_{n+1}(t) = \theta(t)\, g_1(t)\, p_n(t) + \sum_{x=2}^{n+1} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^{x} p_{n-(x-1)}(t) \tag{5.45}$$

and let $x = i+1$, so that $x - 1 = i$

$$(n+1) p_{n+1} = \theta(t)\, g_1(t)\, p_n(t) + \sum_{i=1}^{n} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^{i+1} p_{n-i}(t) \tag{5.46}$$

**To obtain $p_n(t)$ explicitly, we use**

**Method 1: The iteration technique**

(5.47)

For $\quad n = 1$

$$p_1(t) = \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) p_0(t)$$

$$= \left(p_1 t + \frac{p_2}{c_2}\, \frac{c_2 t}{1+c_2 t}\right)\, p_0(t)$$

$$= (p_1 t\, p_0(t)) + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)\, p_0(t) \qquad (5.48)$$

Substituting 5.44 into 5.48, we obtain

$$p_1(t) = e^{-p_1 t}\, \frac{(p_1 t)^1}{1!}\, \left(\frac{c_2 t}{1+c_2 t}\right)^0 \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + e^{-p_1 t}\, \frac{p_1 t)^0}{0!}\, \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right) \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{1} \frac{e^{-p_1 t}\, (p_1 t)^{1-k}}{(1-k)!}\, \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \qquad (5.49)$$

For $\quad n = 2$

$$2\, p_2(t) = \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)\, p_1(t) + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2\, p_0(t)$$

and,

$$p_2(t) = \left\{\left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}\, \frac{p_0(t)}{2}$$

$$= \left\{(p_1 t)^2 + 2\, p_1 t\, \frac{p_2 t}{1+c_2 t} + \left(\frac{p_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}\, \frac{p_0(t)}{2}$$

$$= \left\{(p_1 t)^2 + 2\, p_1 t\, \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right) + \left(\frac{p_2}{c_2}\right)^2 \left(\frac{c_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}\, \frac{p_0(t)}{2}$$

$$= \frac{(p_1 t)^2}{2}\, p_0(t) + p_1 t\, \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)\, p_0(t) + \left(\frac{p_2}{c_2}+1\right)\, \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2\, \frac{p_0(t)}{2}$$

$$p_2(t) = e^{-p_1 t}\, \frac{(p_1 t)^2}{2}\, \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + e^{-p_1 t}\, p_1 t\, \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right) \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} +$$

$$\frac{1}{2} e^{-p_1 t}\, \left(\frac{p_2}{c_2}+1\right)\, \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2 \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{2} \frac{e^{-p_1 t}\, (p_1 t)^{2-k}}{(2-k)!}\, \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \qquad (5.50)$$

For $\quad n = 3$

$$3p_3(t) = \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)\, p_2(t) + \frac{p_2}{c_2}\, \sum_{x=2}^{3} \left(\frac{c_2 t}{1+c_2 t}\right)^x\, p_{3-x}(t)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)\, p_2(t) + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^2\, p_1(t) + \frac{p_2}{c_2}\, \left(\frac{c_2 t}{1+c_2 t}\right)^3\, p_0(t)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) \left\{\left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\} \frac{p_0(t)}{2} +$$

$$\frac{p_2}{c_2} (\left(\frac{c_2 t}{1+c_2 t}\right)^2 \left\{p_1(t) + \left(\frac{p_2 t}{1+c_2 t}\right)\right\} p_0(t)) + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3 p_0(t)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^3 \frac{p_0(t)}{2} + \frac{p_2}{c_2} \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^2 \frac{p_0(t)}{2} +$$

$$\frac{p_2}{c_2} (\left(\frac{c_2 t}{1+c_2 t}\right)^2 \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) p_0(t) + \left(\frac{c_2 t}{1+c_2 t}\right)^3 p_0(t))$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^3 \frac{p_0(t)}{2} + \frac{3}{2} \frac{p_2}{c_2} \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^2 p_0(t) + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c}\right.$$

$$= (p_1 t)^3 \frac{p_0(t)}{2} + 3\,(p_1 t)^2 \left(\frac{p_2 t}{1+c_2 t}\right) \frac{p_0(t)}{2} + p_1 t \left\{3 \left(\frac{p_2 t}{1+c_2 t}\right)^2 + 3\frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)\right.$$

$$\left\{\left(\frac{p_2 t}{1+c_2 t}\right)^3 + 3\left(\frac{p_2 t}{1+c_2 t}\right) \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2 + 2\frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3\right\} \frac{p_0(t)}{2}$$

$$= (p_1 t)^3 \frac{p_0(t)}{2} + 3\,(p_1 t)^2 \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right) \frac{p_0(t)}{2} + 3\,p_1 t \left\{\left(\frac{p_2}{c_2}\right)^2 \left(\frac{c_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2} \left(\right.\right.$$

$$\left\{\left(\frac{p_2}{c_2}\right)^3 \left(\frac{c_2 t}{1+c_2 t}\right)^3 + 3\left(\frac{p_2}{c_2}\right)^2 \left(\frac{c_2 t}{1+c_2 t}\right)^3 + 2\frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3\right\} \frac{p_0(t)}{2}$$

$$= (p_1 t)^3 \frac{p_0(t)}{2} + 3\,(p_1 t)^2 \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right) \frac{p_0(t)}{2} + 3\,(p_1 t) \left(\frac{p_2}{c_2} + 1\right) \left(\frac{p_2}{c_2}\right) \left(\frac{c_2 t}{1+c_2 t}\right.$$

$$\left\{\left(\frac{p_2}{c_2}\right)^2 + 3\left(\frac{p_2}{c_2}\right) + 2\right\} \left(\frac{p_2}{c_2}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^3 \frac{p_0(t)}{2}$$

$$p_3(t) = e^{-p_1(t)} \frac{(p_1 t)^3}{3!} \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + e^{-p_1(t)} \frac{(p_1 t)^2}{2!} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right) \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} +$$

$$e^{-p_1(t)} \frac{(p_1 t)}{1!} \binom{\frac{p_2}{c_2}+1}{2} \left(\frac{c_2 t}{1+c_2 t}\right)^2 \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{3} \frac{e^{-p_1 t}\,(p_1 t)^{3-k}}{(3-k)!} \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

In general therefore,

$$p_n(t) = \sum_{k=0}^{n} \frac{e^{-p_1 t}\,(p_1 t)^{n-k}}{(n-k)!} \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \qquad n = 1, 2, 3, \ldots \tag{5.51}$$

**Method 2: The pgf technique**

$$H(s,t) = \sum_{n=0}^{\infty} p_n(t)\, s^n$$

$$= e^{-p_1 t(1-s)} \left( \frac{\frac{1}{1+c_2 t}}{1 - \frac{ct}{1+c_2 t} s} \right)^{\frac{p_2}{c_2}}$$

$$= e^{-p_1 t} \left( \frac{1}{1+c_2 t} \right)^{\frac{p_2}{c_2}} e^{p_1 t s} \left( 1 - \frac{ct}{1+c_2 t} s \right)^{-\frac{p_2}{c_2}}$$

But

$$e^{p_1 t s} = 1 + \frac{p_1 t}{1} s + \frac{(p_1 t)^2}{2} s^2 + \frac{(p_1 t)^3}{3} s^3 + \dots$$

$$= \sum_{k=0}^{n} \left\{ \frac{(p_1 t)^{n-k}}{(n-k)!} \right\} s^{n-k}$$

$$\left( 1 - \frac{ct}{1+c_2 t} s \right)^{-\frac{p_2}{c_2}} = 1 + \binom{-\frac{p_2}{c_2}}{1} \frac{-c_2 t}{1+c_2 t} s + \binom{-\frac{p_2}{c_2}}{2} \left( \frac{-c_2 t}{1+c_2 t} \right)^2 s^2 + \dots$$

$$= \sum_{k=0}^{n} \left\{ \binom{-\frac{p_2}{c_2}}{k} \left( \frac{-c_2 t}{1+c_2 t} \right)^k \right\} s^k$$

$$\therefore \quad H(s,t) = \sum_{n=0}^{\infty} \left\{ e^{-p_1 t} \left( \frac{1}{1+c_2 t} \right)^{\frac{p_2}{c_2}} \sum_{k=0}^{n} \frac{(p_1 t)^{n-k}}{(n-k)!} \binom{-\frac{p_2}{c_2}}{k} \left( \frac{-c_2 t}{1+c_2 t} \right)^k \right\} s^n \tag{5.52}$$

And therefore

$$p_n(t) = \sum_{k=0}^{n} e^{-p_1 t} \frac{(p_1 t)^{n-k}}{(n-k)!} \binom{-\frac{p_2}{c_2}}{k} \left( \frac{-c_2 t}{1+c_2 t} \right)^k \left( \frac{1}{1+c_2 t} \right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{n} e^{-p_1 t} \frac{(p_1 t)^{n-k}}{(n-k)!} (-1)^k \binom{-\frac{p_2}{c_2}}{k} \left( \frac{c_2 t}{1+c_2 t} \right)^k \left( \frac{1}{1+c_2 t} \right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{n} e^{-p_1 t} \frac{(p_1 t)^{n-k}}{(n-k)!} \binom{\frac{p_2}{c_2} + k - 1}{k} \left( \frac{c_2 t}{1+c_2 t} \right)^k \left( \frac{1}{1+c_2 t} \right)^{\frac{p_2}{c_2}} \tag{5.53}$$

which is a convolution of a Poisson distribution and negative binomial distribution.

**In the equation (5.23) $a_1 = 0$ and $a_2 = 2$**

$$h(t) = \theta'(t) = p_1 + \frac{p_2}{(1+c_2 t)^2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0$$

where the second hazard function is what we have called Polya-Aeppli hazard function (Wakoli and Ottieno 2015, p. 234)

$$\therefore \quad \theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{p_2}{c_2} \left( 1 - (1 + c_2 t)^{-1} \right)$$

$$\theta_1(t-ts) = p_1 * (t-ts) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\left(1-(1+c_2t-c_2ts)^{-1}\right)$$

The pgf of the convolution is

$$H(s,t) = e^{-p_1t(1-s)}\ e^{\frac{p_2}{c_2}\left(1-(1+c_2t-c_2ts)^{-1}\right)} \tag{5.54}$$

Therefore the sum of hazard functions of the exponential distribution and that of Polya-Aeppli distribution gives rise to the convolution of the Poisson distribution and Polya-Aeppli distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$\begin{aligned}
G(s,t) &= 1-\frac{\theta(t-ts)}{\theta(t)} = 1-\frac{1}{\theta(t)}\ (p_1t-p_1ts+\frac{p_2}{c_2}\left(1-(1+c_2t-c_2ts)^{-1}\right))\\
G'(s,t) &= -\frac{1}{\theta(t)}\ \left(-p_1t-\frac{p_2}{c_2}(-1)\ (-c_2t)^1\ (1+c_2t-c_2ts)^{-2}\right)\\
G''(s,t) &= -\frac{1}{\theta(t)}\ \left(-\frac{p_2}{c_2}(-1)(-2)\ (-c_2t)^2\ (1+c_2t-c_2ts)^{-3}\right)\\
G'''(s,t) &= -\frac{1}{\theta(t)}\ \{-\frac{p_2}{c_2}(-1)(-2)(-3)\ (-c_2t)^3\ (1+c_2t-c_2ts)^{-4}\}\\
\therefore \qquad G^{(x)}(s,t) &= \frac{1}{\theta(t)}\frac{p_2}{c_2}\ x!\ (c_2t)^x\ (1+c_2t-c_2ts)^{-(x+1)}\\
&= \frac{1}{\theta(t)}\frac{p_2}{c_2}\ x!\ (\frac{c_2t}{(1+c_2t-c_2ts)})^x\ \frac{1}{(1+c_2t-c_2ts)}, \qquad x=2,3,\ldots
\end{aligned} \tag{5.55}$$

The pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.56a}$$

$$g_1(t) = \frac{1}{\theta(t)}\ \left(p_1t+\frac{p_2}{c_2}\ \frac{c_2t}{(1+c_2t)^2}\right) \tag{5.56b}$$

$$g_x(t) = \frac{1}{\theta(t)}\ \frac{p_2}{c_2}\ (\frac{c_2t}{(1+c_2t})^x\ \frac{1}{(1+c_2t)} \qquad x=2,3,\ldots \tag{5.56c}$$

And so,

$$\begin{aligned}
\frac{g_x(t)}{g_{x-1}(t)} &= \left(\frac{c_2t}{1+c_2t}\right)^x\ (\frac{(1+c_2t)}{(c_2t)})^{x-1} = \frac{c_2t}{1+c_2t}\\
&= (\frac{c_2t}{1+c_2t}+\frac{0}{x}) \qquad x=2,3,\ldots
\end{aligned} \tag{5.57}$$

which is Panjer's recursive form with

$$a = \frac{c_2 t}{1 + c_2 t} \quad \text{and} \quad b = 0$$

The recursive form for the convolution of compound Poisson distribution is:

$$\begin{aligned}
n p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\, p_{n-x}(t) \\
&= \theta(t)\, g_1(t)\, p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x\, g_x(t)\, p_{n-x}(t) \\
&= \left(p_1 t + \frac{p_2}{c_2} \frac{c_2 t}{(1+c_2 t)^2}\right) p_{n-1}(t) + \sum_{x=2}^{n} x\, \frac{p_2}{c_2} \left(\frac{c_2 t}{(1+c_2 t}\right)^x \frac{1}{(1+c_2 t)}\, p_{n-x}(t) \\
&= \left(p_1 t + \frac{p_2}{c_2} \frac{c_2 t}{(1+c_2 t)^2}\right) p_{n-1}(t) + \\
&\qquad \frac{1}{(1+c_2 t)} \frac{p_2}{c_2} \sum_{x=2}^{n} x \left(\frac{c_2 t}{(1+c_2 t}\right)^x p_{n-x}(t), \qquad n = 1, 2, \ldots
\end{aligned} \tag{5.58}$$

**In the equation (5.23) $a_1 = 0$ and $a_2 \to \infty$**

$$\begin{aligned}
h(t) = \theta'(t) &= p_1 + \lim_{a_2 \to \infty} p_2\, (1 + c_2 t)^{-a_2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0 \\
&= p_1 + \lim_{a_2 \to \infty} p_2 \sum_{k=0}^{\infty} \binom{-a_2}{k} (c_2 t)^k \\
h(t) &= p_1 + p_2\, e^{-bt}
\end{aligned} \tag{5.59}$$

where,

$$b = \lim_{a_2 \to \infty} a_2\, c_2$$

The second hazard function is that of the Gompertz distribution and the sum is known as the Gompertz-Makeham hazard function.

$$\therefore \qquad \theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{p_2}{b} \left(1 - e^{-bt}\right)$$

$$\theta_1(t - ts) = p_1 * (t - ts) \quad ; \quad \theta_2(t - ts) = \frac{p_2}{b} \left(1 - e^{-bt(1-s)}\right)$$

The pgf of the convolution is:

$$H(s,t) = e^{-p_1 t (1-s)}\, e^{-\frac{p_2}{b} \left(1 - e^{-bt(1-s)}\right)} \tag{5.60}$$

Thus the sum of hazard functions of exponential distribution and that of the Gompertz distribution gives rise to a convolution of the Poisson distribution and the Neyman type A (Poisson-Poisson) distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is:

$$\begin{aligned}
G(s,t) &= 1-\frac{\theta(t-ts)}{\theta(t)} = 1-\left\{\frac{p_1 * (t-ts)+\frac{p_2}{b}\left(1-e^{-bt(1-s)}\right)}{\theta(t)}\right\} \\
&= 1-\left\{\frac{p_1 t - p_1 ts + \frac{p_2}{b}\left(1-e^{-bt}\, e^{bts}\right)}{\theta(t)}\right\} \\
&= 1-\left\{\frac{p_1 t - p_1 ts + \frac{p_2}{b}\left(1-e^{-bt}\, e^{bts}\right)}{\theta(t)}\right\} \\
&= 1-\left\{\frac{p_1 t - p_1 ts + \frac{p_2}{b} - \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)}\right\} \\
&= \frac{\theta(t) - p_1 t + p_1 ts - \frac{p_2}{b} + \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)}
\end{aligned}$$

But

$$\theta(t) = p_1 t + \frac{p_2}{b}\left(1-e^{-bt}\right)$$

Therefore,

$$\begin{aligned}
G(s,t) &= \frac{p_1 t + \frac{p_2}{b}\left(1-e^{-bt}\right) - p_1 t + p_1 ts - \frac{p_2}{b} + \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)} \\
&= \frac{-\frac{p_2}{b}\, e^{-bt} + p_1 ts + \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)} \\
&= \frac{p_1 ts + \frac{p_2}{b}\, e^{-bt} \sum_{x=1}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)} \qquad (5.61)
\end{aligned}$$

By differentiating G(s,t)

$$\begin{aligned}
G'(s,t) &= \frac{p_1 t + \frac{p_2}{b}\, e^{-bt} \sum_{x=1}^{\infty} \frac{(bt)^x\, s^{x-1}}{(x-1)!}}{\theta(t)} \\
&= \frac{p_1 t + \frac{p_2}{b}\, e^{-bt}\, (bt) \sum_{x=1}^{\infty} \frac{(bt)^{x-1}\, s^{x-1}}{(x-1)!}}{\theta(t)} \\
G''(s,t) &= \frac{\frac{p_2}{b}\, e^{-bt}\, (bt)^2 \sum_{x=2}^{\infty} \frac{(bt)^{x-2}\, s^{x-2}}{(x-2)!}}{\theta(t)} \\
G^x(s,t) &= \frac{\frac{p_2}{b}\, e^{-bt}\, (bt)^x \sum_{x=2}^{\infty} \frac{(bt)^{x-x}\, s^{x-x}}{(x-x)!}}{\theta(t)}
\end{aligned}$$

$$= \frac{\frac{p_2}{b}\, e^{-bt}\, (bt)^x}{\theta(t)} \tag{5.62}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.63a}$$

$$\begin{aligned} g_1(t) &= \frac{p_1 t + \frac{p_2}{b}\, e^{-bt}\, bt}{\theta(t)} \\ &= \frac{1}{\theta(t)} \left\{ p_1 t + p_2 t\, e^{-bt} \right\} \end{aligned} \tag{5.63b}$$

$$g_x(t) = \frac{1}{\theta(t)}\, \frac{p_2\, e^{-bt}}{b}\, \frac{(bt)^x}{x!} \qquad x = 2, 3, \ldots \tag{5.63c}$$

By power series expansion and using 5.61, the pmfs of the iid random variables are

(5.64)

$$g_0 = 0 \tag{5.65a}$$

the coefficient of $s^0$

$$g_1(t) = \frac{1}{\theta(t)} \left\{ p_1 t + p_2 t\, e^{-bt} \right\} \tag{5.65b}$$

the coefficient of $s^1$

$$g_x(t) = \frac{1}{\theta(t)}\, \frac{p_2\, e^{-bt}}{b}\, \frac{(bt)^x}{x!} \qquad x = 2, 3, \ldots \tag{5.65c}$$

the coefficient of $s^x$

$$\begin{aligned} \therefore \qquad \frac{g_x(t)}{g_{x-1}(t)} &= \frac{1}{x}\, bt \\ &= 0 + \frac{bt}{x} \qquad\qquad x = 1, 2, \ldots \end{aligned} \tag{5.66}$$

which is in Panjer's recursive form, with parameters $\mathbf{0}$ and $bt$

The recursive form for the convolution of compound Poisson distribution is:

$$\begin{aligned}
(n+1)\,p_{n+1}(t) &= \theta(t)\sum_{i=0}^{n}(i+1)\,g_{i+1}(t)\;p_{n-i}(t) \qquad n=0,1,2,\ldots \\
&= \theta(t)g_1(t)\,p_n(t)+\theta(t)\sum_{i=1}^{n}(i+1)\,g_{i+1}(t)\;p_{n-i}(t) \\
&= \theta(t)\left\{\frac{1}{\theta(t)}\left(p_1t+p_2te^{-bt}\right)\right\}p_n(t)+ \\
&\quad \theta(t)\sum_{i=1}^{n}(i+1)\left\{\frac{1}{\theta(t)}\frac{p_2e^{-bt}}{b}\frac{(bt)^{i+1}}{(i+1)!}\right\}p_{n-i}(t) \\
&= \left(p_1t+p_2te^{-bt}\right)p_n(t)+\sum_{i=1}^{n}(i+1)\left\{\frac{p_2e^{-bt}}{b}\frac{(bt)^{i+1}}{(i+1)!}\right\}p_{n-i}(t) \\
&= \left(p_1t+p_2te^{-bt}\right)p_n(t)+\frac{p_2}{b}\,bt\sum_{i=1}^{n}e^{-bt}\left\{\frac{(bt)^i}{i!}\right\}p_{n-i}(t) \\
&= \left(p_1t+p_2te^{-bt}\right)p_n(t)+p_2t\sum_{i=1}^{n}e^{-bt}\frac{(bt)^i}{i!}\,p_{n-i}(t) \qquad n=0,1,2,\ldots
\end{aligned} \tag{5.67}$$

### 5.4.2 When the first hazard function is that of a Pareto Distribution

**In the equation (5.23) $a_1=1$ and $a_2=1$**

$$h(t)=\frac{p_1}{(1+c_1t)}+\frac{p_2}{(1+c_2t)} \qquad p_1>0,\quad p_2>0,\quad c_1>0,\quad c_2>0 \tag{5.68}$$

The second hazard function is also that of a Pareto.

$$\therefore \qquad \theta_1(t)=\frac{p_1}{c_1}\,In(1+c_1t) \quad ; \quad \theta_2(t)=\frac{p_2}{c_2}\,In(1+c_2t) \tag{5.69}$$

$$\theta_1(t-ts)=\frac{p_1}{c_1}\,In(1+c_1t-c_1ts) \quad ; \quad \theta_2(t-ts)=\frac{p_2}{c_2}\,In(1+c_2t-c_2ts)$$

The pgf of the convolution is

$$\begin{aligned}
H(s,t) &= e^{-\frac{p_1}{c_1}In(1+c_1t-c_1ts)}\,e^{-\frac{p_2}{c_2}In(1+c_2t-c_2ts)} \\
&= \left(\frac{1}{1+c_1t-c_1ts}\right)^{\frac{p_1}{c_1}}\left(\frac{1}{1+c_2t-c_2ts}\right)^{\frac{p_2}{c_2}}
\end{aligned} \tag{5.70}$$

Therefore the sum of hazard functions of two Pareto distributions gives rise to the convolution of two negative binomial distributions.

When $\mathbf{c_1 = c_2 = c,}$ then we have a single hazard function

$$h(t) = \frac{p_1 + p_2}{1 + ct} \qquad \text{where} \qquad p_1 + p_2 > 0 \qquad \text{and} \qquad c > 0 \tag{5.71}$$

Its associated Poisson mixture is a negative binomial distribution with parameters $\frac{p_1+p_2}{c}$ *and* $\frac{1}{1+ct}$

.

whose pgf is

$$H(s,t) = \left( \frac{\frac{1}{1+ct}}{1 - \frac{ct}{1+ct}\, s} \right)^{\frac{p_1+p_2}{c}}$$

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{1}{\theta(t)} \left\{ \frac{p_1}{c_1}\, In(1 + c_1 t - c_1 ts) + \frac{p_2}{c_2}\, In(1 + c_2 t - c_2 ts) \right\}$$

and,

$$G^x(s,t) = \frac{1}{\theta(t)} \left\{ (x-1)!\, \frac{p_1}{c_1}\, (\frac{c_1 t}{1 + c_1 t})^x + (x-1)!\, \frac{p_2}{c_2}\, (\frac{c_2 t}{1 + c_2 t})^x \right\} \tag{5.72}$$

refer to (5.5)

(5.73)

The pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.74a}$$

$$g_x(t) = \frac{1}{x}\, \frac{1}{\theta(t)} \left\{ \frac{p_1}{c_1}\, (\frac{c_1 t}{1 + c_1 t})^x + \frac{p_2}{c_2} \left( \frac{c_2 t}{1 + c_2 t} \right)^x \right\} \tag{5.74b}$$

where,

$$\theta(t) = In \left( (1 + c_1 t)^{\frac{p_1}{c_1}}\, (1 + c_2 t)^{\frac{p_2}{c_2}} \right)$$

$$\frac{g_x(t)}{g_{x-1}(t)} = \left\{ \frac{\frac{p_1}{c_1}\, (\frac{c_1}{1+c_1})^x + \frac{p_2}{c_2} \left( \frac{c_2 t}{1+c_2 t} \right)^x}{\frac{p_1}{c_1}\, (\frac{c_1}{1+c_1})^{x-1} + \frac{p_2}{c_2} \left( \frac{c_2 t}{1+c_2 t} \right)^{x-1}} \right\} \frac{x-1}{x} \tag{5.74c}$$

which is not in Panjer's recursive form, since the term in the curled bracket is not a constant.

However, in the Parameterization where $c_1 = c_2 = c$

$$\frac{g_x(t)}{g_{x-1}(t)} = \left\{ \frac{x-1}{x} \; \frac{\frac{p_1+p_2}{c}\;(\frac{ct}{1+ct})^x}{\frac{p_1+p_2}{c}\;(\frac{ct}{1+ct})^{x-1}} \right\}$$
$$= \left\{ \frac{x-1}{x} \; (\frac{ct}{1+ct}) \right\} \tag{5.75}$$

which is Panjer's recursive model with

$$a = \frac{ct}{1+ct} \qquad and \qquad b = -\frac{ct}{1+ct}$$

The recursive form for the convolution of compound Poisson distribution is:

$$\begin{aligned} np_n &= \theta(t) \sum_{x=1}^{n} x \, g_x(t) \, p_{n-x}(t) \\ &= \theta(t) \sum_{x=1}^{n} x \, \frac{1}{x} \, \frac{1}{\theta(t)} \left\{ \frac{p_1}{c_1} \, (\frac{c_1 t}{1+c_1 t})^x + \frac{p_2}{c_2} \left( \frac{c_2 t}{1+c_2 t} \right)^x \right\} p_{n-x}(t) \\ &= \sum_{x=1}^{n} \left\{ \frac{p_1}{c_1} \, (\frac{c_1 t}{1+c_1 t})^x + \frac{p_2}{c_2} \left( \frac{c_2 t}{1+c_2 t} \right)^x \right\} p_{n-x}(t) \\ &= \frac{p_1}{c_1} \sum_{x=1}^{n} (\frac{c_1 t}{1+c_1 t})^x \, p_{n-x}(t) + \frac{p_2}{c_2} \sum_{x=1}^{n} \left( \frac{c_2 t}{1+c_2 t} \right)^x \, p_{n-x}(t) \qquad n = 1, 2, 3, . \end{aligned} \tag{5.76}$$

**In the equation (5.23) $a_1 = 1$ and $a_2 = \frac{1}{2}$**

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1 t)} + \frac{p_2}{(1+c_2 t)^{\frac{1}{2}}} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0 \tag{5.77}$$

where the second hazard function is that of an exponential-inverse Gaussian distribution.

This sum of hazard functions can be obtained by considering reciprocal inverse Gaussian as the a mixing distribution in an exponential mixture as described below:

Let $\mathbf{X} = \frac{1}{\mathbf{\Lambda}}$ where $\boldsymbol{X}$ is a random variable from an inverse-Gaussian distribution. We wish to determine the distribution of $\mathbf{\Lambda}$.

The pdf of $\boldsymbol{\Lambda}$ is

$$\begin{aligned}
g(\lambda) &= f(x)|J| \\
&= f(x)|\frac{dx}{d\lambda}| \\
\therefore \qquad g(\lambda) &= f(x)|-\frac{1}{\lambda^2}|
\end{aligned}$$

The pdf of an inverse Gaussian distribution is

$$f(x) = \left(\frac{\phi}{2\,\pi\,x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi\,(x-\mu)^2}{2\,\mu^2\,x}\right\} \qquad x > 0;,\ \phi > 0,\ -\infty < \mu < \infty \tag{5.78}$$

The pdf of the reciprocal inverse Gaussian distribution is

$$\begin{aligned}
g(\lambda) &= \left(\frac{\phi\lambda^3}{2\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi(\frac{1}{\lambda}-\mu)^2}{2\,\mu^2\,\frac{1}{\lambda}}\right\}\frac{1}{\lambda^2} \\
&= \left(\frac{\phi}{2\pi\lambda}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi\lambda}{2\mu^2}\left(\frac{1}{\lambda^2}-\frac{2\mu}{\lambda}+\mu^2\right)\right\} \\
&= \left(\frac{\phi}{2\pi\lambda}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi}{2\mu^2\lambda}\left(1-2\mu\lambda+(\mu\lambda)^2\right)\right\} \\
&= \left(\frac{\phi}{2\,\pi\,\lambda}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi(1-\mu\,\lambda)^2}{2\,\mu^2\,\lambda}\right\} \qquad \lambda > 0
\end{aligned} \tag{5.79}$$

The survival function of an exponential mixture is the Laplace transform of the mixing distribution.

That is

$$\begin{aligned}
S(t) &= L_{\Lambda}(t) \\
&= \int_0^{\infty} e^{-t\,\lambda}\,g(\lambda)\,d\lambda \\
&= \int_0^{\infty} e^{-t\,\lambda}\,(\frac{\phi}{2\,\pi\,\lambda})^{\frac{1}{2}} \exp\left\{-\frac{\phi(1-\mu\,\lambda)^2}{2\,\mu^2\,\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^{\infty} \lambda^{-\frac{1}{2}} \left\{-t\,\lambda - \frac{\phi(1-\mu\,\lambda)^2}{2\,\mu^2\,\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^{\infty} \lambda^{-\frac{1}{2}} \exp\left\{-t\,\lambda - \frac{\phi(1-2\mu\,\lambda+\mu^2\,\lambda^2)}{2\,\mu^2\,\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^{\infty} \lambda^{-\frac{1}{2}} \exp\left\{-t\,\lambda - \frac{\phi}{2\mu^2\,\lambda} + \frac{\phi}{\mu} - \frac{\phi\,\lambda}{2}\right\} d\lambda
\end{aligned}$$

$$= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}} \exp\left\{-(\frac{\phi}{2}+t)\,\lambda - \frac{\phi}{2\mu^2\,\lambda} + \frac{\phi}{\mu}\right\} d\lambda$$

$$= (\frac{\phi}{2\,\pi})^{\frac{1}{2}}\, e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}} \exp\left\{-(\frac{\phi}{2}+t)\,\lambda - \frac{\phi}{2\mu^2}\,\frac{1}{\lambda}\right\} d\lambda$$

$$= (\frac{\phi}{2\,\pi})^{\frac{1}{2}}\, e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}} \left\{-(\frac{\phi}{2}+t)\,(\lambda + \frac{\phi}{2\mu^2\,(\frac{\phi}{2}+t)}\,\frac{1}{\lambda})\right\} d\lambda$$

$$= (\frac{\phi}{2\,\pi})^{\frac{1}{2}}\, e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}} \exp\left\{-(\frac{\phi}{2}+t)\,(\lambda + \frac{\phi}{\mu^2\,(\phi+2t)}\,\frac{1}{\lambda})\right\} d\lambda$$

Let $\lambda = \sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\, z \qquad \therefore \qquad d\lambda = \sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\, dz$

$$L_\lambda(t) = \left(\frac{\phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\right)^{\frac{1}{2}} \int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}\,(z+\frac{1}{z})\right\}$$

$$= \left(\frac{\phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\, 2\, K_{\frac{1}{2}}\left(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}\right) \tag{5.80}$$

where $K_v(\omega)$ is the modified Bessel function of the third kind of order $v$.

But $K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\,\omega}}\, e^{-\omega}$ (Watson, 1952).

In this Parameterization $\omega = \sqrt{\sqrt{\frac{\phi(\phi+2\mathrm{t})}{\mu^2}}}$

Therefore,

$$L_\lambda(t) = e^{\frac{\phi}{\mu}} \left(\frac{\phi}{2\,\pi}\sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\right)^{\frac{1}{2}} 2\sqrt{\frac{\pi}{2\,(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})}} \exp\left\{-(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})\right\}$$

$$= 2\, e^{\frac{\phi}{\mu}} \left(\frac{\phi}{4}\sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}\;\frac{1}{\frac{\phi(\phi+2t)}{\mu^2}}}\right)^{\frac{1}{2}} \exp\left\{-(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})\right\}$$

$$= 2\, e^{\frac{\phi}{\mu}} \left(\frac{\phi}{4}\sqrt{\frac{1}{(\phi+2t)^2}}\right)^{\frac{1}{2}} \exp\left\{-(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})\right\}$$

$$= 2\, e^{\frac{\phi}{\mu}} \left(\frac{\phi}{4\,(\phi+2t)}\right)^{\frac{1}{2}} \exp\left\{-(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})\right\}$$

$$
\begin{aligned}
&= e^{\frac{\phi}{\mu}} \; (\frac{\phi}{\phi+2t})^{\frac{1}{2}} \; \exp\left\{-(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})\right\} \\
&= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \; \exp\left\{-(\frac{\phi^2}{\mu^2}\,(\frac{\phi+2t}{\phi}))^{\frac{1}{2}}\right\} \; e^{\frac{\phi}{\mu}} \\
&= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{\frac{\phi}{\mu}-\frac{\phi}{\mu}\,(\frac{\phi+2t}{\phi})^{\frac{1}{2}}\right\} \\
&= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \; \exp\left\{\frac{\phi}{\mu}-\frac{\phi}{\mu}\,(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}\right\} \\
&= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{\frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}})\right\} \\
L'_{\lambda}(t) &= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{\frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}})\right\} \,.\, \frac{\phi}{\mu}\,\frac{1}{2}\,(1+\frac{2}{\phi}\,t)^{-\frac{1}{2}}\,(-\frac{2}{\phi})+ \\
&\quad \exp\left\{\frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}})\right\}\,(-\frac{1}{2})\,(1+\frac{2}{\phi}\,t)^{-\frac{3}{2}}\,(\frac{2}{\phi}) \\
&= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{\frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}})\right\} \,.\, \Bigg(-\frac{1}{\mu}\,(1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} - \frac{1}{\phi}\,(1+
\end{aligned}
\tag{5.81}
$$

and,

$$
\begin{aligned}
h(t) &= -\frac{L'_{\lambda}(t)}{L_{\lambda}} \\
&= \frac{1}{\mu}\,(1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} + \frac{1}{\phi}\,(1+\frac{2}{\phi}\,t)^{-1} \\
&= \frac{\frac{1}{\mu}}{(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}} + \frac{\frac{1}{\phi}}{(1+\frac{2}{\phi}\,t)} \\
&\equiv \frac{p_1}{1+c_1 t} + \frac{p_2}{(1+c_2\,t)^{\frac{1}{2}}} \\
\theta_1(t) &= \frac{p_1}{c_1}\,In(1+c_1 t) \qquad ; \qquad \theta_2(t) = \frac{2p_2}{c_2}\,\Big((1+c_2 t)^{\frac{1}{2}}-1\Big) \\
\theta_1(t-ts) &= \frac{p_1}{c_1}\,In(1+c_1 t-c_1 ts) \qquad ; \qquad \theta_2(t-ts) = \frac{2p_2}{c_2}\,\Big((1+c_2 t-c_2 ts)^{\frac{1}{2}}-1
\end{aligned}
\tag{5.82}
$$

The pgf of the convolution is

$$
\begin{aligned}
H(s,t) &= e^{-\frac{p_1}{c_1}\,In(1+c_1 t-c_1 ts)}\; e^{-\frac{2p_2}{c_2}\,\left((1+c_2 t-c_2 ts)^{\frac{1}{2}}-1\right)} \\
&= (\frac{\frac{1}{1+c_1 t}}{1-\frac{ct}{1+c_1 t}s})^{\frac{p_1}{c_1}}\; e^{-\frac{2p_2}{c_2}\,\left((1+c_2 t-c_2 ts)^{\frac{1}{2}}-1\right)}
\end{aligned}
\tag{5.83}
$$

The sum of the hazard functions of the exponential-gamma and that of the exponential-inverse Gaussian distribution, therefore, gives rise to the convolution of the negative binomial and the Poisson-inverse Gaussian (Sichel) distributions.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{1}{\theta(t)} \left( \frac{p_1}{c_1} \, In(1 + c_1 t - c_1 ts) + \frac{2p_2}{c_2} \, [(1 + c_2 t - c_2 ts)^{\frac{1}{2}} - 1] \right)$$

$$\therefore \qquad G^x(s,t) = \frac{(x-1)!}{\theta(t)} \{ \frac{p_1}{c_1} \, (\frac{c_1 t}{1 + c_1 t - c_2 ts})^x + \binom{\frac{1}{2} + x - 1 - 1}{x - 1} \frac{p_2}{c_2} \, (\frac{c_2 t}{(1 + c_2 t - c_2 ts)})^x (\frac{1}{(1 + c_2 t - c_2 ts)})^{-\frac{1}{2}} \})$$

$$= \frac{(x-1)!}{\theta(t)} \left\{ \frac{p_1}{c_1} (\frac{c_1 t}{1 + c_1 t - c_2 ts})^x \right\} + \frac{(x-1)!}{\theta(t)} \left\{ p_2 t \left( \frac{1}{2} + x - 1 - 1 x - 1 \right) \, (\frac{c_2 t}{(1 + c_2 t - c_2 ts)})^{x-1} (\frac{1}{(1 + c_2 t - c_2 \ldots} \right. \tag{5.84}$$

Therefore, the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.85a}$$

$$g_x(t) = \frac{1}{x \, \theta(t)} \left\{ \frac{p_1}{c_1} (\frac{c_1 t}{1 + c_1 t})^x + p_2 t \binom{\frac{1}{2} + x - 1 - 1}{x - 1} \, (\frac{c_2 t}{(1 + c_2 t)})^{x-1} (\frac{1}{(1 + c_2 t)})^{\frac{1}{2}} \right\} \tag{5.85b}$$

$$x = 1, 2, 3, \ldots$$

where,

$$\theta(t) = \frac{p_1}{c_1} \, In(1 + c_1 t) + \frac{2p_2}{c_2} \, \left( (1 + c_2 t)^{\frac{1}{2}} - 1 \right) \tag{5.86}$$

Panjer's recursive model does not hold in general Parameterization, unless $\mathbf{p_1 = p_2 = p}$ **say a**

In this Parameterization the Panjer's model is

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{x-1}{x} \, (\frac{ct}{1 + ct}) \tag{5.87}$$

The recursive form for the convolution of compound Poisson distributions is:

$$n \, p_n(t) = \theta(t) \sum_{x=1}^{n} x \, g_x(t) \;\; p_{n-x}(t)$$

$$= \sum_{x=1}^{n} \left\{ \frac{p_1}{c_1} (\frac{c_1 t}{1 + c_1 t})^x + p_2 t \binom{\frac{1}{2} + x - 1 - 1}{x - 1} (\frac{c_2 t}{(1 + c_2 t)})^{x-1} (\frac{1}{(1 + c_2 t)})^{\frac{1}{2}} \right\} p_{n-x}(t \tag{5.88}$$

$$n = 1, 2, 3, \ldots \tag{5.89}$$

**In the equation (5.23) $a_1 = 1$ and $a_2 = 2$**

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1t)} + \frac{p_2}{(1+c_2t)^2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0 \tag{5.90}$$

where the second hazard function will be referred to as Polya-Aeppli hazard function (Wakoli and Ottieno 2015, p. 234).

$$\theta_1(t) = \frac{p_1}{c_1} In(1+c_1t) \quad ; \quad \theta_2(t) = \frac{p_2}{c_2}\left(1-(1+c_2t)^{-1}\right) \tag{5.91}$$

$$\theta_1(t-ts) = \frac{p_1}{c_1} In(1+c_1t-c_1ts) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\left(1-(1+c_2t-c_1ts)^{-1}\right)$$

The pgf of the convolution is

$$\begin{aligned} H(s,t) &= e^{-\left(\frac{p_1}{c_1} In(1+c_1t-c_1ts)\right)} e^{\frac{-p_2}{c_2}\left(1-(1+c_2t-c_2ts)^{-1}\right)} \\ &= \left(\frac{\frac{1}{1+c_1t}}{1-\frac{ct}{1+c_1t}s}\right)^{\frac{p_1}{c_1}} e^{-\frac{p_2}{c_2}\left(1-(1+c_2t-c_2ts)^{-1}\right)} \end{aligned} \tag{5.92}$$

The sum of hazard function of a Pareto distribution and the Polya-Aeppli hazard function, therefore, gives rise to the convolution of the negative binomial distribution and the Polya-Aeppli distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \frac{1}{\theta(t)}\left(\frac{p_1}{c_1} In(1+c_1t-c_1ts) + \frac{p_2}{c_2}\left(1-(1+c_2t-c_1ts\right.\right. \tag{5.93}$$

$$\begin{aligned} G^{(x)}(s,t) &= \frac{(x-1)!}{\theta(t)}\left\{\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t-c_1ts}\right)^x\right\} + \frac{x!}{\theta(t)}\left\{\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t-c_2ts}\right)^x \frac{c_2t}{1+c_2t-c_2} \right. \\ &= \frac{1}{\theta(t)}\left((x-1)!\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t-c_1ts}\right)^x + \frac{p_2}{c_2}x!\left(\frac{c_2t}{(1+c_2t-c_2ts)}\right)^x \frac{1}{(1+c_2t-}\right. \end{aligned} \tag{5.94}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.95a}$$

$$g_x(t) = \frac{1}{\theta(t)}\frac{1}{x}\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t}\right)^x + \frac{1}{\theta(t)}\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x \frac{1}{1+c_2t} \tag{5.95b}$$

$$for \quad x = 1, 2, ..$$

Again Panjer's recursive model is not satisfied, unless $c_1 = c_2 = c$, in which Parameterization

$$\frac{g_x(t)}{g_{x-1}(t)} = \left\{\frac{x-1}{x} \frac{\frac{p_1+p_2}{c}\ (\frac{ct}{1+ct})^x}{\frac{p_1+p_2}{c}\ (\frac{ct}{1+ct})^{x-1}}\right\}$$
$$= \frac{x-1}{x} \left(\frac{ct}{1+ct}\right) \tag{5.96}$$

The recursive form for the convolution of compound Poisson distributions is

$$np_n(t) = \theta(t) \sum_{x=1}^{n} x\, g_x(t)\, p_{n-x}(t)$$
$$= \theta(t) \sum_{x=1}^{n} x \frac{1}{\theta(t)} \left(\frac{1}{x} \frac{p_1}{c_1} (\frac{c_1 t}{1+c_1 t})^x + \frac{p_2}{c_2} (\frac{c_2 t}{(1+c_2 t)})^x \frac{1}{(1+c_2 t)}\right) p_{n-x}(t)$$
$$= \frac{p_1}{c_1} \sum_{x=1}^{n} \left(\frac{c_1 t}{1+c_1 t}\right)^x p_{n-x}(t) + \frac{p_2}{c_2} \sum_{x=1}^{n} x \left(\frac{c_2 t}{(1+c_2 t)}\right)^x \frac{1}{(1+c_2 t)})\, p_{n-x}(t) \tag{5.97}$$

### 5.4.3 In the equation (5.23) $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1 t)^{\frac{1}{2}}} + \frac{p_2}{(1+c_2 t)^{\frac{1}{2}}} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0 \tag{5.98}$$

Both hazard functions belong to the exponential-inverse Gaussian distributions.

$$\therefore \qquad \theta_1(t) = \frac{2p_1}{c_1} \left((1+c_1 t)^{\frac{1}{2}} - 1\right) \qquad ; \qquad \theta_2(t) = \frac{2p_2}{c_2} \left((1+c_2 t)^{\frac{1}{2}} - 1\right)$$
$$\theta_1(t-ts) = \frac{2p_1}{c_1} \left((1+c_1 t - c_1 ts)^{\frac{1}{2}} - 1\right) \qquad ; \qquad \theta_2(t-ts) = \frac{2p_2}{c_2} \left((1+c_2 t - c_1 ts)^{\frac{1}{2}}\right.$$

The pgf of the convolution is

$$H(s,t) = e^{-\frac{2p_1}{c_1}\left((1+c_1 t - c_1 ts)^{\frac{1}{2}} - 1\right)}\, e^{-\frac{2p_2}{c_2}\left((1+c_2 t - c_1 ts)^{\frac{1}{2}} - 1\right)} \tag{5.99}$$

and therefore the sum of two hazard functions of exponential-inverse Gaussian distributions gives rise to the convolution of two Poisson-inverse Gaussian distributions.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{1}{\theta(t)} \left\{\frac{2p_1}{c_1} \left((1+c_1 t - c_1 ts)^{\frac{1}{2}} - 1\right) + \frac{2p_2}{c_2} \left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right)\right\}$$

$$G^x(s,t) = \frac{1}{\theta(t)}\,(x-1)!\,\{p_1 t\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_1 t}{(1+c_1 t - c_1 ts})^{x-1}\,(1+c_1 t - c_1 ts)^{-\frac{1}{2}} +$$

$$p_2 t\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2 t}{(1+c_2 t - c_2 ts})^{x-1}\,(1+c_2 t - c_2 ts)^{-\frac{1}{2}}\}$$

$$= \frac{1}{\theta(t)}\,(x-1)!\,\binom{-\frac{1}{2}}{x-1}\,\{p_1 t\,(-\frac{c_1 t}{(1+c_1 t - c_1 ts})^{x-1}\,(1+c_1 t - c_1 ts)^{-\frac{1}{2}} +$$

$$p_2 t\,(-\frac{c_2 t}{(1+c_2 t - c_2 ts})^{x-1}\,(1+c_2 t - c_2 ts)^{-\frac{1}{2}}\} \qquad (5.100)$$

and the pmfs of the iid random variables are

$$g_0(t) = 0$$

$$g_x(t) = \frac{1}{\theta(t)}\,\frac{1}{x}\,\{p_1 t\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_1 t}{(1+c_1 t})^{x-1}\,(1+c_1 t)^{-\frac{1}{2}} +$$

$$p_2 t\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2 t}{(1+c_2 t})^{x-1}\,(1+c_2 t)^{-\frac{1}{2}}\}$$

In the Parameterization where $\mathbf{c_1 = c_2 = c}$

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{x-1}{x}\,\frac{\binom{-\frac{1}{2}}{x-1}}{\binom{-\frac{1}{2}}{x-2}}\,(-\frac{c_2 t}{(1+c_2 t})$$

$$= (\frac{x-1}{x})\,(\frac{\frac{3}{2}-x}{x-1})\,(\frac{-c_2 t}{1+c_2 t})$$

$$= \frac{c_2 t}{(1+c_2 t)} - \frac{3\,c_2 t}{2\,(1+c_2 t)}\,\frac{1}{x}$$

which is in Panjer's recursive form with

$$a = \frac{c_2 t}{(1+c_2 t)} \qquad and \qquad b = -\frac{3\,c_2 t}{2\,(1+c_2 t)}$$

The recursive form for the convolution of the compound Poisson distribution is:

$$n\,p_n(t) = \theta(t)\sum_{x=1}^{n} x\,g_x(t)\;p_{n-x}(t)$$

$$= \sum_{x=1}^{n}\{p_1 t\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_1 t}{(1+c_1 t})^{x-1}\,(1+c_1 t)^{-\frac{1}{2}} +$$

$$p_2 t\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2 t}{(1+c_2 t})^{x-1}\,(1+c_2 t)^{-\frac{1}{2}}\}\;p_{n-x}(t)$$

$$= p_1 t\,(1+c_1 t)^{-\frac{1}{2}}\sum_{x=1}^{n}\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_1 t}{(1+c_1 t})^{x-1}\,p_{n-x}(t)$$

$$+ p_2 t\,(1+c_2 t)^{-\frac{1}{2}}\sum_{x=1}^{n}\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2 t}{(1+c_2 t})^{x-1}\,p_{n-x}(t) \qquad n = 1, 2, \ldots$$

### 5.4.4 In the equation (5.23) $a_1 = 2$ and $a_2 = 2$

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1t)^2} + \frac{p_2}{(1+c_2t)^2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 >$$

Both are the Polya-Aeppli hazard functions

.

$$\therefore \qquad \theta_1(t) = \frac{p_1}{c_1}\left(1-(1+c_1t)^{-1}\right) \quad ; \quad \theta_2(t) = \frac{p_2}{c_2}\left(1-(1+c_2t)^{-1}\right)$$

$$\theta_1(t-ts) = \frac{p_1}{c_1}\left(1-(1+c_1t-c_1ts)^{-1}\right) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\left(1-(1+c_2t-c\right.$$

The pgf of the convolution is

$$H(s,t) = e^{-\frac{p_1}{c_1}\left(1-(1+c_1t-c_1ts)^{-1}\right)}\, e^{-\frac{p_2}{c_2}\left(1-(1+c_2t-c_1ts)^{-1}\right)} \tag{5.102}$$

and therefore the sum of two Polya-Aeppli hazard functions gives rise to the convolution of two Polya-Aeppli distributions.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \frac{1}{\theta(t)}\left(\frac{p_1}{c_1}\{1-(1+c_1t-c_1ts)^{-1} + \frac{p_2}{c_2}\{1-(1+c\right.$$

$$\therefore \qquad G^{(x)}(s,t) = \frac{x!}{\theta(t)}\left\{\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t-c_1ts}\right)^x \frac{1}{1+c_1t-c_1ts}\right\} + \frac{x!}{\theta(t)}\left\{\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t-c_2ts}\right)^x \frac{1}{(1+c_2t-c_2ts)}\right\} \tag{5.103}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{5.104a}$$

$$g_x(t) = \frac{1}{\theta(t)}\left\{\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t}\right)^x \frac{1}{1+c_1t}\right\} + \tag{5.104b}$$

$$\frac{1}{\theta(t)}\left\{\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x \frac{1}{(1+c_2t)}\right\} \quad for \quad x = 1, 2, \ldots \tag{5.104c}$$

Panjer's recursive model does not hold, unless $c_1 = c_2 = c$, in which Parameterization

$$\frac{g_x(t)}{g_{x-1}(t)} = \left(\frac{ct}{1+ct}\right)^x \left(\frac{(1+ct)}{(ct)}\right)^{x-1} = \frac{ct}{1+ct} \qquad x = 2, 3, \ldots$$

$$= (\frac{ct}{1+ct} + \frac{0}{x}) \qquad for \qquad x = 2, 3, \ldots$$

which is Panjer's form with

$$a = \frac{ct}{1+ct} \quad \text{and} \quad b = 0$$

The recursive form for the convolution of the compound Poisson distribution is

$$np_n(t) = \theta(t) \sum_{x=1}^{n} x\, g_x(t)\, p_{n-x}(t) \qquad n = 1, 2 \ldots$$

$$= \theta(t) \sum_{x=1}^{n} x \frac{1}{\theta(t)} \left( \frac{p_1}{c_1} (\frac{c_1 t}{1+c_1 t})^x \frac{1}{1+c_1 t} + \frac{p_2}{c_2} \left( \frac{c_2 t}{1+c_2 t} \right)^x \frac{1}{1+c_2 t} \right) p_{n-x}(t)$$

$$= \sum_{x=1}^{n} \left( \frac{p_1}{c_1} x (\frac{c_1 t}{1+c_1 t})^x \frac{1}{1+c_1 t} + \frac{p_2}{c_2} x \left( \frac{c_2 t}{1+c_2 t} \right)^x \frac{1}{1+c_2 t} \right) p_{n-x}(t)$$

$$= \frac{p_1}{c_1} \frac{1}{1+c_1 t} \sum_{x=1}^{n} x \left( \frac{c_1 t}{1+c_1 t} \right)^x p_{n-x}(t) + \frac{p_2}{c_2} \frac{1}{1+c_2 t} \sum_{x=1}^{n} x \left( \frac{c_2 t}{1+c_2 t} \right)^x p_{n-x}( \qquad (5.105)$$

## 5.5 Concluding remarks

Sums of two hazard functions gives rise to convolutions of infinitely divisible mixed Poisson distributions which are also convolutions of compound Poisson distributions.
The sums can be extended to more than two hazard functions.
The sum of the hazard function of exponential distribution and that of Pareto distribution is the same as the hazard function of exponential-shifted gamma distribution.
Similarly, the sum of the hazard function of Pareto and exponential-inverse Gaussian is the same as the hazard function of exponential-reciprocal inverse Gaussian.

It is easier to express the convolutions in terms of pgfs and recursive forms rather than obtaining pmfs explicitly.

Panjer's recursive model holds for Hofamman hazard function when:

- one of the two hazard functions is a constant.
- when $\mathbf{a_1 = a_2, \quad p_1 = p_2 \quad and \quad c_1 = c_2}$

In chapter 2 and 3 exponential mixtures were expressed either explicitly or in terms of special functions, thus modified Bessel function of the third kind and confluent hyper-geometric function. However, in chapter 4 it has been determined that there is a link between exponential and Poisson mixtures but through the hazard functions that expressed explicitly and are members of the family of Hofmann distributions.

Further work is to identify other families of hazard functions of exponential mixtures, which are not necessarily members of the family of Hofmann distributions, and whose sums of hazard functions give rise to convolutions of Poisson mixtures.

# Chapter 6

# HAZARD FUNCTIONS OF CONTINUOUS COMPOUND DISTRIBUTIONS

## 6.1 Introduction

Laplace transforms of probability distributions play a key role in obtaining exponential mixtures. In this chapter hazard functions are obtained using Laplace transforms of fixed and random sums of independent continuous random variables.

An attempt has also been made to derive the associated mixed Poisson distributions, where possible.

## 6.2 Convolutions

Let

$$z_N = Y_1 + Y_2 + \ldots.. + Y_N$$

where $\mathbf{Y_i's}$ are iid continuous random variables and $\mathbf{N}$ is fixed.

Suppose

$$L_Z(t) = \text{the Laplace transform of } z \tag{6.1}$$

and,

$$L_Y(t) = \text{the Laplace transform of } Y_i \text{ for } i = 1, 2, 3, \ldots, N$$

Then,

$$L_Z(t) = [L_Y(t)]^N$$

If the $Y_i$ is exponential with parameter $\lambda$, then

$$\begin{aligned} L_Z(t) &= [\frac{\lambda}{\lambda + t}]^N \\ &= \lambda^N \ (\lambda + t)^{-N-1} \end{aligned}$$

implying that

$$L_z'(t) = -N \ \lambda^N \ (\lambda + t)^{-N-1}$$

and the hazard function of $z$ is

$$h_z(t) = -\frac{L_z'(t)}{L_Z(t)}$$

$$= \frac{N \lambda^N (\lambda + t)^{-N-1}}{\lambda^N (\lambda + t)^{-N}}$$
$$= \frac{N}{\lambda + t}$$
$$= \frac{N}{\lambda (1 + \frac{t}{\lambda})} \tag{6.2}$$

which is Hofmann hazard function with parameters

$$p = \frac{N}{\lambda}, \qquad c = \frac{1}{\lambda}, \qquad a = 1$$

This is a hazard function of a Pareto (exponential - gamma) distribution.

If $Y_i$ is gamma with parameters $\kappa$ and $\lambda$, then

$$L_Z(t) = \left[ \left( \frac{\lambda}{\lambda + t} \right)^{\kappa} \right]^N$$
$$= \lambda^{\kappa N} (\lambda + t)^{-\kappa N}$$

Therefore,

$$L_z^{'}(t) = \lambda^{\kappa N} (-\kappa N) (\lambda + t)^{-\kappa N - 1}$$

and,

$$h_z(t) = -\frac{L_z^{'}(t)}{L_Z(t)}$$
$$= \frac{\kappa N \lambda^{\kappa N} (\lambda + t)^{-\kappa N - 1}}{\lambda^{\kappa N} (\lambda + t)^{-\kappa N}}$$
$$= \frac{\kappa N}{\lambda + t}$$
$$= \frac{\kappa N}{\lambda (1 + \frac{t}{\lambda})}$$

which is Hofmann hazard function with parameters

$$p = \frac{\kappa N}{\lambda}, \qquad c = \frac{1}{\lambda}, \qquad a = 1$$

This is also a hazard function of a Pareto distribution whose associated mixed Poisson distribution is a negative binomial distribution.

(6.3)

## 6.3 Compound Distributions

Let

$$Z_N = Y_1 + Y_2 + \ldots.. + Y_N$$

where $\mathbf{Y_i's}$ are iid continuous random variables and $\mathbf{N}$ is also a random variable independent of $\mathbf{Y_is}$.

Suppose

$$\begin{aligned} L_Z(t) &= \text{the Laplace transform of } Z \\ L_Y(t) &= \text{the Laplace transform of } Y_i \text{ for } i = 1, 2, 3, \ldots, N \end{aligned} \tag{6.4}$$

and,

$$F(t) = F_N(t) = \text{the pgf of } N$$

and,

$$\begin{aligned} L_Z(t) &= E[e^{-tz}] \\ &= EE[e^{-tz} \mid N] \\ &= EE[e^{-t(Y_1+Y_2+\ldots..+Y_N)} \mid N] \\ &= E\left\{E(e^{-tY_1})\ E(e^{-tY_2}).\therefore.E(e^{-tY_N})\right\} \\ &= E\left\{L_Y(t)\right\}^N \\ &= F[L_Y(t)] \end{aligned} \tag{6.5}$$

which is a pgf of the Laplace transform

$$\tag{6.6}$$

Compound distributions have many natural applications, for example the insurance application. In an individual insurance setting, aggregate claims are modeled during a fixed policy period for an insurance policy. In this setting, more than one claim is possible, for example, auto insurance and property and casualty insurance. In a group insurance setting, the aggregate claims are modeled during a fixed policy period for a group of insureds that are independent. In other words, distributions that can either model the total claims for an individual insured or a group of independent risks over a fixed period such that the claim frequency is uncertain (no claim, one claim or multiple claims) are discussed.

The random variable $z_N$ is said to have a compound distribution if (1) the number of terms $N$ is uncertain, (2) the random variables $Y_i$ are independent and identically distributed (with common distribution Y) and (3) each $Y_i$ is independent of $N$.

The sum $z_N$ as defined above is sometimes called a random sum. In this case, the variable $N$ represents the number of claims generated by an individual policy or a group of independent insureds over a policy period. The variable $Y_i$ represents the $i^{th}$ claim and $z_N$ represents the aggregate claims over the fixed policy period.

## 6.4 Compound Binomial Distribution

If $\mathbf{N}$ is a binomial distribution, then $\boldsymbol{z}$ is said to have a compound binomial distribution.
Suppose $\boldsymbol{N}$ is binomial with parameters $\boldsymbol{n}$ and $\boldsymbol{p}$, then

$$L_Z(t) = (q + p\, L_Y(t))^n$$

where $q = 1 - p$

$$\therefore \qquad L'_Z(t) = np\, L'_Y(t)\ \ (q + p\, L_Y(t))^{n-1} \tag{6.7}$$

and,

$$\begin{aligned} h_z(t) &= \frac{-L'_Z(t)}{L_Z(t)} \\ &= \frac{-np\, L'_Y(t)\ \ (q + p\, L_Y(t))^{n-1}}{(q + p\, L_Y(t))^n} \\ &= \frac{-np\, L'_Y(t)}{q + pL_Y(t)} \end{aligned} \tag{6.8}$$

### 6.4.1 The Parameterization when $Y_i s$ are Exponential

In $\boldsymbol{Y_i}$ is independent and identically distributed as exponential ($\boldsymbol{Exp(\lambda)}$), with parameter $\boldsymbol{\lambda}$, the Laplace transform of $\boldsymbol{Y}$ is

$$L_Y(t) = \frac{\lambda}{\lambda + t} \qquad\qquad L'_Y(t) = -\frac{\lambda}{(\lambda + t)} \tag{6.9}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned} h(t) &= \frac{np\, \frac{\lambda}{(\lambda+t)^2}}{q + p\, \frac{\lambda}{\lambda+t}} \\ &= \frac{np\, \lambda}{(\lambda + t)(\lambda + qt)} \\ &= \frac{A}{\lambda + t} + \frac{B}{\lambda + qt} \end{aligned}$$

by partial fraction technique

$$\begin{aligned} np\lambda &= A(\lambda + qt) + B(\lambda + t) \\ &= A\lambda + Aqt + B\lambda + Bt \\ \therefore \qquad A + B &= np \end{aligned}$$

and,

$$\begin{aligned} Aq + B &= 0 \\ A - Aq &= np \\ Ap &= np \\ \therefore \quad A &= n \end{aligned}$$

and,

$$B = np - n = -nq$$

$$\therefore \quad h_z(t) = \frac{n}{\lambda + t} - \frac{nq}{\lambda + qt} \tag{6.10}$$

Alternatively

$$\begin{aligned} L_Z(t) &= F[L_Y(t)] \\ &= [q + p\, L_Y(t)]^n \\ &= [q + p\, \frac{\lambda}{\lambda + t})]^n \\ &= (\lambda + t)^{-n}\, (\lambda + qt)^n \end{aligned}$$

$$\therefore \quad L'_Z(t) = -n(\lambda + t)^{-n-1}(\lambda + qt)^n + nq(\lambda + t)^{-n}(\lambda + qt)^{n-1} \tag{6.11}$$

Therefore,

$$\begin{aligned} h_z(t) &= \frac{-L'_Z(t)}{L_Z(t)} \\ &= \frac{n(\lambda + t)^{-n-1}(\lambda + qt)^n - nq(\lambda + t)^{-n}(\lambda + qt)^{n-1}}{(\lambda + t)^{-n}\, (\lambda + qt)^n} \\ &= \frac{n}{\lambda + t} - \frac{nq}{\lambda + qt} \end{aligned} \tag{6.12}$$

the difference of hazard functions of Pareto distributions

(6.13)

### 6.4.2 The case with gamma iid random variables

In the Parameterization $\mathbf{Y_i's}$ are identical and independently distributed gamma random variables with parameters $\kappa$ and $\lambda$ and therefore

$$g(y) = \frac{\lambda^\kappa}{\Gamma\kappa}\, e^{-\lambda y}\, y^{\kappa - 1}; \qquad y > 0;\ \lambda > 0;\ \kappa > 0 \tag{6.14}$$

The Laplace transform of $Y$ is

$$L_Y(t) = \int_0^\infty e^{-t\,y}\, g(y)\, dy$$

$$= \int_0^\infty e^{-ty} \frac{\lambda^\kappa}{\Gamma\kappa} e^{-\lambda y} y^{\kappa-1} dy$$

$$= \frac{\lambda^\kappa}{\Gamma\kappa} \int_0^\infty y^{\kappa-1} e^{-(\lambda+t)y} dy$$

$$= \frac{\lambda^\kappa}{\Gamma\kappa} \frac{\Gamma\kappa}{(\lambda+t)^\kappa}$$

$$= (\frac{\lambda}{\lambda+t})^\kappa$$

$$\therefore \quad L'_Y(t) = \frac{-\kappa\, \lambda^\kappa}{(\lambda+t)^{\kappa+1}} \tag{6.15}$$

The hazard function of $z_N(t)$ is

$$h(t) = \frac{np\, \frac{\kappa\, \lambda^\kappa}{(\lambda+t)^{\kappa+1}}}{q + p\, (\frac{\lambda}{\lambda+t})^\kappa}$$

$$= \frac{1}{(\lambda+t)} \frac{np\kappa\, \lambda^\kappa}{(q(\lambda+t)^\kappa + p\lambda^\kappa)}$$

$$= \frac{1}{(\lambda+t)} \frac{np\kappa}{\left(q(1+\frac{1}{\lambda}t)^\kappa + p\right)}$$

$$= \frac{\kappa n}{(\lambda+t)} - \frac{n\kappa q\, (\lambda+t)^{\kappa-1}}{((\lambda+t)^\kappa q + p\lambda^\kappa)} \tag{6.16}$$

using partial fractions.

Alternatively

The Laplace transform of $z_N(t)$ is

$$L_Z(t) = \left(q + p\, (\frac{\lambda}{\lambda+t})^\kappa\right)^n$$

$$= ((\lambda+t)^\kappa q + p\lambda^\kappa)^n\ (\lambda+t)^{-\kappa n}$$

$$\therefore \quad L'_Z(t) = n\,((\lambda+t)^\kappa q + p\lambda^\kappa)^{n-1}\ \kappa q(\lambda+t)^{\kappa-1}\ ($$

The hazard function of $z_N(t)$ is

$$h(t) = \frac{-n\,((\lambda+t)^\kappa q + p\lambda^\kappa)^{n-1}\ \kappa q(\lambda+t)^{\kappa-}}{((\lambda+t)^\kappa}$$

$$= \frac{\kappa n(\lambda+t)^{-\kappa n-1}\,((\lambda+t)^\kappa q + p\lambda^\kappa)^n}{((\lambda+t)^\kappa q + p\lambda^\kappa)^n\ (\lambda+t)^{-\kappa n}} -$$

$$= \frac{\kappa n}{(\lambda+t)} - \frac{n\kappa q\, (\lambda+t)^{\kappa-1}}{((\lambda+t)^\kappa q + p\lambda^\kappa)} \tag{6.17}$$

The compound binomial distribution is applicable to aggregate claims by a group of independent insureds, with $N$ distributed as binomial. The variable $Y_i$ represents the $i^{th}$ claim and $z_N$ represents the aggregate claims over the fixed policy period.

## 6.5 Compound Geometric Distribution

If $\mathbf{N}$ is a geometric with parameter $\mathbf{p}$ and if it is independent of $\mathbf{Y_i's}$ then its probability generating function is given by

$$F_N(t) = \frac{p}{1-qt} \tag{6.18}$$

The Laplace transform of $z_N(t)$ is

$$\begin{aligned} L_Z(t) &= \frac{p}{1-q\,L_Y(t)} \\ \therefore \qquad L'_Z(t) &= \frac{pq\,L'_Y(t)}{(1-q\,L_Y(t))^2} \end{aligned} \tag{6.19}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned} h(t) &= -\frac{L'_Z(t)}{L_Z(t)} \\ &= \frac{\frac{pq\,L'_Y(t)}{(1-q\,L_Y(t))^2}}{\frac{p}{1-q\,L_Y(t)}} \\ &= \frac{q\,L'_Y(t)}{(1-q\,L_Y(t))} \end{aligned} \tag{6.20}$$

### 6.5.1 The case with exponential iid random variables

In $Y_i$ is independent and identically distributed as exponential ($Exp(\lambda)$), with parameter $\lambda$, the Laplace transform is

$$\begin{aligned} L_Y(t) &= \frac{\lambda}{\lambda+t} \\ \therefore \qquad L'_Y(t) &= -\frac{\lambda}{(\lambda+t)^2} \end{aligned} \tag{6.21}$$

The hazard function of $z_N(t)$ is given by

$$\begin{aligned} h(t) &= \frac{q\,\frac{\lambda}{(\lambda+t)^2}}{1-q\frac{\lambda}{\lambda+t}} \\ &= \frac{q\,\lambda}{(\lambda+t)(\lambda(1-q)+t)} \end{aligned}$$

$$= \frac{1}{\lambda p + t} - \frac{1}{\lambda + t} \tag{6.22}$$

using partial fractions.

Alternatively

The Laplace transform of $z_N(t)$ is

$$\begin{aligned} L_Z(t) &= \frac{p}{1 - q(\frac{\lambda}{\lambda+t})} \\ &= \frac{p(\lambda + t)}{\lambda p + t} \\ &= p(\lambda + t)(\lambda p + t)^{-1} \\ L'_Z(t) &= p(\lambda p + t)^{-1} - p(\lambda + t)(\lambda p + t)^{-2} \end{aligned}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned} h(t) &= \frac{-L'_Z(t)}{L_Z(t)} \\ &= \frac{p(\lambda p + t)^{-1} - p(\lambda + t)(\lambda p + t)^{-2}}{p(\lambda + t)(\lambda p + t)^{-1}} \\ &= \frac{1}{\lambda p + t} - \frac{1}{\lambda + t} \end{aligned} \tag{6.23}$$

### 6.5.2 The case with gamma iid random variables

The $\mathbf{Y_i'}$**s** are distributed as gamma with parameters $\kappa$ and $\lambda$ and therefore The Laplace transform of $Y$ is

$$\begin{aligned} L_Y(t) &= (\frac{\lambda}{\lambda + t})^{\kappa} \\ &= \frac{\lambda^{\kappa}}{(\lambda + t)^{\kappa}} \\ L'_Y(t)) &= \frac{-\kappa\lambda^{\kappa}}{(\lambda + t)^{\kappa}} \, \frac{1}{\lambda + t} \end{aligned}$$

The hazard function of $z_N(t)$ is given by

$$\begin{aligned} h(t) &= \frac{q \, \frac{\kappa\lambda^{\kappa}}{(\lambda+t)^{\kappa}} \, \frac{1}{\lambda+t}}{[1 - q \, \frac{\lambda^{\kappa}}{(\lambda+t)^{\kappa}}]} \\ &= \frac{1}{[\lambda + t]} \, \frac{q\kappa\lambda^{\kappa}}{[(\lambda + t)^{\kappa} - q \, \lambda^{\kappa}]} \end{aligned}$$

$$= \frac{\kappa}{(\lambda+t)[1-q(\frac{\lambda}{\lambda+t})^{\kappa}} - \frac{\kappa}{(\lambda+t)}$$

using partial fractions.

Alternatively

The Laplace transform of $z_n(t)$ is

$$L_Z(t) = \frac{p}{1-q\,\frac{\lambda^{\kappa}}{(\lambda+t)^{\kappa}}}$$

$$= p(\lambda+t)^{\kappa}\left((\lambda+t)^{\kappa} - q\,\lambda^{\kappa}\right)^{-1}$$

$$\therefore \quad L'_Z(t) = p\kappa(\lambda+t)^{\kappa-1}\left((\lambda+t)^{\kappa} - q\,\lambda^{\kappa}\right)^{-1} - p(\lambda+t)^{\kappa}\,\kappa(\lambda+t)^{\kappa-1}\,\left((\lambda+t)^{\kappa} - q\,\lambda\right.$$

Therefore, the hazard function of $z_n(t)$ is

$$h(t) = \frac{-p\kappa(\lambda+t)^{\kappa-1}[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}]^{-1} + p(\lambda+t)^{\kappa}\,\kappa(\lambda+t)^{\kappa-1}\,[(\lambda+t)^{\kappa} - q\,\lambda}{p(\lambda+t)^{\kappa}[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}]^{-1}}$$

$$= \frac{p(\lambda+t)^{\kappa}\,\kappa(\lambda+t)^{\kappa-1}\,[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}]^{-2}}{p(\lambda+t)^{\kappa}[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}]^{-1}} - \frac{p\kappa(\lambda+t)^{\kappa-1}[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}}{p(\lambda+t)^{\kappa}[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}]^{-}}$$

$$= \frac{\kappa(\lambda+t)^{\kappa-1}}{[(\lambda+t)^{\kappa} - q\,\lambda^{\kappa}]} - \frac{\kappa}{(\lambda+t)}$$

$$= \frac{\kappa}{(\lambda+t)[1-q(\frac{\lambda}{\lambda+t})^{\kappa}} - \frac{\kappa}{(\lambda+t)}$$

**Remark 6.1.** *The hazard functions of the compound binomial distribution and the compound geometric distribution are a difference between Hofmann hazard functions and therefore they are not convolutions.*

The compound geometric distribution is applicable to aggregate claims by a group of independent insureds, with $N$ the number of claims generated by a portfolio of insurance policies in a fixed time period following the geometric distribution. The variable $Y_i$ represents the $i^{th}$ claim and $z_N$ represents the aggregate claims over the fixed policy period.

## 6.6 Compound Shifted Geometric Distribution

If $\mathbf{N}$ is a shifted geometric with parameter $\mathbf{p}$ and if it is independent of $\mathbf{Y_i'}$s then its probability generating function is given by

$$F_N(t) = \frac{pt}{1-qt}$$

The Laplace transform of $z_N(t)$ is

$$L_Z(t) = \frac{p\,L_Y(t)}{1-q\,L_Y(t)}$$

$$\therefore \qquad L'_Z(t) = \frac{pq\, L'_Y(t)\, L_Y(t)}{(1-q\, L_Y(t))^2} + \frac{p\, L'_Y(t)}{(1-q\, L_Y(t))}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned} h(t) &= \frac{-L'_Z(t)}{L_Z(t)} \\ &= \frac{-q\, L'_Y(t)}{(1-q\, L_Y(t))} + \frac{-L'_Y(t)}{L_Y(t)} \end{aligned}$$

### 6.6.1 The case with exponential iid random variables

In $Y_i$ is independent and identically distributed as exponential ($Exp(\lambda)$), with parameter $\lambda$, the Laplace transform is

$$\begin{aligned} L_Y(t) &= \frac{\lambda}{\lambda+t} \\ \therefore \qquad L'_Y(t) &= -\frac{\lambda}{(\lambda+t)^2} \end{aligned}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned} h(t) &= \frac{q\, \frac{\lambda}{(\lambda+t)^2}}{(1-q\, \frac{\lambda}{\lambda+t})} + \frac{\frac{\lambda}{(\lambda+t)^2}}{\frac{\lambda}{\lambda+t}} \\ &= \frac{q\, \lambda}{(\lambda+t)(\lambda+t-q\, \lambda)} + \frac{\lambda}{(\lambda+t)\lambda} \\ &= \frac{q\, \lambda}{(\lambda+t)(p\lambda+t)} + \frac{1}{(\lambda+t)} \\ &= \frac{1}{(p\lambda+t)} \end{aligned}$$

Alternatively

The Laplace transform of $z_N(t)$ is

$$\begin{aligned} L_Z(t) &= \frac{p\, L_Y(t)}{1-q\, L_Y(t)} \\ &= \frac{p\, \frac{\lambda}{(\lambda+t)}}{1-q\, \frac{\lambda}{(\lambda+t)}} \\ &= \frac{p\lambda}{\lambda+t} \\ &= p\lambda(p\lambda+t)^{-1} \end{aligned}$$

$$\therefore \qquad L'_Z(t) = -p\lambda(p\lambda+t)^{-2}$$

The hazard function of $z_N(t)$ is

$$h(t) = \frac{p\lambda(p\lambda+t)^{-2}}{p\lambda(p\lambda+t)^{-1}} = \frac{1}{(p\lambda+t)}$$

### 6.6.2 The case with gamma iid random variables

The $\mathbf{Y_i'}$s are distributed as gamma with parameters $\kappa$ and $\lambda$ and therefore The Laplace transform of $Y$ is

$$L_Y(t) = (\frac{\lambda}{\lambda+t})^{\kappa}$$

$$\therefore \qquad L'_Y(t) = \frac{-\kappa\,\lambda^{\kappa}}{(\lambda+t)^{\kappa+1}}$$

The hazard function of $z_N(t)$ is

$$h(t) = \frac{q\,\frac{\kappa\,\lambda^{\kappa}}{(\lambda+t)^{\kappa+1}}}{\left(1-q\,\frac{\lambda^{\kappa}}{(\lambda+t)^{\kappa}}\right)} + \frac{\frac{\kappa\,\lambda^{\kappa}}{(\lambda+t)^{\kappa+1}}}{\frac{\lambda^{\kappa}}{(\lambda+t)^{\kappa}}}$$
$$= \frac{q\,\kappa\,\lambda^{\kappa}}{(\lambda+t)\;((\lambda+t)^{\kappa}-q\,\lambda^{\kappa})} + \frac{\kappa\,\lambda^{\kappa}}{(\lambda+t)\;\lambda^{\kappa}}$$
$$= \frac{q\,\kappa\,\lambda^{\kappa}}{(\lambda+t)\;((\lambda+t)^{\kappa}-q\,\lambda^{\kappa})} + \frac{\kappa}{(\lambda+t)}$$

Alternatively

The Laplace transform of $z_N(t)$ is

$$L_Z(t) = \frac{p\,L_Y(t)}{1-q\,L_Y(t)}$$
$$= \frac{p\,(\frac{\lambda}{(\lambda+t)})^{\kappa}}{1-q\,(\frac{\lambda}{(\lambda+t)})^{\kappa}}$$
$$= \frac{p\lambda^{\kappa}}{(\lambda+t)^{\kappa}-q\lambda^{\kappa}}$$
$$= p\lambda^{\kappa}\,((p\lambda+t)^{\kappa}-q\lambda^{\kappa})^{-1}$$

$$\therefore \qquad L'_Z(t) = -p\kappa\lambda^{\kappa}\,((p\lambda+t)^{\kappa}-q\lambda^{\kappa})^{-2}\;(\lambda+t)^{\kappa-1}$$

The hazard function of $z_N(t)$ is

$$h(t) = \frac{p\kappa\lambda^{\kappa}\,((p\lambda+t)^{\kappa}-q\lambda^{\kappa})^{-2}\;(\lambda+t)^{\kappa-1}}{p\lambda^{\kappa}\,((p\lambda+t)^{\kappa}-q\lambda^{\kappa})^{-1}}$$

$$= \frac{\kappa(p\lambda+t)^{\kappa}}{((p\lambda+t)^{\kappa}-q\lambda^{\kappa})}\frac{1}{\lambda+t}$$
$$= \frac{q\,\kappa\,\lambda^{\kappa}}{(\lambda+t)\ ((\lambda+t)^{\kappa}-q\,\lambda^{\kappa})}+\frac{\kappa}{(\lambda+t)}$$

using partial fractions.

The compound shifted geometric distribution is applicable to aggregate claims by a group of independent insureds, with $N$ the number of claims generated by a portfolio of insurance policies in a fixed time period following the shifted geometric distribution. The variable $Y_i$ represents the $i^{th}$ claim and $z_N$ represents the aggregate claims over the fixed policy period.

## 6.7 Compound Negative Binomial Distribution

If **N** is a negative binomial with parameters $\alpha$ and **p** and if it is independent of $\mathbf{Y_i's}$ then its probability generating function is given by

$$F_N(t) = (\frac{p}{1-qt})^{\alpha}, \qquad p+q=1$$

The Laplace transform of $z_N(t)$ is

$$L_Z(t) = (\frac{p}{1-q\,L_Y(t)})^{\alpha}$$
$$\therefore \qquad L'_Z(t) = \alpha\,p^{\alpha}\,q\,L'_Y(t))\ (1-q\,L_Y(t))^{-\alpha-1}$$

The hazard function of $z_N(t)$ is

$$h(t) = \frac{-L'_Z(t)}{L_Z(t)}$$
$$= \frac{-\alpha\,p^{\alpha}\,q\,L'_Y(t))\ (1-q\,L_Y(t))^{-\alpha-1}}{p^{\alpha}\ (1-q\,L_Y(t))^{-\alpha}}$$
$$= \frac{-\alpha\,q\,L'_Y(t))}{(1-q\,L_Y(t))}$$

### 6.7.1 The case with exponential iid random variables

In $Y_i$ is independent and identically distributed as exponential ($Exp(\lambda)$), with parameter $\lambda$, the Laplace transform is

$$L_Y(t) = \frac{\lambda}{\lambda+t}$$
$$\therefore \qquad L'_Y(t) = -\frac{\lambda}{(\lambda+t)^2}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned} h(t) &= \frac{\alpha\, q\, \frac{\lambda}{(\lambda+t)^2}}{\left(1-q\,\frac{\lambda}{\lambda+t}\right)} \\ &= \frac{\alpha\, q\, \lambda}{(p\lambda+t)}\,\frac{1}{(\lambda+t)} \\ &= \frac{\alpha}{t+p\lambda} - \frac{\alpha}{\lambda+t} \end{aligned}$$

using partial fractions.

Alternatively

The Laplace transform of $z_N(t)$ is

$$\begin{aligned} L_Z(t) &= \left(\frac{p}{1-q\,\frac{\lambda}{\lambda+t}}\right)^{\alpha} \\ &= p^{\alpha}\,\left(1-q\,\frac{\lambda}{\lambda+t}\right)^{-\alpha} \\ &= p^{\alpha}\,\left(\frac{\lambda+t-q\lambda}{\lambda+t}\right)^{-\alpha} \\ &= p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha} \\ L_Z'(t) &= -\alpha\, p^{\alpha}\,(t+p\lambda)^{-\alpha-1}\,(\lambda+t)^{\alpha} + \alpha p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha-1} \end{aligned}$$

The hazard function of $z_n(t)$ is

$$\begin{aligned} h(t) &= \frac{\alpha\, p^{\alpha}\,(t+p\lambda)^{-\alpha-1}\,(\lambda+t)^{\alpha} - \alpha p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha-1}}{p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha}} \\ &= \frac{\alpha\, p^{\alpha}\,(t+p\lambda)^{-\alpha-1}\,(\lambda+t)^{\alpha}}{p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha}} - \frac{\alpha p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha-1}}{p^{\alpha}\,(t+p\lambda)^{-\alpha}\,(\lambda+t)^{\alpha}} \\ &= \frac{\alpha}{t+p\lambda} - \frac{\alpha}{\lambda+t} \end{aligned}$$

### 6.7.2 The case with gamma iid random variables

The $\mathbf{Y_i'}$**s** are distributed as gamma with parameters $\kappa$ and $\lambda$ and therefore
The Laplace transform of $Y$ is

$$\begin{aligned} L_Y(t) &= \left(\frac{\lambda}{\lambda+t}\right)^{\kappa} \\ \therefore \qquad L_Y'(t) &= \frac{-\kappa\,\lambda^{\kappa}}{(\lambda+t)^{\kappa+1}} \end{aligned}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned}
h(t) &= \frac{-\alpha\, q\, L'_Y(t))}{(1 - q\, L_Y(t))} \\
&= \frac{\alpha\, q\, \frac{\kappa\, \lambda^\kappa}{(\lambda+t)^{\kappa+1}}}{\left(1 - q\, (\frac{\lambda}{\lambda+t})^\kappa\right)} \\
&= \frac{\alpha\, q\, \kappa\, \lambda^\kappa}{((\lambda+t)^\kappa - q\lambda^\kappa)}\, \frac{1}{(\lambda+t)} \\
&= \frac{\alpha\kappa(\lambda+t)^{\kappa-1}}{((\lambda+t)^\kappa - q\lambda^\kappa)} - \frac{\alpha\kappa}{(\lambda+t)}
\end{aligned}$$

using partial fractions.

Alternatively

The Laplace transform of $z_N(t)$ is

$$\begin{aligned}
L_Z(t) &= (\frac{p}{1 - q\, (\frac{\lambda}{\lambda+t})^\kappa})^\alpha \\
&= p^\alpha\, (1 - q\, (\frac{\lambda}{\lambda+t})^\kappa)^{-\alpha} \\
&= p^\alpha\, (\frac{(\lambda+t)^\kappa - q\lambda^\kappa}{(\lambda+t})^{-\alpha} \\
&= p^\alpha\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha}\, (\lambda+t)^{\alpha\kappa} \\
L'_Z(t) &= -\alpha\kappa p^\alpha\, (\lambda+t)^{\kappa-1}\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha-1}\, (\lambda+t)^{\alpha\kappa} + \\
&\quad \alpha\kappa p^\alpha\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha}\, (\lambda+t)^{\alpha\kappa-1}
\end{aligned}$$

The hazard function of $z_N(t)$ is

$$\begin{aligned}
h(t) &= \frac{\alpha\kappa p^\alpha\, (\lambda+t)^{\kappa-1}\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha-1}\, (\lambda+t)^{\alpha\kappa} - \alpha\kappa p^\alpha\, ((\lambda+t)^\kappa - q\lambda}{p^\alpha\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha}\, (\lambda+t)^{\alpha\kappa}} \\
&= \frac{\alpha\kappa p^\alpha\, (\lambda+t)^{\kappa-1}\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha-1}\, (\lambda+t)^{\alpha\kappa}}{p^\alpha\, ((\lambda+t)^\kappa - q\lambda^\kappa)^{-\alpha}\, (\lambda+t)^{\alpha\kappa}} - \frac{\alpha\kappa p^\alpha\, ((\lambda+t)^\kappa - q\lambda}{p^\alpha\, ((\lambda+t)^\kappa - q\lambda} \\
&= \frac{\alpha\kappa(\lambda+t)^{\kappa-1}}{((\lambda+t)^\kappa - q\lambda^\kappa)} - \frac{\alpha\kappa}{(\lambda+t)}
\end{aligned}$$

**Remark 6.2.** *The hazard function of the compound negative binomial distribution is also a difference between Hofmann hazard functions and therefore it is not a convolution.*

Even though the compound Poisson distribution has many attractive properties, it is not a good model when the variance of the number of claims is greater than the mean of the number of claims. In such situations, the compound negative binomial distribution may be a better fit.

In this case, the compound negative binomial distribution is applicable to aggregate claims by a group of independent insureds and $z_N = Y_1 + Y_2 + \cdots + Y_N$ represents the total aggregate claims generated by a portfolio of policies in the given fixed time period. $N$, which is considered to be the number of claims generated by a portfolio of insurance policies in a fixed time period, follows the negative binomial distribution and $Y_1$ is the amount of the first claim, $Y_2$ is the amount of the second claim and so on.

## 6.8 Compound Shifted Negative Binomial Distribution

In this Parameterization **N** is a shifted negative binomial with parameters $\alpha$ and **p**. Thus

$$F_N(t) = (\frac{pt}{1-qt})^{\alpha}$$

Therefore,

$$L_Z(t) = (\frac{p\, L_Y(t)}{1-q\, L_Y(t)})^{\alpha}$$

### 6.8.1 The case with exponential iid random variables

When $Y_i$ is exponential with parameter $\lambda$, then

$$\begin{aligned} L_Z(t) &= (\frac{p\,\frac{\lambda}{\lambda+t}}{1-q\,\frac{\lambda}{\lambda+t}})^{\alpha} \\ &= \left[\frac{p\lambda}{\lambda+t-q\lambda}\right]^{\alpha} \\ &= \left[\frac{p\lambda}{p\lambda+t}\right]^{\alpha} \\ &= (p\lambda)^{\alpha}\,(p\lambda+t)^{-\alpha} \\ L'_Z(t) &= (p\lambda)^{\alpha}\,(p\lambda+t)^{-\alpha-1}\,(-\alpha) \end{aligned}$$

Therefore,

$$\begin{aligned} h_z(t) &= \frac{\alpha\,(p\lambda)^{\alpha}\,(p\lambda+t)^{-\alpha-1}}{p\lambda)^{\alpha}\,(p\lambda+t)^{-\alpha}} \\ &= \frac{\alpha}{p\lambda+t} \\ &= \frac{\frac{\alpha}{p\lambda}}{1+\frac{t}{p\lambda}} \end{aligned}$$

which is a hazard function of a Pareto distribution with parameters..

The associated mixed Poisson distribution is a negative binomial distribution with parameters..

### 6.8.2 The case with gamma iid random variables

When $\mathbf{Y_i's}$ is gamma with parameters $\kappa$ and $\lambda$ we have

$$\begin{aligned}
L_Z(t) &= \left[\frac{p\,(\frac{\lambda}{\lambda+t})^{\kappa}}{1-q\,(\frac{\lambda}{\lambda+t})^{\kappa}}\right]^{\alpha} \\
&= \left[\frac{p\lambda^{\kappa}}{(\lambda+t)^{\kappa}-q\,\lambda^{\kappa}}\right]^{\alpha} \\
L'_Z(t) &= (p\,\lambda^{\kappa})^{\alpha}\,\frac{d[(\lambda+t)^{\kappa}-q\,\lambda^{\kappa}]^{-\alpha}}{dt} \\
&= (p\,\lambda^{\kappa})^{\alpha}\,(-\alpha)\,[(\lambda+t)^{\kappa}-q\,\lambda^{\kappa}]^{-\alpha-1}\,\kappa\,(\lambda+t)^{\kappa-1} \\
h_z(t) &= \frac{\kappa\,\alpha\,(p\,\lambda^{\kappa})^{\alpha}\,[(\lambda+t)^{\kappa}-q\,\lambda^{\kappa}]^{-\alpha-1}\,(\lambda+t)^{\kappa-1}}{\left[\frac{p\lambda^{\kappa}}{(\lambda+t)^{\kappa}-q\,\lambda^{\kappa}}\right]^{\alpha}} \\
&= \frac{\kappa\,\alpha\,(\lambda+t)^{\kappa-1}}{(\lambda+t)^{\kappa}-q\,\lambda^{\kappa}} \\
&= \frac{\kappa\alpha}{\lambda+t}\times\left\{\frac{1}{1-q\left(\frac{\lambda}{\lambda+t}\right)^{\kappa}}\right\}
\end{aligned}$$

The compound shifted negative binomial distribution is applicable to aggregate claims by a group of independent insureds, with $N$ the number of claims generated by a portfolio of insurance policies in a fixed time period following the shifted negative binomial distribution. The variable $Y_i$ represents the $i^{th}$ claim and $z_N$ represents the aggregate claims over the fixed policy period.

## 6.9 Compound Poisson distribution with continuous iid variables

If $\mathbf{N}$ is a Poisson with parameter $\rho$ and if it is independent of $\mathbf{Y_i's}$, then

$$F_N(t) = e^{-\rho(1-t)}$$

Therefore,

$$\begin{aligned}
L_Z(t) &= e^{-\rho(1-L_Y(t))} \\
\therefore \qquad L'_Z(t) &= \rho\,L'_Y(t)\,e^{-\rho(1-L_Y(t))} \\
h_z(t) &= \frac{-L'_Z(t)}{L_Z(t)} \\
&= \frac{-\rho\,L'_Y(t)\,e^{-\rho(1-L_Y(t))}}{e^{-\rho(1-L_Y(t))}} \\
&= -\rho\,L'_Y(t)
\end{aligned}$$

### 6.9.1 Compound Poisson distribution with exponential iid random variables

When $\boldsymbol{Y_i}$ is exponential with parameter $\boldsymbol{\lambda}$, then

$$L_Y(t) = \frac{\lambda}{\lambda + t}$$

and,

$$\begin{aligned} L_Y'(t) &= -\frac{\lambda}{(\lambda+t)^2} \\ h_Y(t) &= \frac{\rho\,\lambda}{(\lambda+t)^2} \\ &= \frac{\rho\,\lambda}{\lambda^2\,(1+\frac{t}{\lambda})^2} \end{aligned}$$

which is Hofmann hazard function with $p = \frac{\rho}{\lambda}$, $c = \frac{1}{\lambda}$ and $a = 2$

It has been shown in chapter 4 that when the Hofmann hazard function is of the form

$$h(t) = \frac{p}{(1+ct)^2}, \qquad p > 0,\ c > 0$$

then we have the following results associated with mixed Poisson distribution.

The coefficients of $G(s,t)$ were obtained as

$$g_x(t) = \frac{1}{1+ct}\,(\frac{ct}{1+ct})^{x-1}$$

and,

$$\begin{aligned} g_0(t) &= 0 \qquad for \qquad x = 1,2,3,\ldots \\ p_n(t) &= (\frac{ct}{1+ct})^n\, p_0(t)\, \sum_{j=1}^{n} \binom{n-1}{j-1}\, (\frac{pt}{ct\,(1+ct)})^j\, \frac{1}{j!} \end{aligned}$$

which is the Polya-Aeppli distribution and $p_0(t) = e^{-\frac{pt}{1+ct}}$

### 6.9.2 Compound Poisson distribution with gamma iid random variables

When $\mathbf{Y_i's}$ is gamma with parameters $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$, then

$$L_Y(t) = (\frac{\lambda}{\lambda+t})^{\kappa}$$

$$L'_Y(t) = \frac{-\kappa\,\lambda^{\kappa}}{(\lambda+t)^{\kappa+1}}$$

and,

$$\begin{aligned} h_Z(t) &= -\rho\, L'_Y(t) \\ &= \frac{\rho\,\kappa\,\lambda^{\kappa}}{(\lambda+t)^{\kappa+1}} \end{aligned}$$

Alternatively

$$\begin{aligned} Ł_Z(t) &= F_N(L_Y(t)) \\ &= e^{-\rho(1-L_{Y_i(t)})} \\ &= e^{-\rho(1-(\frac{\lambda}{\lambda+t})^{\kappa})} \\ L'_Z(t) &= -\rho\lambda^{\kappa}\kappa\,(\lambda+t)^{-\kappa-1}\,e^{-\rho(1-(\frac{\lambda}{\lambda+t})^{\kappa})} \end{aligned}$$

and,

$$\begin{aligned} h_Z(t) &= \frac{-L'_Y(t)}{L_Y(t)} \\ &= \frac{\rho\lambda^{\kappa}\kappa\,(\lambda+t)^{-\kappa-1}\,e^{-\rho(1-(\frac{\lambda}{\lambda+t})^{\kappa})}}{e^{-\rho(1-(\frac{\lambda}{\lambda+t})^{\kappa})}} \\ &= \rho\lambda^{\kappa}\kappa\,(\lambda+t)^{-\kappa-1} \\ &= \frac{\rho\lambda^{\kappa}\kappa}{(\lambda+t)^{\kappa+1}} \\ &= \frac{\kappa\,p}{\lambda}\,\frac{1}{(1+\frac{t}{\lambda})^{\kappa+1}} \end{aligned}$$

Let,

$$p = \frac{\rho\,\kappa}{\lambda} \qquad c = \frac{1}{\lambda} \qquad a = \kappa+1$$

Therefore,

$$h(t) = \frac{p}{(1+ct)^{r+1}} \qquad p>0, \quad c>0 \quad and \quad r \geqslant -1$$

which is the parameterized Hofmann hazard function

The associated mixed Poisson distribution has been obtained in chapter 4

**Remark 6.3.** *The hazard functions of compound distributions when $\boldsymbol{Y_i s}$ are distributed as exponential are special Parameterizations of hazard function of the same compound distributions when $\boldsymbol{Y_i s}$ are distributed as gamma.*

**Exponential-Hougaard Distribution**

By re-parameterizing (6.9.2) using

$$\kappa = -\alpha, \qquad \lambda = \beta \qquad \rho = \frac{-\delta}{\alpha} \qquad for \quad \alpha \leq 1, \, \delta > 0 \quad and \quad \theta \geq 0$$

the results obtained by Hougaard (1986) and stated by Hesselager et. al. (1998) are obtained

$$\begin{aligned}
L_Z(t) &= e^{-\frac{\delta}{\alpha}\,((\beta+t)^{\alpha}-\beta^{\alpha})} \qquad for \qquad \alpha \in (0,1) \\
L'_Z(t) &= -\delta\,(\beta+t)^{\alpha-1}\, e^{-\frac{\delta}{\alpha}\,((\beta+t)^{\alpha}-\beta^{\alpha})} \\
h(t) &= \frac{\delta}{(\beta+t)^{1-\alpha}} \\
&= \frac{\delta}{\beta^{1-\alpha}\,(1+\frac{1}{\beta}t)^{1-\alpha}}
\end{aligned}$$

**Remark 6.4.** .
*1. The exponential-Hougaard distribution is a compound Poisson distribution whose iid random variables are gamma distributed.*
*2. Hesselager et. al. (1998) have stated the mixing distribution*

$$\mathrm{g}(\theta) = -\mathrm{e}^{\left(-\lambda\,\theta+\mu\,\frac{\lambda^{\alpha}}{\alpha}\right)}\,\frac{1}{\pi\,\theta}\sum_{\mathrm{k}=1}^{\infty}\frac{\Gamma(\mathrm{k}\,\alpha+1)}{\mathrm{k}!}\,(-\mu\,\frac{\theta^{-\alpha}}{\alpha})^{\mathrm{k}}\,\sin(\mathrm{k}\,\alpha\,\pi)$$

*as given by Hougaard (1986)*
*3. The formula for the mixing distribution derived by Hougaard (1986) is complicated. However, knowledge of the Laplace transform of the mixing distribution is sufficient to obtain the associated mixed Poisson distribution.*
*4. The exponential-Hougaard is a parameterization of Hofmann hazard function with parameters*

$$p = \frac{\delta}{\beta^{1-\alpha}} \qquad\qquad c = \frac{1}{\beta} \qquad\qquad a = 1-\alpha$$

### 6.9.3 Associated Mixed Poisson Distribution

We now obtain the associated mixed Poisson distribution using Hougaard's parameters.
The cumulative hazard function is

$$\begin{aligned}
\theta(t) &= \int_0^t \delta\,(\beta+x)^{\alpha-1}\,dx \\
&= \frac{\delta}{\alpha}\,(\beta+x)^{\alpha}\big|_{x=0}^{t}
\end{aligned}$$

$$\theta(t) = \frac{\delta}{\alpha}\ ((\beta + t)^{\alpha} - \beta^{\alpha})$$

$$\theta\ (t - ts) = \frac{\delta}{\alpha}\ ((\beta + t - ts)^{\alpha} - \beta^{\alpha})$$

The pgf of the mixed Poisson distribution is

$$\begin{aligned} H(s,t) &= e^{-\theta\ (t-ts)} = e^{-\frac{\delta}{\alpha}\ ((\beta+t-ts)^{\alpha}-\beta^{\alpha})} \\ &= e^{-\frac{\delta}{\alpha}\ ((\beta-t(s-1))^{\alpha}-\beta^{\alpha})} \\ &= e^{-\frac{\delta\,\beta^{\alpha}}{\alpha}\ \left((1-\frac{t}{\beta}(s-1))^{\alpha}-1\right)} \\ H(s,t) &= e^{-\frac{\delta\,\beta^{\alpha}}{\alpha}\ \left((1-\frac{t}{\beta}(s-1))^{\alpha}-1\right)} \end{aligned}$$

The pgf of the iid random variables is

$$\begin{aligned} G(s,t) &= 1 - \frac{\theta(t-ts)}{\theta(t))} \\ &= 1 - \frac{((\beta + t - ts)^{\alpha} - \beta^{\alpha})}{((\beta + t)^{\alpha} - \beta^{\alpha})} \end{aligned}$$

By power series expansion

$$\begin{aligned} G(s,t) &= \frac{((\beta + t)^{\alpha} - \beta^{\alpha}) - ((\beta + t - ts)^{\alpha} - \beta^{\alpha})}{((\beta + t)^{\alpha} - \beta^{\alpha})} \\ &= \frac{(\beta + t)^{\alpha} - (\beta + t - ts)^{\alpha}}{((\beta + t)^{\alpha} - \beta^{\alpha})} = \frac{(\beta + t)^{\alpha} - (\beta + t)^{\alpha}\ (1 - \frac{ts}{\beta+t})^{\alpha}}{(\beta + t)^{\alpha} - \beta^{\alpha}} \\ &= \frac{1 - (1 - \frac{ts}{\beta+t})^{\alpha}}{1 - (\frac{\beta}{\beta+t})^{\alpha}} \\ &= \frac{1 - \sum\limits_{x=0}^{\infty} \binom{\alpha}{x}\ (-\frac{ts}{\beta+t})^{x}}{1 - (\frac{\beta}{\beta+t})^{\alpha}} \\ &= \frac{-\sum\limits_{x=1}^{\infty} \binom{-\alpha + x - 1}{x}\ (\frac{t\,s}{\beta+t})^{x}}{1 - (\frac{\beta}{\beta+t})^{\alpha}} \\ &= \sum_{x=1}^{\infty} \frac{\binom{-\alpha + x - 1}{x}\ (\frac{t}{\beta+t})^{x}}{(\frac{\beta}{\beta+t})^{\alpha} - 1}\ s^{x} \\ g_x(t) &= \frac{\binom{-\alpha + x - 1}{x}\ (\frac{t}{\beta+t})^{x}}{(\frac{\beta}{\beta+t})^{\alpha} - 1} \end{aligned}$$

which is a zero-truncated negative binomial distribution with parameter $\alpha$; $\quad 0 < \alpha \leqslant 1$

Using the the differentiation method

$$G(s,t) = 1 - \frac{\theta(t-ts}{\theta(t)} = 1 - \frac{(\beta+t-ts)^\alpha - \beta^\alpha}{((\beta+t)^\alpha - \beta^\alpha)}$$

$$G'(s,t) = -\frac{1}{(\beta+t)^\alpha - \beta^\alpha}\Big(\alpha\,(\beta+t-t\,s)^{\alpha-1}\,.\,(-t)\Big)$$

$$G''(s,t) = -\frac{1}{(\beta+t)^\alpha - \beta^\alpha}\Big(\alpha\,(\alpha\,-1)\,(\beta+t-t\,s)^{\alpha-2}\,.\,(-t)^2\Big)$$

$$G'''(s,t) = -\frac{1}{(\beta+t)^\alpha - \beta^\alpha}\Big(\alpha\,(\alpha-1)\,(\alpha-2)\,(\beta+t-t\,s)^{\alpha-3}\,.\,(-t)^3\Big)$$

$$= -\frac{3!}{(\beta+t)^\alpha - \beta^\alpha}\binom{\alpha}{3}\,(\beta+t-t\,s)^\alpha\,(\frac{-t}{(\beta+t-t\,s})^3$$

$$G^x(s,t) = -\frac{x!}{(\beta+t)^\alpha - \beta^\alpha}\binom{\alpha}{x}\,(\beta+t-t\,s)^\alpha\,(\frac{-t}{(\beta+t-t\,s})^x$$

$$g_x(t) = \frac{1}{x!}\,G^x(s,t)|_{s=0} = \frac{1}{x!}\,G^x(s,t)|_{s=0}$$

$$= \frac{(\beta+t)^\alpha}{(\beta+t)^\alpha - \beta^\alpha}\binom{-\alpha+x-1}{x}\,(\frac{t}{(\beta+t})^x$$

$$= \frac{1}{(\frac{\beta}{\beta+t})^\alpha - 1}\binom{-\alpha+x-1}{x}\,(\frac{t}{(\beta+t})^x$$

which is a zero-truncated negative binomial distribution with parameter $\mathbf{r} = -\alpha;\qquad 0 < \alpha \leqslant 1$

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{t}{\beta+t}\,(1-\frac{\alpha+1}{x});\qquad x = 2,3,\ldots$$

The recursive form of the compound Poisson distribution is:

$$(n+1)p_{n+1}(t) = \frac{pt}{(1+ct)^a}\sum_{i=0}^{n}\frac{\Gamma(1-\alpha+i)}{i!\,\Gamma(1-\alpha)}\,(\frac{ct}{1+ct})^i\,p_{n-i}(t)$$

$$= \frac{\delta t}{\beta^{1-\alpha}\,(1+\frac{t}{\beta})^{1-\alpha}}\sum_{i=0}^{n}\frac{\Gamma(1-\alpha+i)}{i!\,\Gamma(1-\alpha)}\,(\frac{t}{\beta+t})^i\,p_{n-i}(t)$$

$$(n+1)p_{n+1}(t) = \frac{\delta t}{(\beta+t)^{1-\alpha}}\sum_{i=0}^{n}\frac{\Gamma(1-\alpha+i)}{i!\,\Gamma(1-\alpha)}\,(\frac{t}{\beta+t})^i\,p_{n-i}(t)$$

For $n = 0$, we obtain $p_1(t)$ as follows:

$$p_1(t) = \frac{\delta t}{(\beta+t)^{1-\alpha}}\,p_0(t)$$

For $\mathbf{n} = 1$, we obtain $\mathbf{p_2(t)}$ as follows:

$$2p_2(t) = \frac{\delta t}{(\beta+t)^{1-\alpha}}\sum_{i=0}^{1}\frac{\Gamma(1-\alpha+i)}{i!\,\Gamma(1-\alpha)}\,(\frac{t}{\beta+t})^i\,p_{1-i}(t)$$

$$
\begin{aligned}
&= \frac{\delta t}{(\beta+t)^{1-\alpha}} \left( p_1(t) + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t}{\beta+t} \, p_0(t) \right) \\
&= \frac{\delta t}{(\beta+t)^{1-\alpha}} \left( \frac{\delta t}{(\beta+t)^{1-\alpha}} + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t}{\beta+t} \right) p_0(t) \\
&= \left( \frac{(\delta t)^2}{(\beta+t)^{2-2\alpha}} + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,\delta t}{(\beta+t)^{2-\alpha}} \right) p_0(t) \\
2p_2(t) &= \left( \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,\delta t}{(\beta+t)^{2-\alpha}} + \frac{(\delta t)^2}{(\beta+t)^{2-2\alpha}} \right) p_0(t) \\
p_2(t) &= \frac{p_0(t)}{2!} \sum_{i=1}^{2} C_{2,i}(\alpha)\, t^{2-i} \frac{(\delta t)^i}{(\beta+t)^{2-i\alpha}}
\end{aligned}
$$

where

$$
C_{2,1}(\alpha) = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}, \qquad C_{2,2} = 1
$$

For $\mathbf{n=2}$, we obtain $\mathbf{p_3(t)}$ as follows:

$$
\begin{aligned}
3p_3(t) &= \frac{\delta t}{(\beta+t)^{1-\alpha}} \sum_{i=0}^{2} \frac{\Gamma(1-\alpha+i)}{i!\,\Gamma(1-\alpha)} \left(\frac{t}{\beta+t}\right)^i p_{2-i}(t) \\
&= \frac{\delta t}{(\beta+t)^{1-\alpha}} [p_2(t) + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t}{\beta+t} \, p_1(t) + \frac{\Gamma(3-\alpha)}{2!\,\Gamma(1-\alpha)} \frac{t^2}{(\beta+t)^2} \, p_0(t) \\
&= \frac{\delta t}{(\beta+t)^{1-\alpha}} [[\frac{1}{2} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,\delta t}{(\beta+t)^{2-\alpha}} + \frac{1}{2} \frac{(\delta t)^2}{(\beta+t)^{2-2\alpha}}] + \\
&\quad \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t}{\beta+t} \frac{\delta t}{(\beta+t)^{1-\alpha}} + \frac{\Gamma(3-\alpha)}{2!\,\Gamma(1-\alpha)} \frac{t^2}{(\beta+t)^2} \, p_0(t)] \\
&= [\frac{1}{2} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,(\delta t)^2}{(\beta+t)^{3-2\alpha}} + \frac{1}{2} \frac{(\delta t)^3}{(\beta+t)^{3-3\alpha}} + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,(\delta t)^2}{(\beta+t)^{3-2\alpha}} + \\
&\quad \frac{\Gamma(3-\alpha)}{2!\,\Gamma(1-\alpha)} \frac{t^2\,\delta t}{(\beta+t)^{3-\alpha}}] \, p_0(t) \\
&= \frac{1}{2} [\frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)} \frac{t^2\,\delta t}{(\beta+t)^{3-\alpha}} + \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,(\delta t)^2}{(\beta+t)^{3-2\alpha}} + 2\,(1-\alpha) \frac{t\,(\delta t)^2}{(\beta+t)^{3-}} \\
&\quad \frac{(\delta t)^3}{(\beta+t)^{3-3\alpha}}] \, p_0(t) \\
&= \frac{1}{2} [C_{3,1} \frac{t^2\,\delta t}{(\beta+t)^{3-\alpha}} + (C_{2,1}(\alpha) + 2\,(1-\alpha)\, C_{2,2}(\alpha)) \frac{t\,(\delta t)^2}{(\beta+t)^{3-2\alpha}} + C_{3,3} \\
3p_3(t) &= \frac{1}{2} [C_{3,1} \frac{t^2\,\delta t}{(\beta+t)^{3-\alpha}} + C_{3,2}(\alpha) \frac{t\,(\delta t)^2}{(\beta+t)^{3-2\alpha}} + C_{3,3} \frac{(\delta t)^3}{(\beta+t)^{3-3\alpha}}] \, p_0(t) \\
p_3(t) &= \frac{p_0(t)}{3!} \sum_{i=1}^{3} C_{3,i}(\alpha)\, t^{3-i} \frac{(\delta t)^i}{(\beta+t)^{3-i\alpha}}
\end{aligned}
$$

where,

$$C_{3,i}(\alpha) = \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}$$
$$C_{3,3}(\alpha) = 1$$

For $\mathbf{n=3}$, we obtain $\mathbf{p_4(t)}$ as follows:

$$4p_4(t) = \frac{\delta t}{(\beta+t)^{1-\alpha}} \sum_{i=0}^{3} \frac{\Gamma(1-\alpha+i)}{i!\,\Gamma(1-\alpha)} \left(\frac{t}{(\beta+t)}\right)^i p_{3-i}(t)$$
$$= \frac{\delta t}{(\beta+t)^{1-\alpha}} [p_3(t) + \frac{\Gamma(2-\alpha)}{1!\,\Gamma(1-\alpha)} \left(\frac{t}{(\beta+t)}\right) p_2(t) +$$
$$\Gamma(3-\alpha)2!\,\Gamma(1-\alpha) \left(\frac{t}{(\beta+t)}\right)^2 p_1(t) + \frac{\Gamma(4-\alpha)}{3!\,\Gamma(1-\alpha)} \left(\frac{t}{(\beta+t)}\right)^3 p_0(t)]$$
$$= [\frac{1}{6} \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)} \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + \frac{1}{6} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} + \frac{1}{3}(1-\alpha) \frac{t\,(\delta}{(\beta+t}$$
$$\frac{1}{6} \frac{(\delta t)^4}{(\beta+t)^{4-4\alpha}} + \frac{1}{2} \left(\frac{\Gamma(2-\alpha)}{1!\,\Gamma(1-\alpha)}\right)^2 \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + \frac{1}{2}(1-\alpha) \frac{(t)^2\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} +$$
$$\frac{\Gamma(3-\alpha)}{2!\,\Gamma(1-\alpha)} \left(\frac{t\,(\delta t)^2}{(\beta+t)}\right)^{4-2\alpha} + \frac{\Gamma(4-\alpha)}{3!\,\Gamma(1-\alpha)} \left(\frac{t\,\delta t}{(\beta+t)}\right)^{4-\alpha}]\, p_0(t)$$
$$= [\frac{\Gamma(4-\alpha)}{3!\,\Gamma(1-\alpha)} \frac{t\,\delta t}{(\beta+t)^{4-\alpha}} +$$
$$\frac{1}{3!} \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)} \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + \frac{1}{3!}\, 2\,(1-\alpha) \frac{t\,(\delta t)^3}{(\beta+t)^{4-2\alpha}} +$$
$$\frac{1}{3!}\, 3 \left(\frac{\Gamma(2-\alpha)}{1!\,\Gamma(1-\alpha)}\right)^2 \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + \frac{1}{3!}\, 3\, \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)} \left(\frac{t\,(\delta t)^2}{(\beta+t)^{4-2\alpha}}\right.$$
$$\frac{1}{3!} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} + \frac{1}{3!}\, 3\,(1-\alpha) \frac{(t)^2\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} + \frac{1}{3!} \frac{(\delta t)^4}{(\beta+t)^{4-4\alpha}}]\, p_0$$
$$= \frac{1}{3!}\, p_0(t) [\frac{\Gamma(4-\alpha)}{\Gamma(1-\alpha)} \frac{t\,\delta t}{(\beta+t)^{4-\alpha}} + \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)} \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + 2\,(1-\alpha) \frac{t}{(\beta-}$$
$$3 \left(\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}\right)^2 \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + 3\, \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)} \left(\frac{t\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \frac{t\,(\delta t)^3}{(\beta+t)^{4-}}\right.$$
$$(1-\alpha) \frac{(t)^2\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} + \frac{(\delta t)^4}{(\beta+t)^{4-4\alpha}}]$$
$$= \frac{1}{3!}\, p_0(t) [C_{4,1} \frac{t\,\delta t}{(\beta+t)^{4-\alpha}} + C_{3,1} \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + 2\,(1-\alpha) \frac{t\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} +$$
$$3\,(1-\alpha)^2 \frac{t^2\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + 3\, C_{3,1} \left(\frac{t\,(\delta t)^2}{(\beta+t)^{4-2\alpha}} + (1-\alpha) \frac{t\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} +\right.$$
$$3\,(1-\alpha) \frac{(t)^2\,(\delta t)^3}{(\beta+t)^{4-3\alpha}} + \frac{(\delta t)^4}{(\beta+t)^{4-4\alpha}}]$$

$$= \frac{1}{3!}\, p_0(t)[C_{4,1}\ \frac{t\ \delta t}{(\beta+t)^{4-\alpha}} + [C_{3,1} + 2(1-\alpha) + (1-\alpha)^2]\ \frac{t^2\ (\delta t)^2}{(\beta+t)^{4-2\alpha}} +$$

$$3\ C_{3,1}\ (\frac{t\ (\delta t)^2}{(\beta+t)^{4-2\alpha}} + (1-\alpha)\ \frac{t\ (\delta t)^3}{(\beta+t)^{4-3\alpha}} +$$

$$3\ (1-\alpha)\ \frac{(t)^2\ (\delta t)^3}{(\beta+t)^{4-3\alpha}} + \frac{(\delta t)^4}{(\beta+t)^{4-4\alpha}}]$$

$$4p_4(t) = \frac{\delta t}{(\beta+t)^{1-\alpha}}\ \frac{p_0(t)}{3!} \sum_{i=1}^{3} C_{3,i}(\alpha)\ t^{3-i}\ \frac{(\delta t)^i}{(\beta+t)^{3-i\alpha}} +$$

$$\frac{\delta t}{(\beta+t)^{1-\alpha}}\ \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}\ \frac{t}{\beta+t}\ \frac{p_0(t)}{2!} \sum_{i=1}^{2} C_{2,i}(\alpha)\ t^{2-i}\ \frac{(\delta t)^i}{(\beta+t)^{2-i\alpha}} +$$

$$\frac{\delta t}{(\beta+t)^{1-\alpha}}\ \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}\ (\frac{t}{\beta+t})^2\ \frac{\delta t}{(\beta+t)^{1-i\alpha}}\ \frac{p_0(t)}{2!} +$$

$$\frac{\delta t}{(\beta+t)^{1-\alpha}}\ \frac{\Gamma(4-\alpha)}{\Gamma(1-\alpha)}\ (\frac{t}{\beta+t})^3\ \frac{p_0(t)}{3!}$$

$$p_4(t) = \frac{p_0(t)}{4!} \left( \sum_{i=1}^{4} C_{4,i}(\alpha)\ t^{4-i}\ \frac{(\delta t)^i}{(\beta+t)^{4-i\alpha}} \right)$$

$$\therefore \qquad p_n(t) = \frac{p_0(t)}{n!} \left( \sum_{i=1}^{n} C_{n,i}(\alpha)\ t^{n-i}\ \frac{(\delta t)^i}{(\beta+t)^{n-i\alpha}} \right)$$

where,

$$C_{4,1}(\alpha) = \frac{\Gamma(4-\alpha)}{\Gamma(1-\alpha)}$$
$$C_{n,1}(\alpha) = \frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)}$$
$$C_{4,2}(\alpha) = C_{3,1}(\alpha) + C_{3,2}(\alpha)\,(3-2\ \alpha)$$
$$C_{4,3}(\alpha) = C_{3,2}(\alpha) + C_{3,3}(\alpha)\ \ (3-3\alpha)$$
$$C_{n,i}(\alpha) = C_{n-1,i-1}(\alpha) + C_{n-1,i}(\alpha)\,((n-1) - i\ \alpha)$$
$$C_{n,n}(\alpha) = C_{4,4}(\alpha) = C_{3,3}(\alpha) = C_{2,2}(\alpha) = 1$$

as given by Hougaard et. al. (1997)

The compound Poisson distribution is applicable to aggregate claims by a group of independent insureds, in which case, $z_N = Y_1 + Y_2 + \cdots + Y_N$ represents the total aggregate claims generated by a portfolio of policies in the given fixed time period. $N$ is considered to be the number of claims generated by a portfolio of insurance policies in a fixed time period, such that, $Y_1$ is the amount of the first claim, $Y_2$ is the amount of the second claim and so on. For the model to be more tractable, the following assumptions are necessary:

1. $\boldsymbol{Y_i s}$ are independent and identically distributed.
2. Each $\boldsymbol{Y_i}$ is independent of the number of claims $\boldsymbol{N}$.

When the claim frequency $\boldsymbol{N}$ follows a Poisson distribution with a constant parameter $\boldsymbol{\lambda}$ , the aggregate claims $\boldsymbol{z_N}$ is said to have a compound Poisson distribution.

## 6.10 The Sum of Hazard Functions of Exponential-Hougaard Distribution with other Forms of Hofmann Hazard Function

The sum of two hazard functions of exponential mixtures can be in the form of Hofmann hazard functions; i.e,

$$\boldsymbol{h(t) = \frac{p_1}{(1+c_1 t)^{a_1}} + \frac{p_2}{(1+c_2 t)^{a_2}}}$$

**In the equation (5.23)** $\boldsymbol{a_1 = 1-\alpha}$ **and** $\boldsymbol{a_2 = \frac{1}{2}}$

$$\boldsymbol{h(t) = \theta'(t) = \frac{p_2}{(1+c_1 t)^{1-\alpha}} + \frac{p_2}{(1+c_2 t)^{\frac{1}{2}}} \quad for \quad p_1, p_2 > 0, \quad and \quad c > 0}$$

where the second hazard function is that of an exponential-inverse Gaussian distributions.

$$\boldsymbol{\therefore \qquad \theta_1(t) = \frac{p_1}{\alpha\, c_1}\ ((1+c_1 t)^{\alpha} - 1)}$$

$$\boldsymbol{\theta_1(t-ts) = \frac{p_1}{\alpha\, c_1}\ ((1+c_1 t - c_1 ts)^{\alpha} - 1)}$$

The pgf of the convolution is

$$\boldsymbol{\begin{aligned} H(s,t) &= e^{-\theta_1(t-ts)} e^{-\theta_2(t-ts)} \\ &= e^{-\frac{p_1}{\alpha\, c_1}\ ((1+c_1 t - c_1 ts)^{\alpha} - 1)}\ e^{-\frac{2p_2}{c_2}\ \left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right)} \end{aligned}}$$

Therefore the sum of hazard functions of exponential-Hougaard distribution and that of the exponential-inverse gamma distribution give rise to the convolution of the Poisson-Hougaard distribution and the Poisson-inverse gamma (Sichel) distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$\boldsymbol{G(s,t) = 1 - \frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\, c_1}\ [(1+c_1 t - c_1 ts)^{\alpha} - 1] + \frac{2p_2}{c_2}\ [(1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1]}$$

$$G'(s,t) = -\frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\alpha(-c_1t)(1+c_1t+c_1ts)^{\alpha-1}+\frac{p_2}{c_2}\ (-c_2t)^1[(1+c_2t-c_2ts)^{-}$$

$$G''(s,t) = -\frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\alpha(\alpha-1)(-c_1t)^2(1+c_1t+c_1ts)^{\alpha-2}+$$
$$(-\frac{1}{2})\ \frac{p_2}{c_2}\ (-c_2t)^2(1+c_2t-c_2ts)^{-\frac{3}{2}}]$$

$$G'''(s,t) = -\frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\alpha(\alpha-1)(\alpha-2)(-c_1t)^3(1+c_1t+c_1ts)^{\alpha-3}+$$
$$(-\frac{1}{2})(-\frac{3}{2})\ \frac{p_2}{c_2}\ (-c_2t)^3[(1+c_2t-c_2ts)^{-\frac{5}{2}}]$$

$$\therefore \qquad G^x(s,t) = \frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t+c_1ts})^x\ (1+c_1t+c_1ts)^{\alpha}+$$
$$\frac{[2.(x-1)-1]}{2}\ \frac{[2.(x-2)-1]}{2}\ \frac{[2.(x-3)-1]}{2}\ \cdots$$
$$\frac{[2.3-1]}{2}\ \frac{[2.2-1]}{2}\ \frac{[2.1-1]}{2}\ \frac{p_2}{c_2}\ (c_2t)^x[(1+c_2t-c_2ts)^{-\frac{(2.x-1)}{2}}]$$

$$= \frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t+c_1ts})^x\ (1+c_1t+c_1ts)^{\alpha}]+$$
$$[(x-1)-\frac{1}{2}]\ [(x-2)-\frac{1}{2}]\ [(x-3)-\frac{1}{2}]\ \cdots\cdot$$
$$[3-\frac{1}{2}]\ [2-\frac{1}{2}]\ [1-\frac{1}{2}]\ \frac{p_2}{c_2}\ (c_2t)^x[(1+c_2t-c_2ts)^{\frac{1}{2}-x}$$

$$= \frac{1}{\theta(t)}\ \frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t-c_1ts})^x\ (1+c_1t-c_1ts)^{\alpha}]+$$
$$\frac{(x-1)!}{\theta(t)}\ \binom{(x-1)-\frac{1}{2}}{x-1}\ \frac{p_2}{c_2}\ (c_2t-c_2ts)^x[(1+c_2t-c_2ts)^{\frac{1}{2}-x}$$

$$= \frac{1}{\theta(t)}\ \frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t-c_1ts})^x\ (1+c_1t-c_1ts)^{\alpha}+$$
$$\frac{(x-1)!}{\theta(t)}\ \frac{p_2}{c_2}\ \binom{\frac{1}{2}+x-1-1}{x-1}\ (\frac{c_2t}{(1+c_2t-c_2ts})^x\ (1+c_2t-c_2ts)^{\frac{1}{2}}$$

$$= \frac{(1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t-c_1ts})^x\ (1+c_1t-c_1ts)^{\alpha}]+$$
$$\frac{(x-1)!}{\theta(t)}\frac{p_2}{c_2}\binom{\frac{1}{2}+x-1-1}{x-1}(\frac{c_2t}{(1+c_2t-c_2ts})^{x-1}\ \frac{c_2t}{(1+c_2t-c_2ts}(1+c_2t-$$

The coefficients of $G(s,t)$ are

$$g_0(t) = 0$$

$$g_x(t) = \frac{1}{\theta(t)}\ \frac{p_1}{\alpha\ c_1}\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t})^x\ (1+c_1t)^{\alpha}+$$
$$\frac{1}{\theta(t)}\ \frac{1}{x}\ p_2t\ \binom{-\frac{1}{2}}{x-1}\ (-\frac{c_2t}{(1+c_2t})^{x-1}\ (1+c_2t)^{-\frac{1}{2}}$$

where,

$$\theta(t)=\frac{p_1}{\alpha\, c_1}\;((1+c_1t)^{\alpha}-1)+\frac{2p_2}{c_2}\;\left((1+c_2t)^{\frac{1}{2}}-1\right)$$

The recursive form for the convolution of compound Poisson distributions is

$$\begin{aligned}
n\,p_n(t) &= \theta(t)\;\sum_{x=1}^{n}\;x\,g_x(t)\;\;p_{n-x}(t)\\
&= \theta(t)\;\sum_{x=1}^{n}\;x\,[\frac{1}{\theta(t)}\,\frac{p_1}{\alpha\,c_1}\,\binom{\alpha}{x}\,(\frac{-c_1t}{1+c_1t})^{x}\,(1+c_1t)^{\alpha}+\\
&\quad \frac{1}{\theta(t)}\,\frac{1}{x}\,p_2t\,\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2t}{(1+c_2t})^{x-1}\,(1+c_2t)^{-\frac{1}{2}}]\,p_{n-x}(t)\\
&= \sum_{x=1}^{n}\;x\,\frac{p_1}{\alpha\,c_1}\,\binom{\alpha}{x}\,(\frac{-c_1t}{1+c_1t})^{x}\,(1+c_1t)^{\alpha}\,p_{n-x}(t)+\\
&\quad \sum_{x=1}^{n}\;p_2t\,\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2t}{(1+c_2t})^{x-1}\,(1+c_2t)^{-\frac{1}{2}}\,p_{n-x}(t)\\
&= \frac{p_1}{\alpha\,c_1}\,(1+c_1t)^{\alpha}\,\sum_{x=1}^{n}\;x\,\binom{\alpha}{x}\,(\frac{-c_1t}{1+c_1t})^{x}\,p_{n-x}(t)+\\
&\quad (1+c_2t)^{-\frac{1}{2}}\,p_2t\,\sum_{x=1}^{n}\,\binom{-\frac{1}{2}}{x-1}\,(-\frac{c_2t}{(1+c_2t})^{x-1}\,p_{n-x}(t)\qquad for\; n=1,2
\end{aligned}$$

### 6.10.1 In the equation (5.23) $a_1 = 1$ and $a_2 = 1-\alpha$

Let us consider a Parameterization when

$$a_1 = 1 \qquad\qquad a_2 = 1-\alpha \qquad\qquad c_1 = c_2 = c$$

Therefore the sum of two hazard functions of exponential mixtures in the form of Hofmann hazard functions is $\mathbf{h(t)=\theta'(t)}$ and therefore

$$h(t)=\frac{p_1}{(1+ct)}+\frac{p_2}{(1+ct)^{1-\alpha}}\quad for\quad p_1,p_2>0,\quad and\quad c>0$$

which is the sum of hazard function of Pareto and that of the exponential-Hougaard distribution.

This sum of hazard functions can be obtained by considering the hazard function of the Benktander type II distribution, which basically is an exponential mixture. The family of Benktander distributions is based on mean excess loss also known as Mean Residual Lifetime (MRL).

The Mean Residual Life is the expected additional lifetime given that a component has survived until time $\mathbf{z}$. More specifically, if the random variable $X$ represents the life of a component, then the Mean Residual Life is given by

$$m(z)=E(X-z|X>z)$$

$$Prob(X > z) = 1 - F(z)$$

$$\therefore \qquad 1 - F(z) = \int_z^\infty f(x)\, dx$$

$$and \qquad 1 = \int_z^\infty \frac{f(x)}{1 - F(z)}\, dx$$

The distribution of Mean Residual Lifetime $\mathbf{X - z}$ is therefore

$$g(z) = \frac{f(x)}{1 - F(z)} \qquad for \quad X > z$$

The Mean Residual Life is given by

$$\begin{aligned} m(z) &= E(X - z | X > z) \\ &= \int_z^\infty (X - z) \frac{f(x)}{1 - F(z)}\, dx \\ &= \frac{1}{1 - F(z)} \int_z^\infty (X - z)\, f(x)\, dx \end{aligned}$$

Integrating by parts,

Let

$$\begin{aligned} u &= X - z \qquad \Rightarrow \\ dv &= f(x)\, dx \\ &= -\frac{d\, S(x)}{dx} \\ &= -S'(x) \qquad \Rightarrow \end{aligned}$$

Therefore,

$$\int_z^\infty (X - z)\, f(x)\, dx = -(X - z)\, S(x)|_z^\infty + \int_z^\infty S(x)\, dx$$

$$= 0 + \int_z^\infty S(x)\, dx$$

$$m(z) = \frac{1}{1 - F(z)} \int_z^\infty (X - z)\, f(x)\, dx$$

$$= \int_z^\infty \frac{1 - F(x)}{1 - F(z)}\, dx$$

$$m'(z) = \frac{d}{dz}\, m(z) = \frac{d}{dz}\, [\int_z^\infty \frac{1 - F(x)}{1 - F(z)}\, dx]$$

$$= [\frac{d}{dz} \frac{1}{1-F(z)}] \int_z^\infty (1-F(x))\, dx +$$

$$\frac{1}{1-F(z)} \frac{d}{dz} \int_z^\infty (1-F(x))\, dx]$$

$$= \frac{F'(z)}{(1-F(z))^2} \int_z^\infty (1-F(x))\, dx + \frac{1}{1-F(z)} [(1-F(x)]|_z^\infty .1$$

$$= \frac{F'(z)}{1-F(z)} \int_z^\infty \frac{(1-F(x))}{(1-F(z))}\, dx + \frac{1}{1-F(z)} [0-(1-F(z)]$$

$$= \frac{F'(z)}{1-F(z)} m(z) - 1$$

$$m'(z) = h(z)\, m(z) - 1$$

Therefore,

$$h(z) = \frac{1+m'(z)}{m(z)}$$

For Benktander distribution type II,

$$m(z) = \frac{z^{1-b}}{a}, \qquad 0 \le b \le 1$$

$$m'(z) = \frac{1-b}{a} z^{-b}$$

$$h(z) = \frac{1+\frac{1-b}{a} z^{-b}}{\frac{z^{1-b}}{a}} = \frac{a\, z^b + (1-b)}{z}$$

$$h(z) = \frac{1-b}{z} + \frac{a}{z^{1-b}}$$

Using the notations of Hesselager et. al. (1998), let $a = \mu$, $b = \alpha$ and $z = x + \lambda$ so that $\frac{dz}{dx} = 1$

Therefore the Mean Residual Lifetime of the Benktander type II distribution is

$$m(x) = \frac{(x+\lambda)^{1-\alpha}}{\mu} \qquad \alpha \in (0,1)$$

and the hazard function is

$$h(x) = \frac{1-\alpha}{x+\lambda} + \frac{\mu}{(x+\lambda)^{1-\alpha}}$$

Let,

$$p_1 = \frac{1-\alpha}{\lambda}$$

$$p_2 = \frac{\mu}{\lambda^{1-\alpha}}$$
$$h(t) = \frac{p_1}{(1+ct)} + \frac{p_2}{(1+ct)^{1-\alpha}} \quad for \quad p_1, p_2 > 0, \quad and \quad c > 0$$

**Remark 6.5.** . *The hazard function of the Benktander type II distribution is the sum of the hazard functions of the exponential-gamma (Pareto) distribution and the exponential-Hougaard distribution.*

$$\begin{aligned} h(t) &= \theta'(t) \\ &= \theta_1'(t) + \theta_2'(t) \\ &= \frac{p_1}{(1+ct)} + \frac{p_2}{(1+ct)^{1-\alpha}} \\ \therefore \qquad \theta_1(t) &= \frac{p_1}{c_1}\ In(1+c_1 t) \\ \theta_1(t-ts) &= \frac{p_1}{c_1}\ In(1+c_1 t - c_1 ts) \end{aligned}$$

The pgf of the convolution is

$$\begin{aligned} H(s,t) &= e^{-\theta_1(t-ts)} e^{-\theta_2(t-ts)} \\ &= e^{-\frac{p_1}{c_1}\ In(1+c_1 t - c_1 ts)}\ e^{-\frac{p_2}{\alpha\ c_2}\ [(1+c_2 t - c_2 ts)^{\alpha} - 1]} \end{aligned}$$

Therefore the sum of the hazard function of the exponential-gamma (Pareto) distribution and that of the hazard function of the exponential-Hougaard distribution give rise to the convolution of the negative binomial distribution and the Poisson-Hougaard distribution

.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$\begin{aligned} G(s,t) &= 1 - \frac{\theta_1(t-ts) + \theta_2(t-ts)}{\theta(t)} \\ \therefore \qquad G(s,t) &= 1 - \frac{\frac{p_1}{c_1}\ In(1+c_1 t - c_1 ts) + \frac{p_2}{\alpha\ c_2}\ [(1+c_2 t - c_2 ts)^{\alpha} - 1]}{\theta(t)} \\ G^{'}(s,t) &= -\frac{1}{\theta(t)}\ [\frac{p_1}{c_1}\ (-c_1 t)\ (1+c_1 t + c_1 ts)^{-1} + \frac{p_2}{\alpha\ c_2} \alpha(-c_2 t)(1+c_2 t + c_2 t \\ G^{''}(s,t) &= -\frac{1}{\theta(t)}\ [\frac{p_1}{c_1}\ (-1)(-c_1 t)^2\ (1+c_1 t + c_1 ts)^{-2} + \\ &\quad \frac{p_2}{\alpha\ c_2} \alpha(\alpha-1)(-c_2 t)^2 (1+c_2 t + c_2 ts)^{\alpha-2}] \\ G^{'''}(s,t) &= -\frac{1}{\theta(t)}\ [\frac{p_1}{c_1}\ (-1)(-2)(-c_1 t)^3\ (1+c_1 t + c_1 ts)^{-3} + \end{aligned}$$

$$\frac{p_2}{\alpha\ c_2}\alpha(\alpha-1)(\alpha-2)(-c_2t)^3(1+c_2t+c_2ts)^{\alpha-3}]$$

$$G^x(s,t) = \frac{1}{\theta(t)}\ [\frac{p_1}{c_1}\ (x-1)!(\frac{c_1t}{1+c_1t+c_1ts})^x - \frac{p_2}{\alpha\ c_2}\ x!\ \binom{\alpha}{x}\ (\frac{-c_2t}{1+c_2t+c_2ts})^x\ (1+c_2t+c_2ts)^\alpha]$$

The coefficients of $G(s,t)$ are

$$g_0 = 0$$

$$g_x(t) = \frac{1}{x!}G(s,t)|_{s=0}$$

$$= \frac{1}{\theta(t)}\ [\frac{p_1}{c_1}\ \frac{1}{x}\ (\frac{c_1t}{1+c_1t})^x - \frac{p_2}{\alpha\ c_2}\ \binom{\alpha}{x}\ (\frac{-c_2t}{1+c_2t})^x\ (1+c_2t+c_2ts)^\alpha]$$

The recursive form for the convolution of compound Poisson distributions is:

$$n\ p_n(t) = \theta(t)\ \sum_{x=1}^{n}\ x\ g_x(t)\ \ p_{n-x}(t)$$

$$= \sum_{x=1}^{n}\ x\ [\frac{p_1}{c_1}\ \frac{1}{x}\ (\frac{c_1t}{1+c_1t})^x - \frac{p_2}{\alpha\ c_2}\ \binom{\alpha}{x}\ (\frac{-c_2t}{1+c_2t})^x\ (1+c_2t+c_2ts)^\alpha]$$

$$= \frac{p_1}{c_1}\ \sum_{x=1}^{n}(\frac{c_1t}{1+c_1t})^x - \frac{p_2}{\alpha\ c_2}\ (1+c_2t+c_2ts)^\alpha\ \sum_{x=1}^{n}\ x\ \binom{\alpha}{x}\ (\frac{-c_2t}{1+c_2t})^x$$

$$and \qquad p_0(t) = e^{-\frac{p_1}{c_1}\ In(1+c_1t)+\frac{p_2}{\alpha\ c_2}\ (1+c_2t)^\alpha-1]}$$

**Non-central chi-squared distribution**

A special Parameterization in this category is when

$$1-\alpha = 2 \quad \Rightarrow \quad \alpha = -1$$

Therefore,

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1t)} + \frac{p_2}{(1+c_2t)^2} \quad \text{for} \quad p > 0, \quad \text{and} \quad c > 0$$

which is the sum of the hazard function of exponential-gamma and the Polya-Aeppli hazard function.

However, this can also be obtained using the Laplace transform of the non-central chi-squared distribution as follows:

The random variable **Y** is said to have a non-central chi-squared distribution with $n$ degrees of freedom and non-central parameter $\theta = \sum_{i=1}^{n}\ \mu_i^2$ if

$$Y = X_1^2 + X_2^2 + \ldots.. + X_n^2$$

where $\mathbf{X_i's}$ are independent variables but not identical and

$$X_i's \sim N(\mu_i, 1) \qquad for \quad i = 1, 2, \ldots, n.$$

**Derivation of the moment generating function (mgf) of Y**

If $\mathbf{X \sim N(\mu, 1)}$, then the mgf of $X^2$ is given by

$$M_{X^2}(t) = E[e^{tX^2}]$$

$$= \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[x^2-2\mu x+\mu^2]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[(t-\frac{1}{2})x^2+\mu x-\frac{1}{2}\mu^2]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[\frac{2t-1}{2}\, x^2+\mu x-\frac{1}{2}\mu^2]} dx$$

$$= e^{-\frac{\mu^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[\frac{2t-1}{2}\, x^2+\mu x]} dx$$

$$= e^{-\frac{\mu^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[-\frac{1}{2}\, (1-2t)\, x^2+\mu x]} dx$$

$$= e^{-\frac{\mu^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[-\frac{1}{2}\, [(2t-1)x^2-2\mu x]} dx$$

$$= e^{-\frac{\mu^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[-\frac{(2t-1)}{2}\, [x^2-\frac{2\mu}{1-2t}x]} dx$$

$$= e^{-\frac{\mu^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[-\frac{(2t-1)}{2}\, [(x-\frac{\mu}{1-2t})^2-(\frac{\mu}{1-2t})^2]} dx$$

$$= e^{[-\frac{\mu^2}{2}+\frac{(2t-1)}{2}\,\frac{\mu^2}{(1-2t)^2}]} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[-\frac{(2t-1)}{2}\, [(x-\frac{\mu}{1-2t})^2]} dx$$

$$= e^{[-\frac{\mu^2}{2}+\frac{\mu^2}{2\,(1-2t)}]} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{[-\frac{1}{2}\,[\frac{(x-\frac{\mu}{1-2t})^2}{\frac{1}{1-2t}}]} dx$$

$$= e^{[-\frac{\mu^2}{2}+\frac{\mu^2}{2\,(1-2t)}]} \int_{-\infty}^{\infty} \frac{2\,\pi}{1-2t} \frac{1}{\sqrt{\frac{2\,\pi}{1-2t}}} e^{[-\frac{1}{2}\,[\frac{(x-\frac{\mu}{1-2t})^2}{\frac{1}{1-2t}}]} dx$$

$$= \frac{1}{\sqrt{1-2t}} e^{[-\frac{\mu^2}{2}+\frac{\mu^2}{2\,(1-2t)}]}\, 1$$

$$= \frac{1}{(1-2t)^{\frac{1}{2}}} \, e^{-\frac{\mu^2}{2}[1-\frac{1}{1-2t}]}$$
$$= \frac{1}{(1-2t)^{\frac{1}{2}}} \, e^{-\frac{\mu^2}{2}[\frac{1-2t-1}{1-2t}]}$$
$$= \frac{1}{(1-2t)^{\frac{1}{2}}} \, e^{-\frac{\mu^2}{2} \, \frac{-2t}{1-2t}}$$
$$= \frac{1}{(1-2t)^{\frac{1}{2}}} \, e^{\frac{t\,\mu^2}{1-2t}} \qquad for \quad 1-2t > 0 \quad \Rightarrow \quad t < \frac{1}{2}$$

Now since $\mathbf{X_i's}$ are independent variables and

$$X_i's \sim N(\mu_i, 1) \qquad for \quad i = 1, 2, \ldots, n$$

then the mgf of $\quad Y = \sum_{i=1}^{n} X_i^2 \quad$ is

$$M_Y(t) = E[e^{tY}]$$
$$= M_{X_1^2}(t) M_{X_2^2}(t) \ldots M_{X_n^2}(t)$$
$$= \frac{1}{(1-2t)^{\frac{n}{2}}} \, e^{\frac{t}{1-2t} \sum_{i=1}^{n} \mu_i^2}$$

Therefore the Laplace transform is

$$L_Y(t) = \frac{1}{(1+2t)^{\frac{n}{2}}} \, e^{\frac{-t}{1+2t} \sum_{i=1}^{n} \mu_i^2}$$
$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \, e^{\frac{-2t}{1+2t} \, \frac{\theta}{2}}$$
$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \, e^{\frac{-\theta}{2} \, \frac{1-2t-1}{1+2t}}$$
$$= \frac{1}{(1+2t)^{\frac{n}{2}}} \, e^{\frac{\theta}{2} \, \frac{1}{1+2t} - 1}$$
$$\therefore \qquad L'_Y(t) = \frac{1}{(1+2t)^{\frac{n}{2}}} \, (\frac{\theta}{2}) \, \frac{d}{dt}[(1+2t)^{-1}] \, e^{[\frac{\theta}{2} \, \frac{1}{1+2t} - 1]} +$$
$$\frac{d}{dt}[(1+2t)^{-\frac{n}{2}}] \, e^{[\frac{\theta}{2} \, \frac{1}{1+2t} - 1]}$$
$$= [\frac{\frac{\theta}{2} \, [-2(1+2t)^{-2}]}{(1+2t)^{\frac{n}{2}}} - \frac{n}{2} \, (1+2t)^{[-\frac{n}{2}-1]} \, 2] \, e^{[\frac{\theta}{2} \, \frac{1}{1+2t} - 1]}$$
$$= [\frac{-\theta \, (1+2t)^{-2}}{(1+2t)^{\frac{n}{2}}} - \frac{n}{(1+2t)^{[\frac{n}{2}+1]}}] \, e^{[\frac{\theta}{2} \, \frac{1}{1+2t} - 1]}$$
$$= [\frac{-\theta}{(1+2t)^{\frac{n}{2}+2}} - \frac{n}{(1+2t)^{[\frac{n}{2}+1]}}] \, e^{[\frac{\theta}{2} \, \frac{1}{1+2t} - 1]}$$

$$= -\frac{1}{(1+2t)^{\frac{n}{2}+1}}[\frac{\theta}{(1+2t)} + n]\ e^{[\frac{\theta}{2}\ \frac{1}{1+2t} - 1]}$$

The hazard function is,

$$\begin{aligned} h(t) &= \frac{-L'_Y(t)}{L_Y(t)} \\ &= \frac{1}{1+2t}[\frac{\theta}{1+2t} + n] \\ &= \frac{\theta}{(1+2t)^2} + \frac{n}{1+2t} \end{aligned}$$

**The Associated Mixed Poisson Distribution**

Let

$$\begin{aligned} p_1(t) &= \theta & \qquad p_2(t) &= n \\ c_1(t) &= 2 & \qquad c_2(t) &= 2 \end{aligned}$$

Therefore,

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1t)} + \frac{p_2}{(1+c_2t)^2} \quad for \quad p > 0, \quad and \quad c > 0$$

which is the sum of the hazard function of exponential-gamma and the Polya-Aeppli hazard function as indicated in chapter 6.

Therefore,

$$\begin{aligned} \theta_1(t) &= \frac{p_1}{c_1}\ In(1+c_1t) & \qquad \theta_2(t) &= \frac{p_2}{c_2}[1-(1+c_2t)^{-1}] \\ \theta_1(t-ts) &= \frac{p_1}{c_1}\ In(1+c_1t-c_1ts) & \qquad \theta_2(t-ts) &= \frac{p_2}{c_2}[1-(1+c_2t-c_1ts)^{-1}] \end{aligned}$$

The pgf of the convolution is

$$H(s,t) = e^{-[\frac{p_1}{c_1}\ In(1+c_1t-c_1ts]}\ e^{\frac{p_2}{c_2}[1-(1+c_2t-c_2ts)^{-1}]}$$

and the pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G^{(x)}(s,t) = \frac{1}{\theta(t)}\ [(x-1)!\ \frac{p_1}{c_1}\ (\frac{c_1t}{1+c_1t-c_1ts})^x + \frac{p_2}{c_2}\ x!\ (\frac{c_2t}{(1+c_2t-c_2ts)})^x\ \frac{1}{(1+c_2t}$$

The coefficients of $G(s,t)$ are

$$g_0(t) = 0$$
$$g_x(t) = \frac{1}{\theta(t)} [\frac{1}{x} \frac{p_1}{c_1} (\frac{c_1 t}{1+c_1 t)^x} + \frac{p_2}{c_2} (\frac{c_2 t}{(1+c_2 t)})^x \frac{1}{(1+c_2 t)}] \quad for \quad x = 1,2,..$$

The recursive form for the convolution of compound Poisson distributions is

$$np_n(t) = \sum_{x=1}^{n} [\frac{p_1}{c_1} (\frac{c_1 t}{1+c_1 t})^x + \frac{p_2}{c_2} x (\frac{c_2 t}{(1+c_2 t)})^x \frac{1}{(1+c_2 t)}] \, p_{n-x}(t) \qquad n = 1,2,3,\ldots$$

### 6.10.2 In the equation (5.23) $a_1 = 1-\alpha$ and $a_2 = 2$

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1 t)^{1-\alpha}} + \frac{p_2}{(1+c_2 t)^2} \quad \text{for} \quad p > 0, \quad \text{and} \quad c > 0$$

which is the sum of the hazard function of exponential-Hougaard and the Polya-Aeppli hazard function.

Therefore,

$$\theta_1(t) = \frac{p_1}{\alpha\, c_1} [(1+c_1 t)^\alpha - 1] \qquad \theta_2(t) = \frac{p_2}{c_2}[1-(1+c_2 t)^{-1}]$$
$$\theta_1(t-ts) = \frac{p_1}{\alpha\, c_1} [(1+c_1 t - c_1 ts)^\alpha - 1] \qquad \theta_2(t-ts) = \frac{p_2}{c_2}[1-(1+c_2 t - c_2 t$$

The pgf of the convolution is

$$H(s,t) = e^{-\theta_1(t-ts)} e^{-\theta_2(t-ts)}$$
$$= e^{-\frac{p_1}{\alpha\, c_1} [(1+c_1 t - c_1 ts)^\alpha - 1]} \, e^{-\frac{p_2}{c_2} [1-(1+c_2 t - c_2 ts)^{-1}]}$$

Therefore the sum of the hazard function of exponential-Hougaard and the Polya-Aeppli hazard function give rise to the convolution of the Poisson-Hougaard distribution and the Polya-Aeppli distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)}$$

$$= 1 - \frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\ [(1+c_1t-c_1ts)^{\alpha}-1] + \frac{p_2}{c_2}[1-(1+c_2t-c_2ts)^{-1}])$$

$$G'(s,t) = -\frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\alpha(-c_1t)(1+c_1t+c_1ts)^{\alpha-1} - \frac{p_2}{c_2}(-1)\ (-c_2t)^1\ (1+c_2t-$$

$$G''(s,t) = -\frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\alpha(\alpha-1)(-c_1t)^2(1+c_1t+c_1ts)^{\alpha-2} - \frac{p_2}{c_2}(-1)(-2)\ (-c_2t)^2\ (1+c_2t-c_2ts)^{-3}]$$

$$G'''(s,t) = -\frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\alpha(\alpha-1)(\alpha-2)(-c_1t)^3(1+c_1t+c_1ts)^{\alpha-3} - \frac{p_2}{c_2}(-1)(-2)(-3)\ (-c_2t)^3\ (1+c_2t-c_2ts)^{-4}]$$

$$\therefore \qquad G^{(x)}(s,t) = \frac{1}{\theta(t)}\ [\frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t+c_1ts})^x\ (1+c_1t+c_1ts)^{\alpha} + \frac{p_2}{c_2}\ x!\ (c_2t)^x\ (1+c_2t-c_2ts)^{-(x+1)}]$$

$$= \frac{1}{\theta(t)}\ (1+c_1t+c_1ts)^{\alpha}\ \frac{p_1}{\alpha\ c_1}\ x!\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t+c_1ts})^x + \frac{1}{\theta(t)}\ \frac{1}{(1+c_2t-c_2ts)}\ \frac{p_2}{c_2}\ x!\ (\frac{c_2t}{(1+c_2t-c_2ts)})^x$$

The coefficients of $G(s,t)$ are

$$g_0(t) = 0$$

$$g_x(t) = \frac{1}{\theta(t)}\ \frac{p_1}{\alpha\ c_1}\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t})^x\ (1+c_1t)^{\alpha} + \frac{1}{\theta(t)}\ \frac{p_2}{c_2}\ (\frac{c_2t}{(1+c_2t})^x\ \frac{1}{(1+c_2t)} \qquad x = 2,3,\ldots$$

The recursive form for the convolution of compound Poisson distribution is:

$$np_n(t) = \theta(t)\ \sum_{x=1}^{n}\ x\ g_x(t)\ p_{n-x}(t) \qquad n = 1,2,\ldots$$

$$= \theta(t)\ \sum_{x=1}^{n}\ x\ [\frac{1}{\theta(t)}\ \frac{p_1}{\alpha\ c_1}\ \binom{\alpha}{x}\ (\frac{-c_1t}{1+c_1t})^x\ (1+c_1t)^{\alpha} + \frac{1}{\theta(t)}\ \frac{p_2}{c_2}\ (\frac{c_2t}{(1+c_2t})^x\ \frac{1}{(1+c_2t)}]\ p_{n-x}(t)$$

### 6.10.3 In the equation (5.23) $a_1 = 1-\alpha$ and $a_2 \to \infty$

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1t)^{1-\alpha}} + \lim_{a\to\infty}\ p_2\ (1+c_2t)^{-a} \quad for \quad p > 0, \quad and \quad c$$

$$
\begin{aligned}
&= \frac{p_1}{(1+c_1t)^{1-\alpha}} + \lim_{a\to\infty} p_2 \sum_{k=0}^{\infty} \binom{-a}{k} (c_2t)^k \\
&= \frac{p_1}{(1+c_1t)^{1-\alpha}} + \lim_{a\to\infty} p_2 \sum_{k=0}^{\infty} \frac{-a\,(-a-1)\,(-a-2)\ldots[-a-(k-1}{k!} \\
&= \frac{p_1}{(1+c_1t)^{1-\alpha}} + \lim_{a\to\infty} p_2 \sum_{k=0}^{\infty} \frac{(-1)^k\,a\,(a+1)\,(a+2)\ldots[a+(k-1)}{k!} \\
&= \frac{p_1}{(1+c_1t)^{1-\alpha}} + \lim_{a\to\infty} p_2 \sum_{k=0}^{\infty} \frac{(-1)^k\,a^k\,(1+\frac{1}{a})\,(1+\frac{2}{a})\ldots[(1+\frac{(k-1)}{a}}{k!} \\
&= \frac{p_1}{(1+c_1t)^{1-\alpha}} + p_2 \sum_{k=0}^{\infty} \lim_{a\to\infty} \frac{(-a)^k\,1.\,(c_2t)^k}{k!} \\
&= \frac{p_1}{(1+c_1t)^{1-\alpha}} + p_2 \sum_{k=0}^{\infty} \lim_{a\to\infty} \frac{(-ac)^k\,(t)^k}{k!}
\end{aligned}
$$

Let,

$$
\begin{aligned}
b &= ac \quad as \quad a \to \infty \\
h(t) &= \frac{p_1}{(1+c_1t)^{1-\alpha}} + p_2 \sum_{k=0}^{\infty} \frac{(-bt)^k}{k!} \\
h(t) &= \frac{p_1}{(1+c_1t)^{1-\alpha}} + p_2\, e^{-bt}
\end{aligned}
$$

where the second hazard function is that of the Gompertz distribution.

Therefore,

$$
\begin{aligned}
\theta_1(t) &= \frac{p_1}{\alpha\, c_1}\,[(1+c_1t)^{\alpha}-1] \\
\theta_1(t-ts) &= \frac{p_1}{\alpha\, c_1}\,[(1+c_1t-c_1ts)^{\alpha}-1] \qquad\qquad \theta_2(t-ts) = \frac{p_2}{b}\,[1-e^{-bt(1-s)}]
\end{aligned}
$$

The pgf of the convolution is:

$$
\begin{aligned}
H(s,t) &= e^{-\theta_1(t-ts)} e^{-\theta_2(t-ts)} \\
&= e^{-\frac{p_1}{\alpha\, c_1}\,[(1+c_1t-c_1ts)^{\alpha}-1]}\; e^{-\frac{p_2}{b}\,[1-e^{-bt(1-s)}]}
\end{aligned}
$$

But

$$
H_1(s,t) = e^{-p_1[t-ts]} = e^{-\lambda_1[1-G_1(s,t)]}
$$

is the pgf of Poisson with parameter $p_1t$ where $G_1(s,t) = s$ is the pgf of the iid random variables.

On the other hand,

$$\begin{aligned} H_2(s,t) &= e^{-\frac{p_2}{b}\,[1-e^{-bt(1-s)}]} \\ &= e^{-\lambda_2[1-G_2(s,t)]} \end{aligned}$$

is also pgf of Poisson with parameter $\frac{p_2}{b}$ where $G_2(s,t) = e^{-bt(1-s)}$ is the pgf of the iid random variables and it is also pgf of Poisson with parameter $bt$.

Therefore the sum of hazard functions of exponential distribution and that of the Gompertz distribution give rise to the convolution of the Poisson distribution and the Poisson-Poisson distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \frac{\frac{p_1}{\alpha\, c_1}\,[(1+c_1t-c_1ts)^{\alpha}-1] + \frac{p_2}{b}\,[1-e^{-bt(1-s)}]}{\theta(t)}$$

which is zero-truncated Poisson distribution with parameter $bt$

Since Gompertz distribution is an exponential mixture, the survival function is the Laplace transform of the mixing distribution and hence,

$$\begin{aligned} S(t) &= L_{\Lambda}(t) = p_0(t) \\ S(t) &= e^{-\int_0^t h(x)\,dx} = e^{-\theta(t)} \\ &= e^{-p_1t+\frac{p_2}{b}\,[e^{-bt}-1]} \\ &= e^{-p_1t}\; e^{-p_1t}\; e^{-\frac{p_2}{b}}\; e^{\frac{p_2}{b}\,e^{-bt}} = e^{-\frac{p_2}{b}} \sum_{j=0}^{\infty} [\frac{p_2}{b}\; e^{-bt}]^j\; \frac{1}{j!} \\ &= e^{-p_1t}\; e^{-\frac{p_2}{b}} \sum_{j=0}^{\infty} [\frac{p_2}{b}]^j\; e^{-btj}\; \frac{1}{j!} = \sum_{j=0}^{\infty} e^{-btj}\; [e^{-\frac{p_2}{b}}\; \frac{[\frac{p_2}{b}]^j}{j!}] \\ &= e^{-p_1t} \sum_{j=0}^{\infty} e^{-btj}\; p_j \\ S(t) &= \sum_{j=0}^{\infty} e^{-t[bj+p_1]}\; p_j \end{aligned}$$

where $\quad p_j = e^{-\frac{p_2}{b}}\, \frac{[\frac{p_2}{b}]^j}{j!} \quad j = 0, 1, 2, \ldots \quad$ is Poisson with parameter $\frac{\mathrm{p}_2}{\mathrm{b}}$

But

$$p_0(t) = \sum_{j=0}^{\infty} e^{-t[bj+p_1]}\; p_j$$

$$p_0'(t) = \sum_{j=0}^{\infty} (-[bj+p_1])^1\, e^{-t[bj+p_1]}\, p_j$$

$$p_0''(t) = \sum_{j=0}^{\infty} (-[bj+p_1])^2\, e^{-t[bj+p_1]}\, p_j$$

Therefore,

$$p_0^{(n)}(t) = \sum_{j=0}^{\infty} (-[bj+p_1])^n\, e^{-t[bj+p_1]}\, p_j$$

$$p_n(t) = (-1)^n \frac{t^n}{n!} \sum_{j=0}^{\infty} (-[bj+p_1])^n\, e^{-t[bj+p_1]}\, p_j$$

$$p_n(t) = \sum_{j=0}^{\infty} (-1)^n \frac{t^n}{n!} (-[bj+p_1])^n\, e^{-t[bj+p_1]}\, p_j$$

$$p_n(t) = \sum_{j=0}^{\infty} \frac{(t[bj+p_1])^n}{n!}\, e^{-t[bj+p_1]}\, e^{-\frac{p_2}{b}}\, \frac{[\frac{p_2}{b}]^j}{j!}; \qquad j = 0, 1, 2, \ldots$$

which is Poisson mixture of Poisson distribution also known as Neyman Type A distribution.

The recursive formula form for the convolution of the compound Poisson distributions is:

$$\begin{aligned}(n+1)\, p_{n+1}(t) &= \theta(t) \sum_{i=0}^{n} (i+1)\, g_{i+1}(t)\;\; p_{n-i}(t) \\ &= \frac{p_2}{b}\, e^{-bt} \sum_{i=0}^{n} (i+1)\, \frac{(bt)^{i+1}}{(i+1)!}\, p_{n-i}(t)\end{aligned}$$

$$(n+1)\, p_{n+1}(t) = p_2 t\, e^{-bt} \sum_{i=0}^{n} \frac{(bt)^i}{i!}\, p_{n-i}(t) \qquad n = 0, 1, 2 \ldots$$

By iteration,

For n=0

$$p_1 t = p_2 t\, e^{-bt}\, p_0(t)$$

For n=1

$$\begin{aligned}2\, p_2 t &= p_2 t\, e^{-bt} \sum_{i=0}^{1} \frac{(bt)^i}{i!}\, p_{1-i}(t) \\ &= p_2 t\, e^{-bt}\, [p_1 t + bt\, p_0(t)] = p_2 t\, e^{-bt}\, [p_2 t\, e^{-bt}\, p_0(t) + bt\, p_0(t)]\end{aligned}$$

$$2\, p_2 t = [(bt)^0 (p_2 t)^2\, e^{-2bt} + (bt)^1\, (p_2 t)^1\, e^{-bt}]\, p_0(t)]$$

$$p_2 t = [\frac{1}{2}\ (bt)^0 (p_2 t)^2\ e^{-2bt} + \frac{1}{2}\ (bt)^1\ (p_2 t)^1\ e^{-bt}]\ p_0(t)]$$

For n=2

$$3\ p_3(t) = p_2 t\ e^{-bt}\ \sum_{i=0}^{2} \frac{(bt)^i}{i!}\ p_{2-i}(t)$$

$$= p_2 t\ e^{-bt}\ [p_2 t + bt\ p_1 t + \frac{(bt)^2}{2!}]\ p_0(t)$$

$$= p_2 t\ e^{-bt}\ [\frac{1}{2}\ (bt)^0 (p_2 t)^2\ e^{-2bt} + \frac{1}{2}\ (bt)^1\ (p_2 t)^1\ e^{-bt} + bt\ p_2 t\ e^{-bt} + \frac{(b}{2}$$

$$= p_2 t\ e^{-bt}\ [\frac{1}{2}\ (bt)^0 (p_2 t)^2\ e^{-2bt} + \frac{3}{2}\ (bt)^1\ (p_2 t)^1\ e^{-bt} + \frac{1}{2!}\ (bt)^2]\ p_0(t)$$

$$= [\frac{1}{2}\ (bt)^0 (p_2 t)^3\ e^{-3bt} + \frac{3}{2}\ (bt)^1\ (p_2 t)^2\ e^{-2bt} + \frac{1}{2}\ (bt)^2 (p_2 t)^1\ e^{-bt}]\ p_0(t$$

$$p_3(t) = [\frac{1}{6}\ (bt)^0 (p_2 t)^3\ e^{-3bt} + \frac{3}{6}\ (bt)^1\ (p_2 t)^2\ e^{-2bt} + \frac{1}{6}\ (bt)^2 (p_2 t)^1\ e^{-bt}]\ p_0(t$$

For n=3

$$4\ p_4(t) = p_2 t\ e^{-bt}\ \sum_{i=0}^{3} \frac{(bt)^i}{i!}\ p_{3-i}(t)$$

$$4\ p_4(t) = p_2 t\ e^{-bt}\ [p_3(t) + bt\ p_2 t + \frac{1}{2!}\ (bt)^2\ p_1 t + \frac{1}{3!}\ (bt)^3\ p_0(t)]$$

$$= p_2 t\ e^{-bt}\ [\frac{1}{6}\ (bt)^0 (p_2 t)^3\ e^{-3bt} + \frac{3}{6}\ (bt)^1\ (p_2 t)^2\ e^{-2bt} + \frac{1}{6}\ (bt)^2 (p_2 t)^1\ e$$

$$bt\ [\frac{1}{2}\ (bt)^0 (p_2 t)^2\ e^{-2bt} + \frac{1}{2}\ (bt)^1\ (p_2 t)^1\ e^{-bt}] + \frac{1}{2!}\ (bt)^2\ p_2 t\ e^{-bt} + \frac{1}{3}$$

$$= p_2 t\ e^{-bt}\ [\frac{1}{6}\ (bt)^0 (p_2 t)^3\ e^{-3bt} + \frac{6}{6}\ (bt)^1\ (p_2 t)^2\ e^{-2bt} +$$

$$\frac{7}{6}\ (bt)^2\ (p_2 t)^1\ e^{-bt} + \frac{1}{6}\ (bt)^3]\ p_0(t)$$

$$p_4(t) = [\frac{1}{24}\ (bt)^0 (p_2 t)^4\ e^{-4bt} + \frac{6}{24}\ (bt)^1\ (p_2 t)^3\ e^{-3bt} +$$

$$\frac{7}{24}\ (bt)^2\ (p_2 t)^2\ e^{-2bt} + \frac{1}{24}\ (bt)^3\ p_2 t\ e^{-bt}]\ p_0(t)$$

For n=4

$$5\ p_5(t) = p_2 t\ e^{-bt}\ \sum_{i=0}^{4} \frac{(bt)^i}{i!}\ p_{4-i}(t)$$

$$5\ p_5(t) = p_2 t\ e^{-bt}\ [p_4(t) + bt\ p_3(t) + \frac{1}{2!}\ (bt)^2\ p_2 t + \frac{1}{3!}\ (bt)^3\ p_1 t + \frac{1}{24}\ (bt)^4\ p_0($$

$$= p_2 t\ e^{-bt}\ [\frac{1}{24}\ (bt)^0 (p_2 t)^4\ e^{-4bt} + \frac{6}{24}\ (bt)^1\ (p_2 t)^3\ e^{-3bt} +$$

$$\frac{7}{24}\ (bt)^2\ (p_2 t)^2\ e^{-2bt} + \frac{1}{24}\ (bt)^3\ p_2 t\ e^{-bt} + \frac{1}{6}\ (bt)^1 (p_2 t)^3\ e^{-3bt} +$$

$$\frac{3}{6}\ (bt)^2\ (p_2 t)^2\ e^{-2bt} + \frac{1}{6}\ (bt)^3 (p_2 t)^1\ e^{-bt} + \frac{1}{4}\ (bt)^2 (p_2 t)^2\ e^{-2bt} +$$

$$\frac{1}{4}\ (bt)^3\ (p_2 t)^1\ e^{-bt} + \frac{1}{3!}\ (bt)^3\ p_2 t\ e^{-bt}\ + \frac{1}{24}\ (bt)^4]\ p_0(t)$$

$$= \frac{1}{24}\ (bt)^0 (p_2 t)^5\ e^{-5bt} + \frac{10}{24}\ (bt)^1\ (p_2 t)^4\ e^{-4bt} +$$
$$\frac{25}{24}\ (bt)^2\ (p_2 t)^3\ e^{-3bt} + \frac{15}{24}\ (bt)^3\ (p_2 t)^2\ e^{-2bt} + \frac{1}{24}\ (bt)^4\ (p_2 t)^1]\ p_0($$

$$= \frac{1}{4!}\ \sum_{k=0}^{5} (\Phi(k,4))\ (bt)^{5-k} (p_2 t)^k\ e^{-kbt}$$

$$p_5(t) = \sum_{k=0}^{5} \frac{1}{5!}\ (\Phi(k,4))\ (bt)^{5-k} (p_2 t)^k\ e^{-kbt}\ p_0(t)$$

In general

$$p_n(t) = \sum_{k=0}^{n} \frac{1}{n!}\ (\Phi(k,n)\ (bt)^{n-k} (p_2 t)^k\ e^{-kbt}\ p_0(t)$$

$$= \sum_{k=0}^{n} (\Phi(k,n)\ \frac{1}{n!}\ (bt)^{n-k} (p_2 t)^k\ e^{-kbt}\ e^{-\frac{p_2}{b}}\ e^{\frac{p_2}{b}\ e^{-bt}}$$

$$= \sum_{k=0}^{n} (\Phi(k,n)\ \frac{1}{n!}\ (bt)^n\ e^{-kbt}\ (\frac{p_2}{b})^k\ e^{-\frac{p_2}{b}}\ e^{\frac{p_2}{b}\ e^{-bt}}$$

$$= \sum_{k=0}^{n} (\Phi(k,n)\ \frac{k!}{k^n}\ e^{\frac{p_2}{b}\ e^{-bt}}\ [\frac{(bkt)^n}{n!}\ e^{-kbt}]\ [\frac{(\frac{p_2}{b})^k}{k!}\ e^{-\frac{p_2}{b}}]$$

As $n \to \infty$

$$p_n(t) = \sum_{k=0}^{\infty} \frac{(btk)^n}{n!} \; e^{btj} \; [\frac{[\frac{p_2}{b}]^k}{k!} \; e^{-\frac{p_2}{b}}] \qquad k = 0, 1, 2 \ldots.$$

which is Poisson mixture of Poisson distribution also known as Neyman Type A distribution.

An alternative way of obtaining the above result is to consider the Makeham Law, which states as follows:

The force of mortality is the Gompertz failure rate plus an age-independent component that accounts for external causes of mortality and therefore the hazard rate function is:

$$\lambda(t) = \alpha \; e^{\beta t} + \mu \qquad \text{where } \alpha, \; \beta, \; \mu > 0$$

or

$$h(t) = p_1 + p_2 \; e^{-bt}$$

where $p_1$ can be considered as a hazard function of the exponential distribution.

It has been established that the cumulative hazard function is

$$\theta(t) = \delta t + \frac{p_2}{b} \; (1 - e^{-bt})$$

and that the exponential mixture with hazard function of the form

$$h(t) = p_1 + \lim_{a \to \infty} \frac{p_2}{(1 + c_2 t)^a} \quad for \quad p > 0, \quad and \quad c > 0$$

characterizes a convolution of Poisson distribution and Poisson-Poisson (Neyman type A)which are infinitely divisible mixed Poisson distributions that are also convolutions of compound Poisson distributions.

## 6.11 Concluding Remarks

Whereas corresponding hazard functions for compound binomial, geometric and negative binomial distributions with exponential iid random variables, are differences of hazard functions of Pareto distributions, the hazard functions for compound binomial, geometric, and negative binomial distributions with gamma iid random variables, are differences of a hazard function of a Pareto distribution and another form of a hazard function.

In the case of compound shifted geometric and shifted negative binomial distributions, with exponential iid random variables, the hazard functions are single hazard functions of Pareto distribution.

For compound shifted geometric and shifted negative binomial distributions, with gamma iid random variables, the hazard functions are sums of a hazard function of a Pareto distribution and a hazard function of another form of a hazard function, while the compound Poisson distribution with exponential iid random random variables, has a single hazard function associated with Polya-Aeppli (mixed Poisson) distribution.

The compound Poisson distribution with gamma iid random variables has hazard function which is a parameterization of the hazard function associated with Hofmann (mixed Poisson) distributions and the hazard function of exponential-Hougaard distribution is also a parameterization of the hazard function associated with Hofmann distributions.

The sums of a hazard function of exponential - Hougaard distribution and other forms of Hofmann hazard functions are considered and associated convolutions of mixed Poisson distributions and obtained.

# Chapter 7

# SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

## 7.1 Introduction

The chapter has a summary of the accomplishments of the thesis, conclusions from results of the study, recommendations on how results could be used and recommendations on the problems for further study.

## 7.2 Summary

Type I and type II exponential mixtures have been constructed using mixing distributions that are not part of the exponential mixtures literature. Moments of the mixtures have been obtained using both the Mellin transform technique and the conditional expectation approach. Mixed Poisson distribution has been defined in terms of the hazard function of an exponential mixture and in so doing the link between exponential and Poisson mixtures has been explored. It has also been shown that a sum of hazard functions of exponential mixtures gives rise to a convolution of infinitely divisible Poisson mixtures, hence a convolution of compound Poisson distributions.

Given the importance of hazard functions of exponential mixtures in the development of these models, Laplace transforms of probability density functions in continuous compound distributions and mean excess loss have been considered as alternative methods of obtaining these hazard functions.

## 7.3 Conclusions

The exponential mixtures constructed are either in explicit form or in terms of special functions, thus the modified Bessel function of third kind and confluent hyper-geometric function. It has been shown that where the Mellin transform method fails, the conditional expectation technique serves as an alternative technique.

The definition of mixed Poisson distribution in this study, in terms of the the hazard function of the exponential mixture, is a generalization of the definition by Walhin and Paris (1999).

Hazard functions of exponential mixtures can be obtained using a various methods.

## 7.4 Recommendations

Whereas exponential distributions can be used in the field of life-testing, with lifetime represented by an exponential random variable and the failure rate (risk) is assumed to be a variable, . The exponential-inverse gamma mixture (Pareto distribution) can be used to model the size component of insurance claims where $X$ is the size of claim by each insured and $Y$ is the mean claim size of each insured such that the conditional distribution is $f(x|y)$. Since the mean claim size is different for the different policyholders, the prior distribution for $y$ can be the inverse gamma.

An application of the compound Poisson distributions could be in the probability distribution of the total cost $S(t) = X_1 + \cdots + X_{N(t)}$, where $N(t)$ is the number of flaws in a roll of length $t$, for example.

In this study the link between exponential and Poisson mixtures was explored for type I exponential mixtures whose hazard functions are in explicit form and have the pattern:

$$h(t) = \frac{p}{(1+ct)^a} \quad for \;\; p > 0, \; c > 0 \text{ and } a \geq 0$$

Further work is to explore the link between exponential and Poisson mixtures for other families of hazard functions of exponential mixtures, which do not necessarily follow the above pattern.

# References


[1] Benktander, G. and Segerdahl, C. (1960). "**On the Analytical Representation of Claim Distributions with Special Reference to Excess of Loss Reinsurance**". *Proceedings of the XVIth International Congress of Actuaries, Brussels*, pp. 626–646.

[2] Bhattacharya, S. K. and Holla, M. S. (1965) "**On a Life Distribution with Stochastic Deriavtions in the Mean**" *Annals of the Institute of Statistical Mathematics,* Vol. 17, pp. 97-104.

[3] Bhattacharya, S. K. and Holla, M. S. (1965) "**On a Discrete Distribution with Special Reference to the Theory of Accident Proneness**" *Journal of the American Statistical Association,* Vol. 60, No. 312, pp. 1060-1066.

[4] Bhattacharya, S. K. (1967) "**Bayesian Approach to Life Testing and Reliability Estimation**" *Journal of the American Statistical Association,* Vol. 62, No. 317, pp. 48-62.

[5] Bhattacharya, S. K. and Kumar, S. (1986) "**E-IG Model Life-Testing Calcutta Statistical Association**" *Bulletin,* Vol. 35, pp. 85-90.

[6] Bhattacharya, S. K., (1966) " **A modified Bessel Function Model in Life Testing**" *Metrika* Vol. 11, pp. 131-144.

[7] Karlis, D. and Xekalaki, E. (2005) "**Mixed Poisson Distributions**" *International Statistical Review*, Vol. 73, No. 1, pp. 35-58.

[8] Drozdenko, M. and Yadrenko M. (2012) "**On some generalizations of mixtures of exponential distributions**" *Seminar Paper*

[9] Feller, W. (1968) " **An Introduction to probability Theory and Its Applications**" *John Wiley and Sons*, Vol. 1, 3rd Edition.

[10] Feller, W. (1971) "**An Introduction to Probability Theory and Its Applications**". *John Wiley and Sons*, Vol. 2.

[11] Frangos N. E. and Karlis, D. (2004) " **Modeling losses using an exponential-Inverse Gaussian distribution** *Insurance: Mathematics and Economics*" Vol. 35, pp. 53–67.

[12] Frangos N. E., and Vrontos, S. (2001) " **Design of optimal bonus-malus systems with a frequency and a severity component on an individual basis in automobile insurance**" *Astin Bulletin* Vol. 33, pp. 1-22.

[13] Gompertz, B. (1825) " **On the nature of the function expressive of the law of human mortality.**" *Philosophical Transactions of the Royal Society of London*, Vol. 115, pp. 513-583.

[14] Hesselager O., Wang and Willmot, G. (1998) " **Exponential and Scale Mixtures and Equilibrium Distributions**"*Scandinavian Actuarial Journal*, Vol. 2, pp. 125 – 142.

[15] Hougaard P. (1986) " **Survival Models for Heterogeneous Populations Derived from Stable Distribution**" *Biometrika,* Vol. 73, No. 2, pp. 387-396.

[16] Johnson, N. L., KEMP, A. W and Kotz, S. (2005) "**Univariate Discrete Distributions**" *Wiley – Inter-science*, 3rd Edition.

[17] Klugman, S. A; Panjer, H. H. and Willmot, G. E. (2008) "**Loss Models: From Data to Decisions**" *3rd Edition, John Wiley and Sons*

[18] McKean, R. V., (2012), "**A Note On Mixed Distributions**" *Casualty Actuarial Society E-Forum, .*

[19] Maceda, E. C. (1948) "**On The Compound and Generalized Poisson Distributions.**" *Annals of Mathematical Statistics*, Vol. 19, pp. 414 – 416

[20] McNolty, F.; Doyle, J. and Hansen E. (Nov., 1980) "**Properties of the Mixed Exponential Failure Process**" *Technometrics, Taylor and Francis, Ltd.* Vol. 22, No. 4 , pp. 555-565

[21] Nadarajah, S. and Kotz, S. (2006) "**Compound mixed Poisson distributions I**" *Scandinavian Actuarial Journal*,Vol. 3, pp. 141-162.

[22] Ospina, A. V. "**A Simple Proof of Feller's Characterizations of the Compound Poisson Distribution.**" *Insurance Mathematics and Economics*,Vol.6, No. 1, pp. 63-64

[23] Panjer, M. (1981) "**Recursive Evaluation of a Family of Compound Distributions**" *ASTIN Bulletin*, Vol. 18,: pp. 57–68.

[24] Walhin, J. F. and Paris, J. (1999) "**Using Mixed Poisson Processes in Connection with Bonus – Malus Systems**" *Astin Bulletin*, Vol 29, No. 1,: pp. 81 – 99.

[25] Wakoli, M. W. and Ottieno, J. A. M. (2015) "**Mixed Poisson Distributions Associated with Hazard Functions of Exponential Mixtures**" *Mathematical Theory and Modeling*, ISSN 2224-5804 (Paper), ISSN 2225-0522 (Online), Vol. 5, No. 6, pp. 209-244.

[26] Wakoli, M. W. and Ottieno, J. A. M. (2015) "**Sums of Hazard Functions of Exponential Mixtures and Associated Convolutions of Mixed Poisson Distributions**" *Mathematical Theory and Modeling*, ISSN 2224-5804 (Paper), ISSN 2225-0522 (Online), Vol. 5, No. 9, pp. 175-197.

[27] Walhin, J.F and Paris, J. (2002) "**A general family of over-dispersed probability laws**" *Belgian Actuarial Bulletin*, Vol. 2, No. 1,: pp. 1 – 8.

[28] Willmot, G. (1986) "**Mixed Compound Poisson Distributions**" *ASTIN Bulletin* Vol. 16,: pp. 559 – 579.